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A Mountain Pass Lemma and its implications regarding the uniqueness of constrained minimizers

Abstract We present a version of the classical Mountain Pass Lemma and explain how to combine it with constraint qualifications to prove that nonlinear programming problems have a unique local minimizer.

1. Introduction

This work presents metric and differential criteria to decide whether the nonlinear programming problem

$$\begin{aligned} &\min f(x) \\ \text{subject to}\ \ &g(x) \in G \end{aligned} \tag{1.1}$$

has a unique local minimizer. Our approach is general in the sense that the domain of the functions f and g and the set G may be a subset of a Banach space or a more general metric space. We discuss uniqueness in depth and generality and provide tools to verify it in specific problems.

Knowing that problem (1.1) has a unique local solution is helpful when we handle it numerically, because if one has this information then he can choose a simpler and more efficient algorithm to solve it. Uniqueness is also important from the modeling point of view. It gives us more assurance that our model is well posed and describes accurately the real world, where we often expect the solutions to be unique and clearly defined. For these practical reasons the statisticians and economists who wrote [32], [39] and [50] developed and studied criteria to prove the uniqueness of local solutions in the context of likelihood maximization and successfully applied their criteria to significant problems in their fields. This paper generalizes and explains the mathematics behind their results, but to appreciate the effectiveness of their approach you should read at least [32].

Uniqueness of local minimizers is related to the connectedness of the sets

$$f_{<c} = \{x \text{ with } f(x) < c\} \qquad \text{and} \qquad f_{\leq c} = \{x \text{ with } f(x) \leq c\},$$

where the concept of connectedness is formalized in the usual topological sense:

Definition 1.1 *Let F be a topological space. We say that $C \subset F$ is connected if for any pair of open sets A and B such that $C \subset A \cup B$ and $A \cap B \cap C = \emptyset$ we have that either $A \cap C = \emptyset$ or $B \cap C = \emptyset$.*

We propose the Connection Lemma as a tool to analyze the connectedness of the sets $f_{<c}$. This lemma is an adaptation of the classical Mountain Pass Lemma. The next theorems relate the uniqueness of local minimizers, the connectedness of the sets $f_{<c}$ and the Connection Lemma:

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Theorem 1.1 *Let F be a metric space and let $f : F \to \mathbb{R}$ be continuous with strict local minimizers. If the sets $f_{\leq c}$ are compact then f has a unique local minimizer if and only if the sets $f_{<c}$ are connected.* □

Theorem 1.2 *Let F be a connected metric space and let $f : F \to \mathbb{R}$ be continuous with strict local minimizers. If the sets $f_{\leq c}$ are compact then the sets $f_{<c}$ are connected if and only if f has no Connection Points.* □

A local minimizer x is strict if it lies in an open set A such that $f(y) > f(x)$ for all $y \in A - \{x\}$. The vast majority of nonlinear programming problems are formulated in metric spaces and this is the level of generality of this work. The results presented here can be extended to more general topological spaces in a natural way, but we only discuss such extensions briefly in the last section in order to make the paper more readable and focused.

To understand theorem 1.2 you must learn what Connection Points are. You can think of them as points whose existence is proved by the Connection Lemma. The exact meaning is not relevant for the moment. Just be sure you understand that

$$\text{If the local minimizers are strict and the sets } f_{\leq c} \text{ are compact then}$$

$$\text{A unique local minimizer} \iff \text{The sets } f_{<c} \text{ are connected} \iff \text{No Connection Points.} \tag{1.2}$$

Under the hypothesis of the theorems above we can prove the uniqueness of local minimizers by showing that there are no Connection Points. This is the Connection Point argument for uniqueness and in the rest of this article we expand and explore it in detail. We prove and generalize the theorems above and explain the need for their hypothesis. We also provide tools to apply these theorems in typical constrained problems.

We introduce Connection Points in this work, but the applied paper [32] that motivated us presents a correct intuitive Connection Point argument and its authors understand the spirit of the Connection Lemma. Unfortunately, they focus on positive definite Hessians and talk superficially about Morse Theory, in what we now see as a red herring. While chasing this red herring we found the work of Hofer [22], [23], Fang [17], Ghoussoub & Preiss [18], Pucci & Serrin [43], [44], [45] and Ekeland & Ghoussoub [15] and realized that the Mountain Pass Lemma is a powerful tool prove the uniqueness of local minimizers. We then formulated our Connection Lemma based on their versions of the Mountain Pass Lemma. As a result, some of our theorems differ from theirs only in technical details, but our proofs and focus are different. We are concerned with the uniqueness of local minimizers for nonlinear programming problems; they follow Birkhoff [4], Ljusternik [29] [30] and Morse [37] and want to prove the existence of multiple solutions for variational problems.

The Connection Lemma is a slightly different version of the Mountain Pass Lemma, which is usually explained in terms of paths connecting points a and b which have $f(a)$ and $f(b)$ below a critical level c. Many authors consider the complete metric space

$$\Gamma(a,b) = \{x \in C([0,1],F) \text{ such that } x(0) = a \text{ and } x(1) = b\}$$

and use Ekeland's variational principle to analyze the minimizers of the functional $\Phi : \Gamma(a,b) \to \mathbb{R}$ given by

$$\Phi(x) = \max_{t \in [0,1]} f(x(t)).$$

From this analysis they prove the existence of Mountain Passes (see [7]). We propose an alternative view. For us the Mountain Pass Lemma is only incidentally related to paths. What really matters is connectedness, not path connectedness. Actually, Mountain Passes are truly a topological phenomenom and hypothesis like the Palais-Smale Condition are nothing but differential lenses to observe them.

We organized this paper in an increasing order of generality and complexity. Section 2 is about the uniqueness of local minimizers in $\mathbb{R}^n$. We state lemmas and theorems and show how they can be combined with constraint qualifications in order to prove the uniqueness of local minimizers for nonlinear programming problems in a finite number of variables. Section 2 motivates and prepares you for the more abstract approach of the next sections. It shows that there are two branches in the analysis of uniqueness of local minimizers: (i) functions with at least one disconnected level set, which certainly have multiple local minimizers (ii) functions with connected level sets may have multiple local minimizers, but their analysis is a bit simpler. In section 3 we handle case (i) for Banach spaces and section 4 covers complete metric spaces. The basic result in these sections is the Connection Lemma. In section 5 we analyze functions with connected level sets. Finally, in section 6 we prove all lemmas and theorems and present some results in more general topological spaces.

2. Uniqueness of local minimizers in $\mathbb{R}^n$

In this section we present basic principles to decide whether the classical nonlinear programming problem

$$\begin{aligned} &\min f(x) \\ \text{subject to } &h_i(x) = 0, \quad i = 1, \ldots m, \\ &g_j(x) \leq 0, \quad j = 1, \ldots, p \end{aligned} \tag{2.1}$$

has a unique local minimizer in the feasible region F. We assume that f is bounded below in F, ie., there exists μ such that $f(x) \geq \mu$ for all $x \in F$. Our goal is to motivate the abstract sections that follow and provide useful results for readers which are only concerned with this problem and do not have the interest or the knowledge to consider more general ones.

We believe the uniqueness question should be approached in two steps: first we must find out whether the sets $f_{<c} = \{x \in F \text{ with} f(x) < c\}$ are connected. If $f_{<c}$ is disconnected then there exist disjoint open sets A_1 and A_2 such that $f_{<c} \subset A_1 \cup A_2$ and f has one local minimizer in A_1 and another in A_2. Therefore, if some set $f_{<c}$ is disconnected then f has multiple local minimizers. Once we pass this test we can handle the case in which all sets $f_{<c}$ are connected.

The Connection Lemma is a fundamental tool to analyze the connectedness of the sets $f_{<c}$. In its simplest form it is exactly the same as the Mountain Pass Lemma and they can be stated as follows:

Lemma 2.1 *If the continuous function $f : \mathbb{R} \to \mathbb{R}$ has two local minimizers u and w then it also has a local maximizer $v \in (u, w)$.* □

People working in partial differential equations and Hamiltonian systems use the local information that u and w are local minimizers and the Mountain Pass Lemma to prove the existence of v, which corresponds to yet another solution for their variational problems. Ambrosetti and Rabinowitz introduced the Mountain Pass Lemma in [1] with this kind of application in mind and it was generalized in [15], [16], [17], [18], [22], [23], [43], [44], [45] and [48] with the same purpose.

We use the Connection Lemma in the other way around: if we prove that f has no local maximizers then lemma 2.1 tells us that f has at most one local minimizer. This can be very hard, but [32] and [39] present concrete situations in which it can be done in $F = \mathbb{R}^n$. However, in general the point v whose existence is proved by the Connection Lemma is not a local maximizer. In favorable situations it will be what Katriel, motivated by Hofer [23], defined as a Mountain Pass Point in [26]:

Definition 2.1 *Let F be a topological space and $f : F \to \mathbb{R}$ a function. The point $x \in F$ with $c = f(x)$ is a Mountain Pass Point if for every neighborhood N of x the set $f_{<c} \bigcap N$ is disconnected.* □

Our approach is slightly different from Hofer's and Katriel's. We are concerned with the connectedness of the sets $f_{<c}$ and its relation to the uniqueness of local minimizers. Our work is based on these definitions:

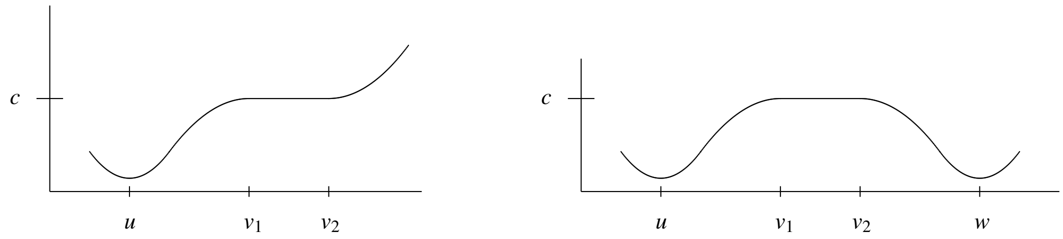
Definition 2.2 *Let F be a topological space and A, B and C be subsets of F. We say that $C = A \cup B$ is a partition of C if A and B are disjoint and not empty. If A and B are open we say that $C = A \cup B$ is an open partition and if A and B are closed we say that $C = A \cup B$ is a closed partition.* □

Definition 2.3 *Let F be a topological space and $f : F \to \mathbb{R}$ a function. The point $x \in F$ is a Connection Point if there exists an open partition $f_{<f(x)} = A \cup B$ such that $x \in \overline{A} \cap \overline{C}$ ($\overline{A}$ is the closure of A).* □

Fig. 1 The strict minimizers u and w and the Connection Point v.

Figure 1 presents the typical Connection Point: a local maximizer in $\mathbb{R}$ or a saddle in $\mathbb{R}^n$ for $n > 1$. Every Connection Point x is a Mountain Pass Point: if N is a neighborhood of x then we can partition $f_{<c} \cap N$ as $(N \cap A) \cup (N \cap B)$. However, there are examples of Mountain Pass Points which are not Connection Points.

Connection Points do not tell the whole story about the uniqueness of local minimizers. Figure 2 shows why the requirement of strict minimizers is essential in theorems 1.1 and 1.2. There are no Connection Points in this figure. However, in the left plot the sets $f_{<c}$ are connected and the degenerate local minimizers form a connected set with a compact closure which is attached to $f_{<c}$. The right plot contains an interval of local minimizers x with $f(x) = c$ connecting distinct components of $f_{<c}$.



A single global minimizer u and the Terrace (v_1, v_2): connected level sets but multiple local minimizers.

The components $A_1 = (-\infty, v_1)$ and $A_2 = (v_2, +\infty)$ of $f(x) < c$ and the Bridge $B = (v_1, v_2)$ connecting them.

Fig. 2 A Terrace and a Bridge.

Figures 1 and 2 describe the worst scenarios for continuous function with compact level sets. We now present technical definitions to analyze them:

Definition 2.4 *Let F be a topological space and $f : F \to \mathbb{R}$ a function. The set P is a Plateau for f at level c if*

(i) $P \subset f^{-1}(c)$,
(ii) *P is connected,*
(iii) *$\overline{P}$ has at least two elements.*
(iv) *The elements of P are local minimizers of f,*

Definition 2.5 *A Plateau P for f at level c is a Terrace if $\overline{P} \cap \overline{f_{<c}} \neq \emptyset$.*

Definition 2.6 *Let $f_{<c} = A_1 \cup A_2$ be an open partition with $\overline{A}_1 \cap \overline{A}_2 = \emptyset$. We say that $B = f_{\leq c} - \overline{A}_1 \cup \overline{A}_2$ is a Bridge if there are no sets B_1 and B_2 such that $B = B_1 \cup B_2$ and $\left(\overline{A}_1 \cup \overline{B}_1\right) \cap \left(\overline{A}_2 \cup \overline{B}_2\right) = \emptyset$.*

The interval $T = (v_1, v_2)$ in the left plot in Figure 2 is the simplest kind of Plateau: a path formed by local minimizers. The Plateau T is a Terrace because $\overline{T}$ touches $\overline{f_{<c}}$. The interval $B = (v_1, v_2)$ in the right plot in Figure 2 is a Bridge. It connects the components A_1 and A_2 of $f_{<c}$ and cannot be split in two parts B_1 and B_2 such that $\overline{A}_1 \cup \overline{B}_1$ and $\overline{A}_2 \cup \overline{B}_2$ yield a closed partition of $f_{\leq c}$.

The Bridges and Terraces above are the simplest ones to visualize. However, using Baire's theorem we can prove the existence of a Plateau $P \subset \mathbb{R}^2$ such that $\overline{P}$ is compact and contains no continuous paths (see [51]). In any case problems with Bridges and Plateaus are odd[1] and we must expect to have difficulties when trying to solve them numerically, as illustrated in the next lemma:

Lemma 2.2 *Let X be a metric space and let $f : X \to \mathbb{R}$ be a continuous function. If A is a Bridge or Plateau then there exists an injective function $p : (0,1) \to A$. Therefore, A has at least the cardinality of $\mathbb{R}$. In particular, if f has a Bridge or Terrace at level c then the set of non strict local minimizers for (1.1) at level c has at least the cardinality of $\mathbb{R}$.*

Terraces destroy the first equivalence in relation (1.2) and Bridges invalidate the second. In reality, for continuous functions with compact level sets we have that:

$$\text{A unique local minimizer} \;\not\Leftarrow\!\!\Rightarrow\; \text{The sets } f_{<c} \text{ are connected} \;\not\Leftarrow\!\!\Rightarrow\; \text{no Connection Points},$$

[1] [27], [28], [36] and [52] illustrate how complex Bridges and Plateaus can be.

but we can still relate Bridges and Connection Points to the change in connectedness of the level sets [2]:

Theorem 2.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. If the sets $f_{<c}$ are bounded, there exists $a \in \mathbb{R}$ such that $f_{<a}$ is disconnected and $c \in (a, +\infty]$ such that $f_{<c}$ is connected then f has a Connection Point or a Bridge in $f^{-1}([a,c))$.* □

As a consequence of the existence of Bridges and Terraces, in order to use Connection Point arguments to prove the uniqueness of local minimizers we show that f satisfies the following:

Definition 2.7 *Let F be a topological space and let $f : F \to \mathbb{R}$ be a continuous function. We say that f satisfies the* **Connectedness Alternative** *if it has a global minimizer z and at least one of these alternatives holds:*

(i) z is the only local minimizer,
(ii) f has a Connection Point,
(iii) f has a Bridge,
(iv) For every $c \in \mathbb{R}$ the sets $f_{<c}$ and $f_{\leq c}$ are connected. There is a local minimizer $w \neq z$ and for all such w's
   (a) If $f(w) > f(z)$ then w is contained in a Terrace.
   (b) If $f(w) = f(z)$ then $f^{-1}(f(z))$ is a Plateau.

A typical theorem would then look like this one:

Theorem 2.2 *If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and the sets $f_{\leq c}$ are bounded then f satisfies the Connectedness Alternative.* □

Cases (iii) and (iv) in the Connectedness Alternative are pathological. They do not happen if f has strict local minimizers. Therefore, the hard part in using theorem 2.2 to prove that (i) holds in specific problems is to rule out the existence of Connection Points (ii). As a consequence, simple criteria to decide whether a given point x is a Connection Point are essential to use theorem 2.2. We now present such criteria. The simplest one is this:

Lemma 2.3 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function with continuous derivatives. If x is a Connection Point then $\nabla f(x) = 0$. If $\nabla f(x) = 0$, f is C^2 and two eigenvalues of $\nabla^2 f(x)$ are negative then x is not a Connection Point.* □

We emphasize that the number of positive or zero eigenvalues is irrelevant in this lemma. Theorem 2.2 and lemma 2.3 are analogous to the second order condition for minimization. The condition "$\nabla^2 f(x)$ with at least two negative eigenvalues" that guarantees that x is not a Connection Point looks like the condition "$\nabla^2 f(x)$ with at least one negative eigenvalue" that guarantees that x is not a local minimizer. In order to extend the analogy with minimization to constrained problems we consider the dual cone associated problem (2.1):

$$C_d(x) = \left\{ \sum_{i=1}^{m} \eta_i \nabla h_i(x) - \sum_{j=1}^{p} \gamma_j \nabla g_j(x) \ \text{ for } \eta_i \in \mathbb{R} \text{ and } \ \gamma_j \geq 0 \ \text{ with } \ \gamma_j g_j(x) = 0 \right\}. \tag{2.2}$$

We can then state a first order condition to rule out Connection Points for constrained problems:

Theorem 2.3 *Suppose the functions f, h_i and g_j in problem (2.1) have continuous first order derivatives and let F be its feasible region. If the point $x_0 \in F$ satisfies the Mangasarian Fromovitz constraint qualification and is a Connection Point for f constrained to F then $\nabla f(x_0) \in C_d(x_0)$.* □

Second order conditions for Connection Points under Mangasarian Fromovitz are more complicated, as they already are for minimization. Consider this two dimensional example:

Example 2.1 The nonlinear programming problem in two variables given by

$$\begin{aligned} \min f(x,y) &= 4x - 2y^2, \\ \text{subject to } g_1(x,y) &= -x \qquad \leq 0, \\ g_2(x,y) &= -x + y^2 \leq 0 \end{aligned}$$

[2] See [15], [17], [18], [23], [43], [44] and [45] for analogous theorems involving Mountain Pass Points.

has a convex feasible region $F = \{(x,y) \text{ with } x \geq y^2\}$ and the point $p = (0,0)$ satisfies the Slater Constraint Qualification (see [3]), which implies Mangasarian Fromovitz. Moreover, for $\gamma_1 = 3$ and $\gamma_2 = 1$,

$$\nabla f(p) = -\gamma_1 \nabla g_1(p) - \gamma_2 \nabla g_2(p) \tag{2.3}$$

and

$$d' \left(\nabla^2 f(p) + \gamma_1 \nabla^2 g_1(p) + \gamma_2 \nabla^2 g_2(p) \right) d = -2\|d\|^2$$

for all d in the subspace orthogonal to $\nabla g_1(p)$ and $\nabla g_2(p)$. However, p is a global minimizer because if $x \geq y^2$ then $f(x,y) \geq 4y^2 - 2y^2 \geq 0 = f(p)$. □

This example shows that it is possible to have minimizers with negative eigenvalues in the Lagrangian's Hessian for some combinations (2.3) under Mangasarian Fromovitz. By adding an extra dimension we get an analogous example for Connection Points:

Example 2.2 The nonlinear programming problem in three variables given by

$$\begin{aligned} \min f(x,y,z) &= 2z - x^2 - 2y^2, \\ \text{subject to } g_1(x,y,z) &= -z \qquad \leq 0, \\ g_2(x,y,z) &= -z + y^2 \leq 0 \end{aligned}$$

has a convex feasible region $F = \{(x,y,z) \text{ with } z \geq y^2\}$ and the point $p = (0,0,0)$ satisfies the Slater Constraint Qualification. Moreover, for $\gamma_1 = \gamma_2 = 1$,

$$\nabla f(p) = -\gamma_1 \nabla g_1(p) - \gamma_2 \nabla g_2(p)$$

and

$$d' \left(\nabla^2 f(p) + \gamma_1 \nabla^2 g_1(p) + \gamma_2 \nabla^2 g_2(p) \right) d = -2\|d\|^2$$

for all d in the two dimensional subspace orthogonal to $\nabla g_1(p)$ and $\nabla g_2(p)$. However, p is a Connection Point because if $f(x,y,z) < 0$ and $g_2(x,y,z) \leq 0$ then

$$-x^2 = f(x,y,z) + 2g_2(x,y,z) < 0 \Rightarrow x \neq 0$$

and the set $\{(x,y,z) \text{ with } f(x,y,z) < 0 = f(p)\}$ has two connected components, one containing the points with $x > 0$ and the other the points with $x < 0$. □

The next theorem shows that the Connection Point x_0 with a Hessian with two negative eigenvalues in the tangent cone in the example above occurs because the derivative of $g = (g_1, g_2)$ at x_0 is singular.

Theorem 2.4 *Suppose the functions f_i and g_j in problem (2.1) have continuous second order derivatives and let F be its feasible region. If $x_0 \in F$ is such that the vectors*

$$D = \{\nabla h_i(x_0), i = 1, \ldots, m\} \bigcup \{\nabla g_j(x_0), j = 1, \ldots, p \ \text{ with } \ g_j(x) = 0\}$$

are linearly independent, $\nabla f(x) = \sum_{i=1}^m \eta_i \nabla h_i(x_0) - \sum_{j=1}^p \gamma_j \nabla g_j(x_0)$, with $\gamma_j \geq 0$ and $\gamma_j \nabla g_j(x_0) = 0$, and there exists a two dimensional subspace $V \subset \mathbb{R}^n$ such that

$$d'v = 0 \qquad \text{and} \qquad v' \left\{ \nabla^2 f(x_0) - \sum_{i=1}^m \eta_i \nabla^2 h_i(x_0) + \sum_{j=1}^p \gamma_j \nabla^2 g_j(x_0) \right\} v < 0$$

for all $v \in V - \{0\}$ and $d \in D$ then x_0 is not a Connection Point for f constrained to F. □

Theorems 2.3 and 2.4 provide local criteria to rule out the existence of Connection Points. To use them we need global topological criteria to verify that the Connectedness Alternative in definition 2.7 holds. A good global criteria was introduced by Palais and Smale in [40] for manifolds modeled in Hilbert Spaces. We are interested in the constrained problems (2.1), which under Mangasarian Fromovitz may have a feasible region F that is not a manifold. In our case what matters is the distance of the gradient to the dual cone in (2.2) and we suggest the following notion of derivative:

Definition 2.8 *Let F be the feasible region of problem (2.1) and C_d the dual cone in (2.2). We define the constrained derivative of f at $x \in F$ by* $\mathrm{cd}f(x) = \mathrm{dist}(\nabla f(x), C_d(x))$. □

We can then state the corresponding version of the Palais-Smale Condition for constrained problems:

Definition 2.9 *Let F be the feasible region of problem (2.1). Given $c \in \mathbb{R}$, we say that this problem satisfies the Palais-Smale condition at level c if every sequence $\{x_n, n \in \mathbb{N}\} \subset F$ such that* $\lim_{n\to\infty} f(x_n) = c$ *and* $\lim_{n\to\infty} \mathrm{cd}f(x_n) = 0$ *has a convergent subsequence.* □

The Palais-Smale Condition is satisfied by functions with bounded level sets. It leads to global criteria to verify the Connectedness Alternative:

Theorem 2.5 *Suppose the functions f, h_i and g_j in problem (2.1) have continuous derivatives and f is bounded below in the feasible region F. If F is connected, the problem satisfies the Palais-Smale condition for all $c \in \mathbb{R}$ and the Mangasarian-Fromovitz constraint qualification holds for all $x \in F$ then f constrained to F satisfies the Connectedness Alternative in definition 2.7.* □

Combining theorems 2.3, 2.4 and 2.5 when we know that f has no Bridges or Terraces in F we get a reasonable technique to prove that problem (2.1) has a unique local solution: we use theorems 2.3 and 2.4 to rule out the existence of Connection Points and then conclude that alternative (i) in definition 2.7 must hold. The applicability of this technique to the particular problem we care about depends on the details of the problem. References [32] and [39] exemplify how this can be done in important problems in statistics and economics. However, in general problems it is already hard to show that F is connected.

3. The Connection Lemma in Banach Spaces

In this section we generalize the criteria presented in the previous section to prove that certain points are not Connection Points for problems

$$\begin{gathered} \min f(x) \\ \text{subject to } x \in C \text{ and } g(x) \in G \end{gathered} \tag{3.1}$$

in which the function f is defined in a Banach space X and g is a function from X to a Banach space Y. We assume that C and G are closed and convex and the functions f and g are strictly differentiable at $x_0 \in C \cap g^{-1}(G)$, in the following sense:

Definition 3.1 *Let X and Y be Banach spaces, f a function from X to Y and $D : X \to Y$ a continuous linear transformation. D is a strict derivative of f at $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$x, y \in \mathbb{B}_\delta(x_0) \Rightarrow \|f(y) - f(x) - D(y-x)\| \leq \varepsilon \|y - x\|.$$

We say that f is strictly differentiable at x_0 if there exists a strict derivative for f at x_0. □

Here $\mathbb{B}_\delta(x_0)$ denotes the closed ball with center x_0 and radius δ. When $x_0 = 0$ we write simply $\mathbb{B}_\delta$.

This section is based on the work of Ljusternik [31], Bartle & Graves [2], [19], [20], Michael [34] and [35] and Borwein [5]. The early work of Ljusternik and Graves were translated to a modern language and generalized in the work of Dontchev [12], [13] and Borwein & Dontchev [6] and lead to the concept of metric regularity, which is related to constraint qualifications in the work of Borwein [5], Ioffe [24] and Cominetti [8]. The theory presented here is analogous to the ones for minimization in the references above. It is also based upon Robinson's Constraint Qualification [47]

$$g(x_0) \in \mathrm{core}(G - dg(x_0)(C - x_0)) \tag{3.2}$$

and Rockafellar's [46] characterization of Clarke's tangent cone:

Definition 3.2 *Let X be a Banach space and $C \subset X$. Given $x_0 \in C$ we say that d is tangent to C at x_0 if for every $\varepsilon > 0$ there exist $\delta, \lambda > 0$ such that if $\|x - x_0\| \leq \delta$ and $t \in [0, \lambda]$ then* $\mathrm{dist}(x + td, C) \leq \varepsilon t$. *We denote by $T_C(x_0)$ the set of vectors tangent to C at x_0.* □

Our theory starts with this primal theorem:

Theorem 3.1 *Let X and Y be Banach spaces and let $C \subset X$ and $G \subset Y$ be closed and convex. Suppose the functions $g : X \to Y$ and $f : X \to \mathbb{R}$ are continuous in X and have strict derivatives $df(x_0)$ and $dg(x_0)$ at $x_0 \in F = C \cap g^{-1}(G)$ and $g(x_0) \in \mathrm{core}(G - dg(x_0)(C - x_0))$. If x_0 is a Connection Point for f constrained to F and $d \in T_C(x_0) \cap dg(x_0)^{-1}(T_G(g(x_0)))$ then $df(x_0)\, d \geq 0$.* □

The dual result corresponding to theorem 3.1 is stated in terms of the sets

$$A^+ = \{x^* \in X^* \text{ such that } x^*(a) \geq 0 \text{ for all } a \in A\}$$

We can then use theorem 3.1 above and theorem 6.3 in [5] to deduce this theorem:

Theorem 3.2 *Let X and Y be Banach spaces. Suppose $C \subset X$ and $G \subset Y$ are closed and convex, $g : X \to Y$ and $f : X \to \mathbb{R}$ are continuous in X and strictly differentiable at $x_0 \in C \cap g^{-1}(G)$ and $g(x_0) \in \mathrm{core}(G - dg(x_0)(C - x_0))$. If x_0 is a Connection Point for f constrained to $C \cap g^{-1}(G)$ then $df(x_0) \in T_C(x_0)^+ + dg(x_0)^* T_G(g(x_0))^+$.* □

Theorems 3.1 and 3.2 provide first order primal and dual criteria to show that x_0 is not a Connection Point. These criteria are analogous to the ones for minimization. Example 2.2 shows that second order criteria to prove that a point is not a Connection Point should require more from the problem. As a consequence, general second order theorems about Connection Points look like this one:

Theorem 3.3 *Let X and Y be Banach spaces, $f \in \mathrm{C}^2(X, \mathbb{R})$, $g \in \mathrm{C}^2(X, Y)$ and let $C \subset X$ and $G \subset Y$ be closed and convex. Suppose $x_0 \in C \cap g^{-1}(G)$ is such that $g(x_0) \in \mathrm{core}(G - dg(x_0)(C - x_0))$ and $df(x_0) = dg(x_0)^* \gamma + \mu$ for $\mu \in T_C(x_0)^+$ and $\gamma \in T_G(g(x_0))^+$. If there exist $d_1, d_2, s_{11}, s_{12}, s_{22} \in X$ such that d_1 and d_2 are linearly independent and, for $1 \leq i \leq j \leq 2$,*

$$d_i \in (C - x_0) \cap (x_0 - C), \tag{3.3}$$

$$s_{ij} \in T_C(x_0) \cap (-T_C(x_0)), \tag{3.4}$$

$$dg(x_0)\, d_i \in (G - g(x_0)) \cap (g(x_0) - G), \tag{3.5}$$

$$dg(x_0)\, s_{ij} + d^2 g(x_0)(d_i, d_j)/2 \in T_G(g(x_0)) \cap (-T_G(g(x_0))) \tag{3.6}$$

and $d^2 f(x_0)(d, d) < \gamma(d^2 g(x_0)(d, d))$ for $d \in \mathrm{Span}\{d_1, d_2\} - \{0\}$ then x_0 is not a Connection Point for f constrained to $C \cap g^{-1}(G)$. □

This theorem illustrates why ruling out Connection Points is easier than classical variational analysis. A Morse Theorist would read this theorem as "If x_0 has Morse index at least two then it is not a Connection Point". He would be correct, but in order to proceed with his theory he would usually restrict himself to Hilbert spaces and functions f with Fredholm second derivatives (see [21] and [49]).

As in the previous section, we need a global criteria to use theorems 3.1, 3.2 and 3.3 in order to prove that f has a unique local minimizer using the Connectedness Alternative in definition 2.7. We propose the following definitions of derivative and Palais Smale condition for problem (3.1):

Definition 3.3 *Let F be the feasible region of problem (3.1). The constrained derivative of f at $x \in F$ is*

$$\mathrm{cd}f(x) = \sup_{d \in T_C(x) \cap \left(dg(x)^{-1}(T_G(g(x)))\right) \text{ with } \|d\|=1} -df(x)\, d.$$

□

Definition 3.4 *Let F be the feasible region of problem (3.1). Given $c \in \mathbb{R}$, we say that this problem satisfies the Palais-Smale condition at level c if every sequence $\{x_n, n \in \mathbb{N}\} \subset F$ such that $\lim_{n \to \infty} f(x_n) = c$ and $\lim_{n \to \infty} \mathrm{cd}f(x_n) = 0$ has a convergent subsequence.* □

Definition 3.4 is different from definition 2.8 due to technical details about metric projections in Banach spaces discussed by Penot in [42]. However, their essence is the same and they are equivalent in reflexive Banach spaces.

Lemma 3.1 *If X is reflexive and problem (3.1) satisfies Robinson's Constraint Qualification (3.2) at x_0 then*

$$\mathrm{cd}f(x_0) = \mathrm{dist}\left(df(x_0),\ T_C(x_0)^+ + dg(x_0)^* T_G(g(x_0))^+\right).$$

□

Finally, the Palais-Smale Condition leads to global criteria to verify the Connectedness Alternative:

Theorem 3.4 *Suppose the sets C and G in problem (3.1) are closed and convex, the functions f and g have continuous Fréchet derivatives and f is bounded below in the feasible region F. If F is connected, the problem satisfies the Palais-Smale condition for all $c \in \mathbb{R}$ and Robinson's Constraint Qualification (3.2) for all $x \in F$ then f constrained to F satisfies the Connectedness Alternative in definition 2.7.* $\square$

4. The Connection Lemma in Complete Metric Spaces

The previous sections presented local criteria to decide whether a given point $x_0 \in F$ is a Connection Point and global criteria that imply the Connectedness Alternative. These criteria follow the nonlinear programming tradition and impose conditions on the derivatives of f. In this section we show that we can obtain similar results replacing derivatives by the "weak slope". The concept of weak slope is due to Degiovanni and Marzochi [11] and Katriel [26] and is discussed in depth by Corvellec et. al. [9] and Ioffe & Schwartzman [25]. As discussed in [10], the "weak slope" is a tool to apply the concept of deformation, which in turn is related to the gradient flow and the early work of Marston Morse [38]. In formal terms, the "weak slope" is defined by:

Definition 4.1 *Let X be a metric space, $f : X \to \mathbb{R}$ a function and $u \in X$. Denote by $|df|(u)$ the supremum of the σ's in $[0, +\infty)$ such that there exists $\delta > 0$ and a continuous function $\varphi : \mathbb{B}_\delta(u) \times [0, \delta] \to X$ satisfying*

$$\mathrm{dist}(\varphi(v,t), v) \leq t \qquad \text{and} \qquad f(\varphi(v,t)) \leq f(v) - \sigma t \tag{4.1}$$

for all $(v,t) \in \mathbb{B}_\delta(u) \times [0, \delta]$. The extended number $|df|(u)$ is called the **weak slope** *of f at u.* $\square$

The next lemma relates the constrained derivative in definition 3.3 to the weak slope:

Lemma 4.1 *Let X and Y be Banach spaces and let $C \subset X$ and $G \subset Y$ be closed and convex. Suppose $f : X \to \mathbb{R}$ and $g : X \to Y$ are continuous and strictly differentiable at $x_0 \in C \cap g^{-1}(G)$. If $g(x_0) \in \mathrm{core}(G - dg(x_0)(C - x_0))$ and f_F is restriction of f to $C \cap g^{-1}(G)$ then $|\mathrm{d}f_F|(x_0) \geq \mathrm{cd}f(x_0)$.* $\square$

This section explains that weak slopes and Connection Points work well together in complete metric spaces. Our first result is analogous to theorem 3.1 from last section:

Lemma 4.2 *Let F be a complete metric space and suppose that F is locally connected at x_0. If $f : X \to \mathbb{R}$ is continuous and $x_0 \in F$ is a Connection Point then $|\mathrm{d}f|(x_0) = 0$.* $\square$

The Palais Smale condition has a natural generalization to metric spaces (see [9] and [25]):

Definition 4.2 *Let F be a metric space and $f : X \to \mathbb{R}$ a continuous function. We say that f satisfies the Metric Palais Smale Condition at level c if every sequence $\{x_k, k \in \mathbb{N}\} \subset X$ such that $f(x_k)$ converges to c and $|df|(x_k) \to 0$ has an accumulation point.* $\square$

Based on the work of Corvellec, Degiovanni and Marzocchi [9] we can state the Connection Lemma for complete metric spaces:

Lemma 4.3 *Let F be a complete metric space and $f : F \to \mathbb{R}$ a continuous function. If $f_{<a}$ is disconnected and there exists $c \in (a, +\infty]$ such that the set $f_{<c}$ is connected and f satisfies the Metric Palais Smale Condition for all $b \in [a, c)$ then f has a Connection Point or a Bridge in $f^{-1}([a, c))$.* $\square$

This lemma is the basic result upon which the theory in the previous sections is built.

5. Functions with connected level sets

As we saw in section 2, we should consider two situations when analyzing the uniqueness of constrained minimizers: (i) the sets $f_{<c} = \{x \in F \text{ with } f(x) < c\}$ are connected for all c, (ii) some sets $f_{<c}$ are disconnected. The previous sections show that case (ii) can be approached via Connection Points and Bridges. In this section we analyze case (i). We show that in this case if f does not have a unique local minimizer then it has a complex set of local minimizers, as described in item (iv) of the Connectedness Alternative in definition 2.7.

We use the following definition:

Definition 5.1 *Let F be a metric space and $x \in C \subset F$. We define*

$$\mathscr{C}(x,C) = \bigcup_{A \subset C \text{ connected with } x \in A} A.$$

□

In words, $\mathscr{C}(x,C)$ is the union of all connected subsets of C which contain x. It is clear that $x \in \mathscr{C}(x,C)$ and that $\mathscr{C}(x,C)$ is connected. We then consider the following set:

Definition 5.2 *Let F be a metric space and $f : F \to \mathbb{R}$. We denote by $f_c^{\min}$ the set of local minimizers x of f such that $f(x) = c$.* □

If $x \in f_c^{\min}$ and $\overline{\mathscr{C}(x, f_c^{\min})} \cap \overline{f_{<c}} \neq \emptyset$ then $\mathscr{C}\big(x, f_c^{\min}\big)$ is a Terrace (see definition 2.5). This is the usual situation for local minimizers when the sets $f_{<c}$ are connected. The exceptional x's belong to

$$D_c = \left\{ x \in f_c^{\min} \ \text{ such that } \ \overline{\mathscr{C}(x, f_c^{\min})} \cap \overline{f_{<c}} = \emptyset \right\}. \tag{5.1}$$

The next lemma shows that if the level sets are connected and the metric Palais Smale condition holds then the sets D_c are actually empty:

Lemma 5.1 *Let F be a metric space, $f : F \to \mathbb{R}$ a continuous function and $c > \inf_{x \in F} f(x)$. If $\overline{f_c^{\min}}$ is compact and $D_c \neq \emptyset$ then $f_{\leq c}$ is disconnected.* □

We can then state the the main theorem in this section:

Theorem 5.1 *Let F be a metric space and $f : F \to \mathbb{R}$ a continuous function. If f is bounded below and, for all $c \in \mathbb{R}$, the set $f_{<c}$ is connected and f satisfies the metric Palais Smale Condition at level c, then either f has unique local minimizer or it satisfies item (iv) in definition 2.7.* □

6. Proofs

In this section we prove the lemmas and theorems stated in the previous sections. The section has six subsections. Each subsection starts with the proofs of the theorems in the corresponding section and ends with the proofs of the lemmas in this section.

6.1 Proofs for section 1.

Proof of theorem 1.1 Since X is compact, f satisfies the metric Palais Smale at all levels and the sets $f_c^{\min}$ in definition 5.2 are relatively compact. Moreover, since the local minimizers of f are strict there are no Plateaus or Terraces, because lemma 2.2 shows that Plateaus contain infinite non strict local minimizers. This rules out possibility (iv) in the Connectedness Alternative in definition 2.7. If all sets $f_{<c}$ are connected then theorem 5.1 and the observations above show that f has a unique local minimizer. On the other hand, if the set $f_{<c}$ is disconnected then there exists an open partition $f_{<c} = A_1 \cup A_2$ and, by the compactness of $f_{\leq c}$, f has at least one local minimizer in A_1 and another in A_2. □

Proof of theorem 1.2 As in the previous proof, f has a lower bound, satisfies the metric Palais-Smale condition at all levels, has no Bridges, Plateaus or Terraces and the sets $f_c^{\min}$ are relatively compact. This rules out possibilities (iii) and (iv) in the Connectedness Alternative in definition 2.7. Therefore, if the sets $f_{<c}$ are connected then theorem 5.1 and the observations above show that f has no Connection Points. On the other hand, lemma 6.8 shows that f has the extension property for all levels and if $f_{<a}$ is disconnected for some a then the hypothesis that F is connected and lemma 6.7 with $c = +\infty$ show that f has a Connection Point. □

6.2 Proofs for section 2.

Proof of theorem 2.1. Theorem 2.1 is a corollary of theorem 2.2 and definition 2.7. □

Proof of theorem 2.2. $\mathbb{R}^n$ is a complete metric space and if $f : \mathbb{R} \to \mathbb{R}$ is continuous and has bounded level sets then it is bounded below and for all c it satisfies the metric Palais Smale Condition 4.2 and the sets $f_c^{\min}$ in definition 5.2 are relatively compact. If all sets $f_{<c}$ are connected then theorem 5.1 implies that f satisfies item (i) or (iv) in definition 2.7. If some level set $f_{<c}$ is disconnected then lemma 6.7 shows that f satisfies item (ii) or (iii) in definition 2.7. □

Proof of theorem 2.3. The Mangasarian-Fromovitz condition is equivalent to Robinson's condition (3.2) for problem (2.1) and theorem 2.3 is a particular case of theorem 3.2. □

Proof of theorem 2.4. In order to simplify the notation let us assume that $g_j(x_0) = 0$ for all j. Let $q : \mathbb{R}^n \to \mathbb{R}^{m+p}$ be given by $q(x) = (h_1(x), \ldots, h_m(x), g_1(x), \ldots, g_p(x))'$ and let $dq(x_0)$ denote its jacobian matrix at x_0. Take a basis $\{v_1, v_2\}$ of V. The linear independence hypothesis implies that, for $1 \leq i \leq j \leq 2$, there exists $s_{ij} \in \mathbb{R}^n$ such that

$$dq(x_0)\, s_{ij} = -\frac{1}{2} \begin{pmatrix} v_i \nabla^2 h_1(x_0)\, v_j \\ \cdots \\ v_i \nabla^2 h_m(x_0)\, v_j \\ v_i \nabla^2 g_1(x_0)\, v_j \\ \cdots \\ v_i \nabla^2 g_p(x_0)\, v_j \end{pmatrix}$$

Theorem 2.4 follows from theorem 3.3 with $X = \mathbb{R}^n$, $G = \{(y,w) \in \mathbb{R}^m \times \mathbb{R}^p$ with $y = 0$ and $w \leq 0\}$, $Y = \mathbb{R}^{m+p}$, $C = \mathbb{R}^n$, $\mu = 0$, $\gamma : \mathbb{R}^{m+p} \to \mathbb{R}$ given by $\gamma(x) = \sum_{i=1}^m \eta_i x_i - \sum_{j=1}^p \gamma_j x_{m+j}$, $d_k = v_k$ and s_{ij} as above. □

Proof of theorem 2.5. The Mangasarian-Fromovitz condition is equivalent to Robinson's condition (3.2) for problem (2.1) and theorem 2.3 is a particular case of theorem 3.4. □

Proof of lemma 2.1. The proof of lemma 2.1 is left to the reader. □

Proof of lemma 2.3. The situation described in lemma 2.3 can be modeled as problem 2.1 with by adding a variable x_{n+1} and the constraint $x_{n+1} = 0$. Lemma 2.3 then follows from theorem 2.4. □

Proof of lemma 2.2 Let B be a Bridge as in definition 2.6. We claim that for every $r \in (0,1)$ the set

$$S_r = \{x \in B \text{ with } \operatorname{dist}(x, A_1) = r \operatorname{dist}(x, A_2)\}$$

is not empty. In fact, if S_r were empty then the sets

$$B_1 = \{x \in B \text{ with } \operatorname{dist}(x, A_1) < r \operatorname{dist}(x, A_2)\} \quad \text{and} \quad B_2 = \{x \in B \text{ with } \operatorname{dist}(x, A_1) > r \operatorname{dist}(x, A_2)\}$$

would be such that

$$\overline{B}_1 \cap \overline{B}_2 \subset S_r = \emptyset = \left(\overline{B}_1 \cap \overline{A}_2\right) = \left(\overline{B}_2 \cap \overline{A}_1\right)$$

and this contradicts definition 2.6. Therefore, for every $r \in (0,1)$ we can choose $p(r)$ in S_r in order to obtain the function claimed in lemma 2.2.

If P is a Plateau, then using its connectivity, the fact that $\overline{P}$ contains at least two points and the same argument above we can build the desired injective function p. □

6.3 Proofs for section 3.

The main tool in this subsection is a version of Ljusternik's [31] implicit function theorem. In problem (3.1), if $x_0 \in F = C \cap g^{-1}(G)$, $y_0 = g(x_0) \in G$, $d \in T_C(x_0)$ and $dg(x_0)\, d \in T_G(y_0)$ then an implicit function argument shows that if x is close to x_0 then there exists a continuous path $\eta(x,t) \subset F$ which is approximately tangent to d for each t. If $df(x_0)\, d$ is negative then, for $c = f(x_0)$, these paths can be used to connect all points in $F \cap f_{<c}$ near x_0 to a single point $x \in F \cap f_{<c}$. As a consequence, x_0 is not a Connection Point by the following lemma:

Lemma 6.1 *Let F be a topological space, $f : F \to \mathbb{R}$ a continuous function and $x_0 \in F$ with $f(x_0) = c$. If there exist a neighborhood N of x_0, a connected set $C \subset f_{<c}$ and, for every $x \in N \cap f_{<c}$, a connected set $C_x \subset f_{<c}$ such that $x \in C_x$ and $C_x \cap C \neq \emptyset$ then x_0 is not a Connection Point.* □

The second order theory is similar. An implicit function argument shows that if there exists a two dimensional subspace D in the tangent space T to $F = C \cap g^{-1}(G)$ in which $d^2 f(x_0)(.,.) - \gamma\big(d^2 g(x_0)(.,.)\big)$ is negative and $d^2 g(x_0)(D,D) \subset dg(x_0)T$ then there exists a (deformed) circle $\mathscr{C} \subset F$ near D such that $f(x) < c = f(x_0)$ in $\mathscr{C}$ and we can connect each x near x_0 in $F \cap f_{<c}$ to $\mathscr{C}$ via a path contained in $F \cap f_{<c}$ and lemma 6.1 implies that x_0 is not a Connection Point.

In the rest of this subsection we formalize this intuitive introduction. The essence of our arguments is known since the work of Ljusternik in the 1930's, but our presentation uses Michael's Selection Theorem [35], which was developed in the 1950's. Michael himself has used his theorem to extend the results of Bartle and Graves and more recently Páles [41] and Borwein & Dontchev [6] have also used similar arguments. We add a few details to this previous work in order to prove the theorems in section 3. We borrow the following definition and lemma from Borwein [5]:

Definition 6.1 *Let X and Y be Banach spaces and V a metric space. We say that $g : X \times V \to Y$ is partially strictly differentiable in x at $(x_0, v_0) \in X \times V$ if the partial Fréchet derivative $dg(x)x_0, v_0$ exists and for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $v, x, y \in \mathbb{B}_\delta(x_0)$ then $\|g(y,v) - g(x,v) - dg(x)x_0, v_0\,(y-x)\| \leq \varepsilon \|y-x\|$.* □

Lemma 6.2 *Let X and Y be Banach spaces and let $C \subset X$ and $G \subset Y$ be closed and convex and V a metric space. Suppose $g : X \times V \to Y$ is continuous in X and partially strictly differentiable in x at $(x_0, v_0) \in (C \times V) \cap g^{-1}(G)$. If $g(x_0, v_0) \in \mathrm{core}(G - d_x g(x_0, v_0)(C - x_0))$ then there exists $\rho, \delta > 0$ such if $x \in C \cap \mathbb{B}_\delta(x_0)$ and $v \in \mathbb{B}_\delta(v_0)$ then*

$$\mathrm{dist}(x, R(v)) \leq \rho\, \mathrm{dist}(g(x,v),\ G)\,, \tag{6.1}$$

where

$$R(v) = \{c \in C \text{ with } g(c,v) \in G\}\,. \tag{6.2}$$

□

Borwein's result, Michael's Selection Theorem and the next lemma lead to our implicit function theorem 6.1 stated below. After presenting these lemma and theorem we prove the theorems and lemmas in section 3.

Lemma 6.3 *Let X and Y be Banach spaces, V a metric space and $U, C \subset X$ and $G \subset Y$, with U open. Let $r : V \to \mathbb{R}$, $c : V \to X$ and $\varphi : X \times V \to Y$ be continuous functions. Define*

$$R(v) = \{c \in C \text{ with } \varphi(c,v) \in G\} \tag{6.3}$$

and suppose that there exists $\rho \in \mathbb{R}$ such that, for all $c \in U \cap C$ and $v \in V$,

$$\mathrm{dist}(c, R(v)) \leq \rho\, \mathrm{dist}(\varphi(c,v),\ G)\,. \tag{6.4}$$

Consider the multi valued function $A : V \rightrightarrows X$ given by

$$A(v) = \{c(v)\} \ \text{ if } r(v) \leq 0 \qquad \text{and} \qquad A(v) = U \cap R(v) \cap \mathrm{int}\big(\mathbb{B}_{r(v)}(c(v))\big) \ \text{ if } r(v) > 0. \tag{6.5}$$

If the sets $A(v)$ are not empty then $B : V \rightrightarrows X$ defined by $B(v) = \overline{A(v)}$ is lower semi continuous. □

Theorem 6.1 *Let X and Y be Banach spaces. Suppose $C \subset X$ and $G \subset Y$ are closed and convex and $g : X \to Y$ is continuous in X and strictly differentiable at $x_0 \in C \cap g^{-1}(G)$. If $g(x_0) \in \mathrm{core}(G - dg(x_0)(C - x_0))$ then there exist $\delta, \rho > 0$ such that for every metric space M and continuous function $h : M \to C \cap \mathbb{B}_\delta(x_0)$ there exists a continuous function $\tilde{h} : M \to C$ such that $g\big(\tilde{h}(m)\big) \in G$ for $m \in M$ and*

$$\|\tilde{h}(m) - h(m)\| \leq \rho\, \mathrm{dist}(g(h(m)), G)\,. \tag{6.6}$$

□

Corollary 6.1 *Let X and Y be Banach spaces. Suppose $C \subset X$ and $G \subset Y$ are closed and convex and $g : X \to Y$ is continuous in X and strictly differentiable at $x_0 \in C \cap g^{-1}(G)$. If $g(x_0) \in \mathrm{core}(G - dg(x_0)(C - x_0))$ then there exist $\delta, \rho > 0$ such that for every metric space M and continuous functions $h_x : M \to C \cap \mathbb{B}_\delta(x_0)$ and $h_y : M \to G \cap \mathbb{B}_\delta(g(x_0))$ there exist continuous functions $\tilde{h}_x : M \to C$ and $\tilde{h}_y : M \to G$ such that, for all $m \in M$,*

$$g\big(\tilde{h}_x(m)\big) = \tilde{h}_y(m), \tag{6.7}$$

$$\|\tilde{h}_x(m) - h_x(m)\| + \|\tilde{h}_y(m) - h_y(m)\| \leq \rho\, \|g(h_x(m)) - h_y(m)\|. \tag{6.8}$$

□

Corollary 6.2 *Let X and Y be Banach spaces. Suppose $C \subset X$ and $G \subset Y$ are closed and convex and $g : X \to Y$ is continuous in X and strictly differentiable at $x_0 \in F = C \cap g^{-1}(G)$. If $g(x_0) \in \mathrm{core}(G - dg(x_0)(C - x_0))$ then given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x_1, x_2 \in F \cap \mathbb{B}_\delta(x_0)$ there exists a continuous $\varphi : [0,1] \to C$ with*

$$\|\varphi(t) - tx_2 - (1-t)x_1\| + \|g(\varphi(t)) - tg(x_2) - (1-t)g(x_1)\| \leq \varepsilon t(1-t)\|x_2 - x_1\|. \tag{6.9}$$

and $g(\varphi(t)) \in G$ for $t \in [0,1]$. □

Proof of Theorem 3.1. Theorem 3.1 follows from definition 4.1, lemma 4.2 and the next lemma and theorem, which will be proved below:

Lemma 6.4 *Let X and Y be Banach spaces. Suppose $C \subset X$ and $G \subset Y$ are closed and convex. If the function $g : X \to Y$ is continuous and strictly differentiable at $x_0 \in C \cap g^{-1}(G)$ and $g(x_0) \in \mathrm{core}(G - dg(x_0)(C - x_0))$ then there exists $\delta_0 > 0$ such that if $\delta \in (0, \delta_0)$ then $C \cap g^{-1}(G) \cap \mathbb{B}_\delta(x_0)$ is path connected.* □

Theorem 6.2 *Let X and Y be Banach spaces. Suppose $C \subset X$ and $G \subset Y$ are closed and convex and $g : X \to Y$ and $f : X \to \mathbb{R}$ are continuous in X and have strict derivatives $df(x_0)$ and $dg(x_0)$ at $x_0 \in F = C \cap g^{-1}(G)$. If $g(x_0) \in \mathrm{core}(G - dg(x_0)(C - x_0))$ and there exists $d \in T_C(x_0)$ such that $\|d\| = 1$ and $dg(x_0)d \in T_G(g(x_0))$ then the weak slope of f constrained to F at x_0 is at least $-df(x_0)d$.* □

Proof of theorem 3.2. If we take $K = T_C(x_0)$ and $S = T_G(g(x_0))$ then theorem 3.1 implies that

$$df(x_0) \in \left(K \cap dg(x_0)^{-1}(S)\right)^+.$$

The fourth line of the proof of corollary 6.2 in page 35 of [5] and equation (67) in the statement of theorem 6.3 in page 34 of [5] show that $df(x_0) \in K^+ + dg(x_0)^* S^+$. □

Proof of theorem 3.3. Let us write $F = C \cap g^{-1}(G)$, $y_0 = g(x_0)$, $c = f(x_0)$, $S = \mathrm{Span}(s_{11}, s_{12}, s_{22})$ and $D = \mathrm{Span}(d_1, d_2)$. Equation (3.4) implies that $S \subset T_C(x_0) \cap (-T_C(x_0))$ and equation (3.6) shows that there exists a symmetric quadratic function $Q : D \times D \to S$ with $Q(d_i, d_j) = s_{ij}$ for $1 \leq i \leq j \leq 2$ and

$$dg(x_0)Q(d,d) + d^2g(x_0)(d,d)/2 \in T_G(y_0) \cap (-T_G(y_0)).$$

If K is convex and $x \in K$ then $K - x \subset T_K(x)$. Thus, equations (3.3) – (3.6) imply that $D \subset T_C(x_0) \cap (-T_C(x_0))$ and $dg(x_0)D \subset T_G(y_0) \cap (-T_G(y_0))$ and the positivity of μ in $T_C(x_0)$. The the positivity of γ in $T_G(y_0)$ yield

$$\mu(d) = \mu(Q(d,d)) = \gamma(dg(x_0)d) = \gamma(dg(x_0)Q(d,d) + d^2g(x_0)(d,d)/2) = 0. \tag{6.10}$$

By the hypothesis, $d^2f(x_0)(d,d) < \gamma(d^2g(x_0)(d,d))$ for $d \in D - \{0\}$ and there exists $\kappa \in (0,1)$ such that,

$$d^2f(x_0)(d,d) - \gamma\left(d^2g(x_0)(d,d)\right) \leq -8\kappa\|d\|^2. \tag{6.11}$$

We now use the equations above and corollaries 6.1 and 6.2 to define $\varepsilon, \sigma \in (0,1)$, a continuous function $\psi : D \cap \mathbb{B}_\sigma \to X$, and for each $x \in \mathbb{B}_\sigma(x_0)$, continuous functions $\varphi_x : [0,1] \to X$ and $\omega_x : [0,1] \to X$ such that for $d \in D \cap \mathbb{B}_\sigma - \{0\}$, $t \in [0,1]$ and $x \in F \cap f_{<c} \cap \mathbb{B}_{\varepsilon\sigma^2}(x_0)$,

$$\psi(d) \in F \cap f_{<c}, \qquad \varphi(x,t) \in F \cap f_{<c}, \qquad \omega_x(t) \in F \cap f_{<c}, \tag{6.12}$$

$$\varphi_x(0) = x, \qquad \omega_x(0) = \varphi_x(1), \qquad \omega_x(1) = \psi(d_x), \tag{6.13}$$

where $d_x \in D$ has norm σ. The set $H = \psi(D \cap \mathbb{B}_\sigma - \{0\})$ is connected and $H \subset F \cap f_{<c}$. Moreover, if $x \in F \cap f_{<c} \cap \mathbb{B}_{\varepsilon\sigma^2}(x_0)$ then the path $\{\varphi_x(t),\ t \in [0,1]\} \cup \{\omega_x(t),\ t \in [0,1]\}$ connects x to H via $F \cap f_{<c}$. Therefore, if we find functions ψ, φ_x and ω_x as above then theorem 3.3 will follow from lemma 6.1.

We now explain how to build ψ, φ_x and ω_x. According to corollaries 6.1 and 6.2 there exists $\delta, \rho > 0$ such that for every metric space M and continuous functions $h_x : M \to C \cap \mathbb{B}_\delta(x_0)$ and $h_y : M \to G \cap \mathbb{B}_\delta(y_0)$ there exist continuous functions $\tilde{h}_x : M \to C$ and $\tilde{h}_y : M \to G$ such that (6.7) and (6.8) are satisfied for $m \in M$ and if $x, y \in C \cap g^{-1}(G) \cap \mathbb{B}_\delta(x_0)$ then there exists a continuous path $\pi_{x,y} : [0,1] \to C \cap g^{-1}(G)$ such that

$$\|\pi_{x,y}(t) - ty - (1-t)x\| \leq t(1-t)\|x - y\|. \tag{6.14}$$

Let us define

$$\xi_f(x,w) = f(x+w) - f(x) - df(x)\,w - d^2 f(x_0)\,(w,w)/2, \tag{6.15}$$
$$\xi_g(x,w) = g(x+w) - g(x) - dg(x)\,w - d^2 g(x_0)\,(w,w)/2. \tag{6.16}$$

The continuity of f's and g's second derivatives and the fact that Q is a quadratic in the finite dimensional space D imply that there exists $\lambda > 1$ and $\nu \in (0,\delta)$ such that if $x,y \in \mathbb{B}_\nu(x_0)$ and $d,e \in D$ then

$$\|\,df(x)\,\| + \|\,dg(x)\,\| \leq \lambda, \tag{6.17}$$
$$\|\,Q(d,d) - Q(e,e)\| \leq \lambda\|d-e\|\,(\|d\| + \|e\|)\,, \tag{6.18}$$
$$\|\,df(x) - df(y)\,\| + \|\,dg(x) - dg(y)\,\| \leq \lambda\|y-x\|. \tag{6.19}$$

Using the constant κ in (6.11) let us define

$$\varepsilon = \frac{\min\{1,\kappa\}}{90\,(1+\rho)\,\lambda^2}. \tag{6.20}$$

The hypothesis on d_i and s_{ij} and lemma 6.5 (presented at the end of this subsection) applied to the subspaces

$$S_x = \left\{ \sum_{1\leq i\leq j\leq 2} a_{ij} Q(d_i,d_j),\;\; a_{ij} \in \mathbb{R} \right\} \subset X,$$
$$S_y = \left\{ \sum_{1\leq i\leq j\leq 2} a_{ij}\left(dg(x_0)\,Q(d_i,d_j) + d^2 g(x_0)\,(d_i,d_j)/2\right),\;\; a_{ij} \in \mathbb{R} \right\} \subset Y$$

imply that there exists a positive σ such that

$$\sigma \leq \min\left\{\nu, \frac{\varepsilon}{6(\lambda+\varepsilon)^2}\right\} \tag{6.21}$$

and if $x \in \mathbb{B}_\sigma(x_0)$, $w \in \mathbb{B}_\sigma$ and $d \in D \cap \mathbb{B}_\sigma$ then the following bounds are satisfied:

$$\operatorname{dist}(\,(x+d) + Q(d,d), C) \leq \varepsilon\|d\|^2, \tag{6.22}$$
$$\operatorname{dist}\big(\,(g(x) + dg(x_0)\,d) + \big(dg(x_0)\,Q(d,d) + d^2 g(x_0)\,(d,d)/2\big), G\big) \leq \varepsilon\|d\|^2, \tag{6.23}$$
$$\|\xi_f(x,w)\| + \|\xi_g(x,w)\| \leq \varepsilon\|w\|^2. \tag{6.24}$$

Let M be the metric space $(C \cap \mathbb{B}_\sigma(x_0)) \times (D \cap \mathbb{B}_\sigma)$. Using (6.22), Lemma 6.3 with $X = X$, $Y = X$, $V = M$, $U = X$, $C = C$, $G = C$, $r(x,d) = 2\varepsilon\|d\|^2$, $c(x,d) = x + d + Q(d,d)$ and $\varphi(c,x,d) = c$ and Michaels's selection theorem with the multi valued function $T : M \rightrightarrows X$ given by $T(x,d) = \mathbb{B}_{2\varepsilon\|d\|^2}(x + d + Q(d,d)) \cap C$ we obtain a continuous function $\chi : M \to X$ such that

$$\|\chi(x,d)\| \leq 2\varepsilon\|d\|^2, \tag{6.25}$$
$$h_x(x,d) = x + d + Q(d,d) + \chi(x,d) \in C. \tag{6.26}$$

Analogously, there exists a continuous $\upsilon : M \to Y$ such that

$$\|\upsilon(x,d)\| \leq 2\varepsilon\|d\|^2, \tag{6.27}$$
$$h_y(x,d) = g(x) + dg(x_0)\,d + d^2 g(x_0)\,(d,d)/2 + dg(x_0)\,Q(d,d) + \upsilon(x,d) \in G. \tag{6.28}$$

To evaluate the bound (6.8), notice that $w = Q(d,d) + \chi(x,d)$ satisfies

$$\|w\| \leq \|Q(d,d)\| + \|\chi(x,d)\| \leq (\lambda+\varepsilon)\,\|d\|^2, \tag{6.29}$$

$\|x + d - x_0\| \leq \sigma + \|d\| \leq 2\sigma$ and

$$g(h_x(x,d)) - h_y(x,d) = g(x+d) + dg(x+d)\,w + d^2 g(x_0)\,(w,w)/2 + \xi_g(x+d,w) - h_y(x,d) =$$
$$= \xi_g(x,d) + (dg(x+d) - dg(x_0))\,Q(d,d) + dg(x+d)\,\chi(x,d) + \xi_g(x+d,w) - \upsilon(x,d) + d^2 g(x_0)\,(w,w)/2.$$

Now using (6.19)–(6.25), (6.27) and remembering that $\lambda > 1$ and $\|d\| \leq \sigma \leq \varepsilon/6(\lambda+\varepsilon)^2$, we get that

$$\|g(h_x(x,d)) - h_y(x,d)\| \leq \left(\varepsilon + 2\sigma\lambda^2 + 2\lambda\varepsilon + \varepsilon(\lambda+\varepsilon)^2\|d\|^2 + 2\varepsilon + \lambda\,(\lambda+\varepsilon)^2\,\|d\|^2\right)\|d\|^2$$

and

$$\|g(h_x(x,d)) - h_y(x,d)\| \leq 7\lambda\varepsilon\|d\|^2. \tag{6.30}$$

Equations (6.8) and (6.30) and the way δ was defined show that there exist continuous function $\tilde{h}_x : M \to C$ and $\tilde{h}_y : M \to G$ such that $\tilde{h}_y(x,d) = g\left(\tilde{h}_x(x,d)\right)$ and

$$\|\tilde{h}_x(x,d) - h_x(x,d)\| + \|\tilde{h}_y(x,d) - h_y(x,d)\| \leq 7\lambda\rho\varepsilon\|d\|^2. \tag{6.31}$$

As a consequence of the last line, if we define

$$\tilde{\chi}(x,d) = \tilde{h}_x(x,d) - x - d - Q(d,d), \tag{6.32}$$
$$\tilde{\upsilon}(x,d) = \tilde{h}_y(x,d) - g(x) - dg(x_0)\,d - d^2g\,(x_0)\,(d,d)\,/2 - dg(x_0)\,Q(d,d) \tag{6.33}$$

then using (6.25) – (6.33) we obtain

$$\|\tilde{\chi}(x,d)\| \leq 8\lambda\rho\varepsilon\|d\|^2 \qquad \text{and} \qquad \|\tilde{\upsilon}(x,d)\| \leq 8\lambda\rho\varepsilon\|d\|^2, \tag{6.34}$$
$$\tilde{h}_x(x,d) = x + d + Q(d,d) + \tilde{\chi}(x,d), \tag{6.35}$$
$$\tilde{h}_y(x,d) = g(x) + dg(x_0)\,d + d^2g\,(x_0)\,(d,d) + dg(x_0)\,Q(d,d)/2 + \tilde{\upsilon}(y,d). \tag{6.36}$$

We claim that if $\|x - x_0\| \leq \sigma$ and $\|d\| \leq \sigma$,

$$(dg(x) - dg(x_0))\,d = 0 \tag{6.37}$$

and κ is the constant in (6.11) then

$$f(\tilde{h}_x(x,d)) \leq f(x) - 2\kappa\|d\|^2. \tag{6.38}$$

In fact, equation (6.35) show that, for w as in (6.29),

$$f\left(\tilde{h}_x(x,d)\right) = f(x) + df(x)\,(d + Q(d,d) + \tilde{\chi}(x,d)) + d^2f\,(x_0)\,(w+d, w+d)\,/2 + \xi_f(x, w+d).$$

It follows from (6.37) that

$$f\left(\tilde{h}_x(x,d)\right) = f(x) + df(x_0)\,(d + Q(d,d)) + d^2f\,(x_0)\,(d,d)\,/2 + \xi\,(x, w+d) \tag{6.39}$$

where

$$\xi\,(x,w) = df(x)\,\tilde{\chi}(x,d) + (df(x) - df(x_0))\,Q(d,d) + \left(d^2f\,(x_0)\,(w+d,w+d) - d^2f\,(x_0)\,(d,d)\right)/2 + \xi_f(x,w+d).$$

The bounds (6.17) – (6.24), (6.29) and (6.34) lead to $\|w+d\| \leq \|w\| + \|d\| \leq 2\|d\|$ and

$$\|\xi\,(x,w)\| \leq \left(8\lambda^2\rho\varepsilon + \lambda^2\sigma + 3\lambda^2(\lambda+\varepsilon)\|d\| + 4\varepsilon\right)\|d\|^2 \leq 2\kappa\|d\|^2. \tag{6.40}$$

Moreover, equation (6.10) leads to

$$df(x_0)\,(d + Q(d,d)) = \mu\,(d + Q(d,d)) + \gamma(dg(x_0)\,(d + Q(d,d))) = \gamma(dg(x_0)\,Q(d,d)) = -\gamma\left(d^2g\,(x_0)\,(d,d)\right)\right)/2.$$

Equation (6.38) follows from (6.11) and (6.39) – (6.40) and the last equation.

Since (6.37) is satisfied for all d when $x = x_0$, equation (6.38) shows that $\psi(d) = \tilde{h}_x(x_0,d)$ is such that $f(\psi(d)) \leq f(x_0) - 2\kappa\|d\|^2 < c$ for $d \in D \cap \mathbb{B}_\sigma$. Therefore, ψ is as promised in (6.12). Since D has dimension two, for every $x \in \mathbb{B}_\sigma(x_0)$ there exists $d = d_x$ with $\|d_x\| = \sigma$ satisfying (6.37). If follows from (6.38) that $\varphi_x(t) = \tilde{h}_x(x, td_x)$ is such that $f(\varphi_x(t)) < c$ if $f(x) < c$. Moreover, equations (6.34) and (6.35) show that $\varphi_x(0) = \tilde{h}_x(x,0) = x$. Therefore, φ_x is also as promised in (6.12) – (6.13).

Finally, we define and $\omega_x(t) = \pi_{\varphi_x(1),\psi(d_x)}(t)$, where $\pi_{x,y}$ is the path in (6.14). By the way $\pi_{x,y}$ was defined $\omega_x(t)$ satisfies (6.13) and $\omega_x(t) \in F$ for $t \in [0,1]$. To complete this proof we now show that if $f(x) < c$ and $\|x - x_0\| \leq \varepsilon\sigma^2$ then $f(\omega_x(t)) < c$ for $t \in [0,1]$. In fact, the bounds (6.19) and (6.38) lead to

$$\begin{aligned} f(\omega_x(t)) &\leq f(\psi(d_x)) + \lambda \|\omega_x(t) - \psi(d_x)\| \\ &\leq f(x) - 2\kappa\sigma^2 + \lambda \|\omega_x(t) - t\psi(d_x) - (1-t)\varphi_x(1)\| + \lambda(1-t)\|\varphi_x(1) - \psi(d_x)\| \end{aligned}$$

and (6.14) leads to

$$f(\omega_x(t)) \leq c - 2\kappa\sigma^2 + \lambda(1-t^2)\|\varphi_x(1) - \psi(d_x)\| = c - 2\kappa\sigma^2 + \lambda\|\tilde{h}_x(x,d_x) - \tilde{h}_x(x_0,d_x)\|. \tag{6.41}$$

Equations (6.34) and (6.35) and the assumption that $\|x - x_0\| \leq \varepsilon\sigma^2$ and $\|d_x\| = \sigma$ lead to

$$\|\tilde{h}_x(x,d_x) - \tilde{h}_x(x_0,d_x)\| \leq \|x - x_0\| + \|\tilde{\chi}(x,d_x)\| + \|\tilde{\chi}(x_0,d_x)\| \leq 15\lambda(1+\rho)\varepsilon\sigma^2.$$

This equation, (6.20) and (6.41) show that $f(\omega_x(t)) \leq c - \kappa\sigma^2$ and ω_x is as claimed in (6.12). □

Proof of theorem 3.4. Let f_F be the restriction of f to the feasible region F. Lemma 4.1 and the hypothesis that f_F satisfies the Palais Smale condition in definition 3.4 imply that f_F satisfy the Metric Palais Smale condition in definition 4.2 for all $c \in \mathbb{R}$. As a consequence, lemmas 6.9 and 6.18 show that f_F satisfies the Topological Palais Smale condition and the sets $f_c^{\min}$ in definition 5.2 are relatively compact for all c. If $f_{<a}$ is connected for all a then theorem 5.1 shows that f_F satisfies item (i) or (iv) in the Connectivity Alternative. On the other hand, if $f_{<a}$ is disconnected for some a then lemma 6.8 and lemma 6.7 with $c = +\infty$ show that f_F satisfies item (ii) or item (iii) of the Connectivity Alternative. Either way f_F satisfies the Connectivity Alternative. □

Proof of theorem 6.1. The function $\theta : X \times X \to Y$ given by $\theta(x,v) = g(v) + dg(x_0)(x - v)$ is continuous in $X \times X$ and partially strictly differentiable in x at (x_0,x_0), with partial derivative in x given by $d_x\theta(x_0,x_0) = dg(x_0)$. Therefore,

$$\theta(x_0,x_0) = g(x_0) \in \operatorname{core}(G - dg(x_0)(C - x_0)) = \operatorname{core}(G - d_x\theta(x_0,x_0)(C - x_0))$$

and lemma 6.2 applied to $g = \theta$ and $V = X$ yields $\mu,\sigma > 0$ such that

$$\operatorname{dist}(x, R(v)) \leq \mu \operatorname{dist}(g(v) + dg(x_0)(x - v),\ G) \tag{6.42}$$

for

$$R(v) = \{\ c \in C \text{ with } g(v) + dg(x_0)(c - v) \in G\ \} \tag{6.43}$$

and $x \in C \cap \mathbb{B}_\sigma(x_0)$ and $v \in \mathbb{B}_\sigma(x_0)$. Since g is strictly differentiable at x_0 there exists $\tau \in (0,\sigma/4)$ such that if $\|x - x_0\| \leq 4(1+\mu)\tau$ and $\|w - x_0\| \leq 4(1+\mu)\tau$ then

$$\|g(w) - g(x) - dg(x_0)(w - x)\| \leq \frac{1}{2(1+\mu)}\|w - x\|. \tag{6.44}$$

The continuity of g yields $\delta \in (0,\tau)$ such that if $\|x - x_0\| \leq 4(1+\mu)\delta$ then

$$\chi(x) = \operatorname{dist}(g(x), G) \leq \frac{\tau}{2(1+\mu)}. \tag{6.45}$$

Given a metric space M and a continuous function $h : M \to C \cap \mathbb{B}_\delta(x_0)$ we define $h_0 = h$ as the first element of a Cauchy sequence $\{h_k, k \in \mathbb{N}\}$ in the complete metric space $\mathrm{C_b}(M,C)$ which converges to a function $\tilde{h}$ as claimed in theorem 6.1. The following equations are obvious for $k = 0$ and $m \in M$:

$$\operatorname{dist}(\ g(h_k(m)), G) \leq 2^{-k}\chi(h(m)), \tag{6.46}$$

$$\|h_k(m) - h(m)\| \leq 2\left(1 - 2^{-k}\right)(1+\mu)\,\chi(h(m)). \tag{6.47}$$

We now define h_{k+1} assuming that (6.46) – (6.47) are satisfied. Notice that the use of (6.43) – (6.44) with x replaced by $h_k(m)$ is justified because (6.45) and (6.47) yield

$$\|h_k(m) - x_0\| \leq \|h_k(m) - h(m)\| + \|h(m) - x_0\| \leq 2(1+\mu) \times \frac{\tau}{2(1+\mu)} + \delta = \tau + \delta < 4\tau < \sigma. \tag{6.48}$$

The sets $U = \mathrm{int}\,(\mathbb{B}_\sigma(x_0))$, $V = M$ and the functions

$$c(m) = h_k(m), \tag{6.49}$$
$$r(m) = 2^{-k}\,(1+\mu)\,\chi(h(m)), \tag{6.50}$$
$$\varphi(x,m) = g(h_k(m)) + dg(x_0)\,(x - h_k(m))\,. \tag{6.51}$$

satisfy the hypothesis of lemma 6.3, because (6.42) implies (6.4) and we claim that the sets A in (6.5) corresponding to c, r and φ above are not empty. In fact, if $R(m)$ is the set corresponding to c, r and φ in (6.3) and if $r(m) > 0$ then (6.46) shows that

$$\mathrm{dist}(c(m),R(m)) = \mathrm{dist}(h_k(m),R(m)) \leq \mu\,\mathrm{dist}(\varphi(h_k(m)),G) \leq \mu 2^{-k}\chi(h(m)) < 2^{-k}\,(1+\mu)\,\chi(h(m)) = r(m).$$

and there exists $x \in R(m)$ with $\|x - c(m)\| < r(m)$. This x belongs to the set $A(m)$ in (6.5). The sets $B(m) = \overline{A(m)}$ are closed and convex and we conclude from lemma 6.3 and Michael's Selection Theorem that there exists a continuous function $h_{k+1} : M \to C$ such that

$$\|h_{k+1}(m) - c(m)\| = \|h_{k+1}(m) - h_k(m)\| \leq r(m) = 2^{-k}\,(1+\mu)\,\chi(h(m)), \tag{6.52}$$
$$\eta_k(m) = g(h_k(m)) + dg(x_0)\;(h_{k+1}(m) - h_k(m)) \in G.$$

Adding (6.52) to (6.47) we conclude that (6.47) holds for $k+1$ and (6.48) allows us to use (6.44) to conclude that $g(h_{k+1}(m)) = \eta_k(m) + \xi_k(m)$ with

$$\|\xi_k(m)\| \leq \frac{1}{2\,(1+\mu)}\|h_{k+1}(m) - h_k(m)\| \leq 2^{-(k+1)}\chi(h(m)).$$

Combining the last two equations and the fact that $\eta_k(m) \in G$ we obtain (6.46) for $k+1$. The bound (6.52) shows that h_k is a Cauchy sequence. This completes the inductive construction of h_k, which converge to $\tilde{h}$ such that $\tilde{h}(m) \in G$ by (6.46) and, by (6.47),

$$\|\tilde{h}(m) - h(m)\| \leq 2\,(1+\mu)\,\chi(h(m)) = 2\,(1+\mu)\,\mathrm{dist}(\,g(h(m)),G)\,.$$

The bound (6.6) follows from this equation with $\rho = 2\,(1+\mu)$. □

Proof of corollary 6.1. Let us define $y_0 = g(x_0)$. The function $\gamma : X \times Y \to Y$ given by $\gamma(x,y) = g(x) - y$ is continuous and strictly partially differentiable at (x_0,y_0) and satisfies $d_x\gamma(x_0,y_0) = dg(x_0)$ and $d_y\gamma(x_0,y_0) = -I$. Therefore, the hypothesis $y_0 \in \mathrm{core}(G - dg(x_0)\,(C - x_0))$ implies that

$$0 \in \mathrm{core}(\{0\} - d_y\gamma(x_0,y_0)\,(G - y_0) - d_x\gamma(x_0,y_0)\,(C - x_0))$$

Theorem 6.1 for $X = X \times Y$ with the norm $\|(x,y)\| = \|x\| + \|y\|$, $Y = Y$, $C = C \times G$, $G = \{0\}$, $g = \gamma$, yields $\delta', \rho > 0$ such that for any metric space M and continuous functions $h : M \to (C \times G) \cap \mathbb{B}_{\delta'}((x_0,y_0))$ there exists a continuous $\tilde{h} : M \to C \times G$ such that

$$g\big(\tilde{h}_x(m)\big) - \tilde{h}_y(m) \,\in\, \{0\}, \tag{6.53}$$
$$\|\tilde{h}_x(m) - h_x(m)\| + \|\tilde{h}_y(m) - h_y(m)\| \,\leq\, \rho\,\|g(h_x(m)) - h_y(m)\|. \tag{6.54}$$

Taking $\delta = \delta'/2$ we have that $(C \cap \mathbb{B}_\delta(x_0)) \times (G \cap \mathbb{B}_\delta(y_0)) \subset (C \times G) \cap \mathbb{B}_{\delta'}((x_0,y_0))$ and if h_x and h_y are continuous functions from M to $C \cap \mathbb{B}_\delta(x_0)$ and $G \cap \mathbb{B}_\delta(y_0)$ respectively then the range of $h(m) = (h_x(m),h_y(m))$ is contained in $(C \times G) \cap \mathbb{B}_{\delta'}((x_0,y_0))$. Thus theorem 6.1 shows that there exist $\tilde{h}_x : M \to C$ and $\tilde{h}_y : M \to G$ that satisfy (6.53) and (6.54) for all m. Equations (6.7) and (6.8) follow from (6.53) and (6.54) and we are done. □

Proof of corollary 6.2. Let δ' and ρ be the numbers yielded by corollary 6.1. By the strict differentiability of g at x_0 there exists $\delta \in (0,\delta')$ such that if $x, y \in \mathbb{B}_\delta(x_0)$ then $g(x_1), g(x_2) \in \mathbb{B}_\delta(g(x_0))$ and

$$4\rho\,\|g(x) - g(y) - dg(x_0)\,(x - y)\| \leq \varepsilon\|x - y\|. \tag{6.55}$$

Given $x_1, x_2 \in C \cap \mathbb{B}_\delta(x_0)$, let us now define $M = [0,1]$, $h_x(t) = tx_2 + (1-t)x_1$ and $h_y(t) = tg(x_2) + (1-t)g(x_1)$, apply corollary 6.1 and take $\varphi(t) = \tilde{h}_x(t)$. Equation (6.8) shows we will done if we prove that

$$2\rho\,\|\,\psi(t)\| \leq \varepsilon t\,(1-t)\,\|x_2 - x_1\|, \tag{6.56}$$

for $\psi(t) = g(h_x(t)) - h_y(t)$. To verify this bound, notice that

$$\psi(t) = t\,(g(h_x(t)) - g(x_2) - (1-t)dg(x_0)\,(x_1 - x_2)) + (1-t)\,(g(h_x(t)) - g(x_1) - tdg(x_0)\,(x_2 - x_1)))\,. \quad (6.57)$$

The bound (6.55) for $x = h_x(t) = x_2 + (1-t)(x_1 - x_2)$ and $y = x_2$ implies that

$$4\rho\,\|g(h_x(t)) - g(x_2) - (1-t)dg(x_0)\,(x_1 - x_2)\,\| \leq \varepsilon(1-t)\|x_1 - x_2\|. \quad (6.58)$$

The same argument with $x = h_x(t) = x_1 + t(x_2 - x_1)$ and $y = x_1$ yields

$$4\rho\,\|g(h_x(t)) - g(x_1) - tdg(x_0)\,(x_2 - x_1))\,\| \leq \varepsilon t\|x_1 - x_2\|.$$

The bound (6.56) follows from (6.57) – (6.58) and the last equation. □

Proof of theorem 6.2. Let us write $y_0 = g(x_0)$ and let δ and ρ be given by theorem 6.1. For $\zeta \in \mathbb{R}$, define $M_\zeta = \left(C \cap \mathbb{B}_\zeta(x_0)\right) \times [0,\zeta]$. Given $\varepsilon > 0$, we will find $\tau > 0$ and a function $\varphi_\varepsilon : M_\tau \to C \cap g^{-1}(G)$ such that

$$\|\varphi_\varepsilon(x,t) - x\| \leq t \qquad \text{and} \qquad f(\varphi_\varepsilon(x,t)) \leq f(x) + (df(x_0)\,d + \varepsilon)\,t. \quad (6.59)$$

By definition 4.1, this proves that $|df|(x_0) \geq -df(x_0)\,d$. Let us define

$$\begin{aligned} \xi_f(x,w) &= f(x+w) - f(x) - df(x_0)\,w, \\ \xi_g(x,w) &= g(x+w) - g(x) - dg(x_0)\,w. \end{aligned}$$

The continuity of g, the strict differentiability of f and g at x_0 and the hypothesis that $d \in T_C\,(x_0)$ and $dg(x_0)\,d \in T_G\,(y_0)$ imply that there exists $\sigma \in (0,\delta)$ such that if $x \in \mathbb{B}_\sigma(x_0)$, $y \in \mathbb{B}_\sigma(y_0)$ and $t \in [0,\sigma]$ then

$$\operatorname{dist}(x + td, C) + \operatorname{dist}(y + t\,dg(x_0)\,d, G) \leq \mu t, \quad (6.60)$$

$$\|\xi_g(x,w)\| + \|\xi_f(x,w)\| \leq \mu\|w\|, \quad (6.61)$$

for

$$\mu = \frac{\varepsilon}{20\,(1+\rho)\,(1+\|dg(x_0)\,\|)\,(1+\|df(x_0)\,\|)}. \quad (6.62)$$

The argument used to derive (6.25) – (6.28) yields continuous functions $h_x : M_\sigma \to C$ and $\upsilon : M_\sigma \to G$ with

$$\|h_x(x,t) - x - td\| \leq 2\mu t \qquad \text{and} \qquad \|\upsilon(y,t) - y - t\,dg(x_0)\,d\| \leq 2\mu t. \quad (6.63)$$

By the continuity of g there exists $\tau \in (0,\sigma)$ such that if $\|x - x_0\| \leq \tau$ then $\|g(x) - y_0\| \leq \sigma$. Let $\tilde{h}_x : M_\tau \to C$ and $\tilde{h}_y : M_\tau \to G$ be functions obtained applying corollary 6.1 with h_x constrained to M_τ and $h_y : M_\tau \to G \cap \mathbb{B}_\sigma(x_0)$ given by $h_y(x,t) = \upsilon(g(x),t)$. Let us now estimate the right hand side of (6.8) for $(x,t) \in M_\tau$:

$$\begin{aligned} \|g(h_x(x,t)) - h_y(x,t)\| &\leq \|g(h_x(x,t)) - g(x+td)\| + \|g(x+td) - g(x) - tdg(x_0)\,d\| + \|h_y(x,t) - g(x) - t\,dg(x_0)\,d\| \\ &\leq (\|dg(x_0)\,\| + 1)\,\|h_x(x,t) - x - td\| + \|\xi_g(x,td)\| + \|\upsilon(g(x),t) - g(x) - t\,dg(x_0)\,d\| \end{aligned}$$

and equations (6.61) – (6.63) lead to

$$\|g(h_x(x,t)) - h_y(x,t)\| \leq \frac{\varepsilon}{4\,(1+\rho)\,(1+\|df(x_0)\,\|)}t$$

and (6.8) leads to

$$\|\tilde{h}_x(x,t) - h_x(x,t)\| \leq \frac{\varepsilon}{4\,(1+\|df(x_0)\,\|)}t. \quad (6.64)$$

Defining

$$\alpha = 2\mu + \frac{\varepsilon}{4\,(1+\|df(x_0)\,\|)},$$

$\tilde{t} = (1-\alpha)t$ and $\varphi_\varepsilon : M_\tau \to C \cap g^{-1}(G)$ by $\varphi_\varepsilon(x,t) = \tilde{h}_x(x,\tilde{t})$ we get from (6.63) – (6.64) that

$$\|\varphi_\varepsilon(x,t) - x - \tilde{t}d\| \leq \|\tilde{h}_x(x,\tilde{t}) - h_x(x,\tilde{t})\| + \|h_x(x,\tilde{t}) - x - \tilde{t}d\| \leq \alpha\tilde{t}. \quad (6.65)$$

This implies that $\|\varphi_\varepsilon(x,t)-x\| \leq (1+\alpha)\tilde{t} = (1-\alpha^2)t$ and φ_ε satisfies the first bound in (6.59). To prove the second bound, notice that $\mu < \varepsilon/2$, $\alpha\,(\|df(x_0)\|+1) < \varepsilon/2$ and (6.61) and (6.65) show that

$$\begin{aligned} f(\varphi_\varepsilon(x,t)) - f(x) &= f(x+\tilde{t}d) - f(x) + df(x_0)\,(\varphi_\varepsilon(x,\tilde{t}) - x - \tilde{t}d) + \xi_f(x+\tilde{t}d, \varphi_\varepsilon(x,\tilde{t}) - x - \tilde{t}d) \\ &\leq \tilde{t}\,df(x_0)\,d + \|\xi_f(x,\tilde{t}d)\| + (\|df(x_0)\|+1)\,\|\varphi_\varepsilon(x,\tilde{t}) - x - \tilde{t}d\| \\ &\leq \tilde{t}\,(df(x_0)\,d + \mu + \alpha\,(\|df(x_0)\|+1)) \leq t\,(df(x_0)\,d + \varepsilon)\,. \end{aligned}$$

This proves the second bound in (6.59). □

Proof of lemma 3.1. Let A be the closed convex cone $A = T_C\,(x_0) \cap dg(x_0)^{-1}\,T_G\,(g(x_0))$. Theorem 6.3 in page 34 of [5] states that $A^+ = T_C\,(x_0)^+ + dg(x_0)^*\,T_G\,(g(x_0))^+$. According to definition 3.3, this lemma will be proved if we show that

$$\sup_{d\in A\cap\mathbb{B}_1} -df(x_0)\,d = \operatorname{dist}\big(df(x_0), A^+\big)\,. \tag{6.66}$$

Applying corollary 3.2 in page 91 of [42] with $C = A^+$ and $w = df(x_0)$ and using the reflexivity of X we obtain $y \in A\cap\mathbb{B}_1$ and $x \in A^+$ such that $x(y) = 0$ and

$$\operatorname{dist}\big(df(x_0), A^+\big) = \|\,x - df(x_0)\,\| = y\,(\,x - df(x_0)\,) = x(y) - df(x_0)\,y = -df(x_0)\,y \leq \sup_{d\in A\cap\mathbb{B}_1} -df(x_0)\,d.$$

On the other hand, if $x \in A^+$ then

$$\|\,x - df(x_0)\,\| = \sup_{d\in\mathbb{B}_1}\,(\,x(d) - df(x_0)\,d\,) \geq \sup_{d\in A\cap\mathbb{B}_1}\,(x(d) - df(x_0)\,d) \geq \sup_{d\in A\cap\mathbb{B}_1} -df(x_0)\,d.$$

Therefore,

$$\operatorname{dist}\big(df(x_0), A^+\big) = \inf_{x\in A^+} \|\,df(x_0) - x\,\| \geq \sup_{d\in A\cap\mathbb{B}_1} -df(x_0)\,d.$$

The last three equations imply (6.66). □

Proof of lemma 6.1. The proof is by contradiction. If x_0 is a Connection Point then there is an open partition $f_{<c} = A_1 \cup A_2$ with $x_0 \in \overline{A}_1 \cap \overline{A}_2$. This implies that there exist $a_1 \in N\cap A_1$ and $a_2 \in N\cap A_2$. The set $C_{a_1} \cup C$ is connected because C_{a_1} and C are connected and $C_{a_1} \cap C \neq \emptyset$. Since $f_{<c} = A_1 \cup A_2$ is an open partition and $C \subset f_{<c}$, this implies that $C \subset C_{a_1} \cup C \subset A_1$. By the same reason, $C \subset A_2$. This contradicts the fact that A_1 and A_2 are disjoint. □

Proof of lemma 6.2. This lemma is a particular version of theorem 4.2 in page 23 of [5]. □

Proof of Lemma 6.3. We show that if $\lim_{k\to\infty} v_k \to v_\infty$ and $x_\infty \in B(v_\infty)$ then there exists a subsequence v_{n_k} of v_k and $x_{n_k} \in B(v_{n_k})$ such that $\lim_{k\to\infty} x_{n_k} = x_\infty$. If $r(v_\infty) = 0$ the only element of $B(v_\infty)$ is $x_\infty = c(v_\infty)$ then for any choice $x_k \in B(v_k)$ the sequence $\{x_k, k\in\mathbb{N}\}$ converges to x_∞, because $r(v_k) \to r(v_\infty) = 0$, $c(v_k) \to c(v_\infty) = x_\infty$ and $\|x_k - c(v_k)\| \leq r(v_k)$. On the other hand, if $r(v_\infty) > 0$ then given $x_\infty \in B(v_\infty)$ there exists a sequence $\{z_m, m\in\mathbb{N}\}$ with $z_m \in A(v_\infty)$ and $\lim_{k\to\infty} z_m = x_\infty$. We take a random $x_{k_0} \in A(v_0)$ and $k_0 = 0$ as the first member of an increasing sequence $\{k_n, n\in\mathbb{N}\}$ of indices that will yield the $x_{k_n} \in B(v_{k_n})$ converging to x_∞. Let us then suppose that we have defined $k_0,\ldots,k_{n-1}$ with $x_{k_j} \in A(k_j)$ and

$$\|x_{k_j} - x_\infty\| \leq \delta_j = \frac{1}{j+1}\big(1 + \|x_{k_0} - x_\infty\|\big)\,. \tag{6.67}$$

Let m_n be such that if $m \geq m_n$ then

$$2\|z_{m_n} - x_\infty\| < \delta_{n+1}. \tag{6.68}$$

By the continuity of φ at (z_{m_n}, v_∞) there exists $p_n \geq k_{n-1}$ such that if $k \geq p_n$ then

$$2\rho\,\|\,\varphi(z_{m_n}, v_k) - \varphi(z_{m_n}, v_\infty)\| < \delta_{n+1}. \tag{6.69}$$

Definitions (6.3) – (6.4) imply that $\varphi(z_{m_n}, v_\infty) \in G$. Thus, $\operatorname{dist}(\varphi(z_{m_n}, v_\infty), G) = 0$ and (6.4) yields

$$\operatorname{dist}(z_{m_n}, R(v_k)) \leq \rho\operatorname{dist}(\varphi(z_{m_n}, v_k),\ G) \leq \rho\,\|\,\varphi(z_{m_n}, v_k) - \varphi(z_{m_n}, v_\infty)\| + \rho\operatorname{dist}(\varphi(z_{m_n}, v_\infty),\ G) =$$

$$\rho\,\|\,\varphi(z_{m_n}, v_k) - \varphi(z_{m_n}, v_\infty)\|$$

for $k \geq p_n$. Definition (6.3) implies that for each $k \geq k_0$ there exists $y_k \in C$ such that

$$\varphi(y_k, v_k) \in G \qquad \text{and} \qquad \|y_k - z_{m_n}\| \leq \rho \|\varphi(z_{m_n}, v_k) - \varphi(z_{m_n}, v_\infty)\|. \tag{6.70}$$

The last equation and (6.68) (6.69) imply that for all $k \geq p_n$

$$\|y_k - x_\infty\| \leq \|y_k - z_{m_n}\| + \|z_{m_n} - x_\infty\| \leq \delta_{n+1}. \tag{6.71}$$

Since $z_{m_n} \in A(v_\infty)$ we have that $r(v_\infty) - \|z_{m_n} - c(v_\infty)\| > 0$. We also have that $z_{m_n} \in U$, which is open. Therefore, $\mathrm{dist}(z_{m_n}, A^c) > 0$ and we can take $k = k_n \geq p_n$ such that

$$\rho \|\varphi(z_{m_n}, v_{k_n}) - \varphi(z_{m_n}, v_\infty)\| + (r(v_\infty) - r(v_{k_n})) + \|c(v_\infty) - c(v_{k_n})\| < \mu_m, \tag{6.72}$$

for

$$\mu_m = \min \left\{ \frac{1}{2} \mathrm{dist}(z_{m_n}, U^c), \; r(v_\infty) - \|z_{m_n} - c(v_\infty)\| \right\},$$

because the left hand side of (6.72) converges to 0 as $k_n \to \infty$ and $\mu_m > 0$. As a consequence of the last two equations and (6.70), $x_{k_n} = y_{k_n}$ satisfies

$$\|x_{k_n} - c(v_{k_n})\| \leq \|y_{k_n} - z_{m_n}\| + \|z_{m_n} - c(v_\infty)\| + \|c(v_\infty) - c(v_{k_n})\| < r(v_{k_n}).$$

This shows that $x_{k_n} \in \mathbb{B}_{r(v_{k_n})}(c(v_{k_n}))$ by (6.5). The same equations show that

$$\mathrm{dist}(x_{k_n}, U^c) \geq \mathrm{dist}(z_{m_n}, U^c) - \|x_{k_n} - z_{m_n}\| \geq 2\mu_m - \rho \|\varphi(z_{m_n}, v_{k_n}) - \varphi(z_{m_n}, v_\infty)\| \geq \mu_m > 0.$$

Therefore, $x_{k_n} \in U \cap \mathbb{B}_{r(v_{k_n})}(c(v_{k_n})) \cap R(v_{k_n}) = A(v_{k_n})$. Finally, the second inequality in (6.71) shows that x_{k_n} satisfies (6.67) and the inductive construction of x_{k_n} is complete. $\square$

Proof of lemma 6.4. Let δ' be the number and φ the function yielded by corollary 6.2 with $\varepsilon = 1/2$. By the continuity of g there exists $\delta_0 \in (0, \delta')$ such that if $\|x - x_0\| \leq \delta_0$ then $g(x) \in \mathbb{B}_{\delta'}(g(x_0))$. If $\|x - x_0\| \leq \delta \leq \delta_0$ then equation (6.9) implies that $\|\varphi(t) - t x_0 - (1-t)x\| \leq t\,(1-t)\,\|x_0 - x\|/2$. As a consequence,

$$\|\varphi(t) - x_0\| \leq (1-t)\,(1+t/2)\,\|x_0 - x\| \leq (1-t/2)\,\|x - x_0\|$$

and $\varphi(t) \in \mathbb{B}_\delta(x_0)$. Therefore, for every $\delta \in (0, \delta_0)$ the set $N_\delta = C \cap g^{-1}(G) \cap \mathbb{B}_\delta(x_0)$ is such that if $x \in N_\delta$ then there exists a continuous path contained in N_δ connecting x to x_0. $\square$

Lemma 6.5 *Let X be a Banach space, $C \subset X$ and $x_0 \in C$. If $S \subset T_C(x_0)$ is a finite dimensional subspace of X then given $\varepsilon > 0$ there exists δ such that for all $x \in \mathbb{B}_\delta(x_0)$ and $d \in S$ with $\|d\| \leq \delta$ we have* $\mathrm{dist}(x + d, C) \leq \varepsilon \|d\|$. $\square$

Proof of lemma 6.5. Since S is finite dimensional, given $\varepsilon > 0$ there exists $V = \{v_1, \ldots, v_n\} \subset S$ with $\|v_i\| = 1$ such that if $d \in S$ then there exists $v_d \in V$ such that $2\|d - \|d\| v_d\| \leq \varepsilon \|d\|$. Since V is finite and $V \subset T_C(x_0)$, there exists δ such that if $\|x - x_0\| \leq \delta$, $t \in [0, \delta]$ and $v \in V$ then $2\,\mathrm{dist}(x + tv, G) < \varepsilon t$. Now given $d \in S$ with $\|d\| \leq \delta$ and $x \in \mathbb{B}_\delta(x_0)$ we have

$$2\,\mathrm{dist}(x + d, G) \leq 2\|d - \|d\| v_d\| + 2\,\mathrm{dist}(x + \|d\| v_d, G) \leq \varepsilon \|d\| + \varepsilon \|d\| = 2\varepsilon \|d\|.$$

Therefore, if $x \in \mathbb{B}_\delta(x_0)$, $d \in S$ and $\|d\| \leq \delta$ then $\mathrm{dist}(x + d, G) \leq \varepsilon \|d\|$. $\square$

6.4 Proofs for section 4.

Instead of dealing only with metric spaces, we present here a more general theory, for normal spaces, i.e., spaces in which closed sets can be separated by open sets. We **do not** require these spaces to be Hausdorff. If F is normal then the Hausdorff property is not important for our purposes because we can argue using the closure $\overline{x}$ of the point x instead of x itself. Normality, however, is essential because we often need to replace the condition $A \cap B \cap C = \emptyset$ in definition 1.1 by $A \cap B = \emptyset$ when C is closed:

Lemma 6.6 *Let F be a normal topological space. If $C \subset F$ is closed then C is disconnected if and only if there exist open sets B_1 and B_2 such that $B_1 \cap B_2 = \emptyset$, $C \subset B_1 \cup B_2$, $B_1 \cap C \neq \emptyset$ and $B_2 \cap C \neq \emptyset$.* □

These are the properties and results about normal spaces which we will use:

Definition 6.2 *Let F be a topological space and $f : F \to \mathbb{R}$. We say that f has the Extension Property at level c if for every closed partition $f_{\leq c} = U_1 \cup U_2$ there exists a decreasing sequence $\{c_n, n \in \mathbb{N}\} \subset \mathbb{R}$ with $\lim_{n\to\infty} c_n = c$ and open partitions $f_{<c_n} = A_{1n} \cup A_{2n}$ with $U_1 \subset A_{1n}$, $U_2 \subset A_{2n}$ and $\overline{A}_{1n} \cap \overline{A}_{2n} = \emptyset$.* □

In words, f has the Extension Property at level c if every closed partition of $f_{\leq c}$ can be extended to open partitions of $f_{<c+\varepsilon}$ for ε arbitrarily small. This is all there is to the Connection Lemma in topological spaces and we claim that the same holds for the Mountain Pass Lemma. Here is the general Connection Lemma:

Lemma 6.7 *Let F be a topological space and $f : F \to \mathbb{R}$ a continuous function. If $f_{<a}$ is disconnected and there exists $c \in (a, +\infty]$ such that the set $f_{<c}$ is connected and f has the Extension Property for $b \in [a, c)$ then f has a Connection Point or a Bridge in $f^{-1}([a, c))$.* □

The hard part in using the Connection Lemma 6.7 is, of course, verifying that f has the Extension Property. In Morse Theory and Mountain Pass articles that assume differentiability the extension property is derived from the properties of the flow of the gradient field. Roughly, they consider a solution $\psi(x,t)$ of the differential equation

$$\frac{\partial \psi}{\partial t}(x,t) = -\nabla f(\psi(x,t)) \qquad \text{and} \qquad \psi(x,0) = x,$$

take $A_{in} = \{x \in F \text{ with } \psi(x, 1/n) \in U_i\}$ in definition 6.2 and use the Extension Property in one way or another. They need the Palais Smale condition to handle the zeroes of the gradient, which lead to stationary points in the gradient flow. References [9], [15], [25] do the same in Complete Metric Spaces using the weak slope. Therefore, the known versions of the Palais Smale Condition can be interpreted as requirements under which it is possible to define a well behaved gradient flow outside neighborhoods of a set of critical points. We propose something similar for topological spaces:

Definition 6.3 *Let F be a topological space $c \in \mathbb{R}$ and $f : F \to \mathbb{R}$. We say that $K \subset F$ is fc-compact if for every closed set $C \subset K$ with $\inf_{x\in C} f(x) > c$ there exists $d > c$ such that $C \subset f_{>d}$.* □

Definition 6.4 *Let F be a topological space, $f : F \to \mathbb{R}$ a continuous function and $c \in \mathbb{R}$. We say that f satisfies the Topological Palais Smale Condition at level $c \in \mathbb{R}$ if there exists a fc-compact set K_c such that for every open set $A \supset K_c$ there exists $d > c$ and a continuous function $\psi : f_{<d} \times [0,1] \to F$ such that, for $x \in f_{<d}$,*

$$\psi(x,0) = x, \tag{6.73}$$

$$f(\psi(x,t)) \leq f(x) \text{ for all } t \in [0,1], \tag{6.74}$$

$$f(\psi(x,1)) \geq c \Rightarrow \psi(x,t) \in A \text{ for all } t \in [0,1]. \tag{6.75}$$

□

If f is continuous then every compact set is fc-compact and condition 6.4 is satisfied if $f_{\leq d}$ is compact and $c < d$ (take $K_c = f_{\leq d}$ and $\psi(x,t) = x$). In the differentiable setting we could take K_c as the set of critical points $x \in f^{-1}(c)$ and [9], [15] and [25] explain how to extend this idea to complete metric spaces. We believe definition 6.4 captures the minimal features of the other Palais Smale Conditions which allow as to prove the existence of critical points in a topological setting. As the other definitions, it requires a flow with no stationary points outside neighborhoods of a critical fc-compact set. It is justified by the following lemma:

Lemma 6.8 *Let F be a normal topological space, $c \in \mathbb{R}$ and $f : F \to \mathbb{R}$ a continuous function. If f satisfies the Topological Palais Smale Condition at level c then it has the Extension Property at this level.* □

Therefore, when combined with lemma 6.7 the Topological Palais Smale Condition leads to the same results obtained under the usual Palais Smale Condition. Moreover, the following lemma shows that the topological condition is implied by the differentiable ones when derivatives (or their generalizations) are available:

Lemma 6.9 *Let F be a complete metric space and $f : F \to \mathbb{R}$ a continuous function. If f satisfies the Metric Palais Smale Condition 4.2 at level c then it also satisfies the Topological Palais Smale Condition at this level, with a set K_c compact and contained in $f^{-1}(c)$ and $\overline{f_c^{\min}} \subset K_c$.* □

(The set $f_c^{\min}$ in the previous lemma is defined in (5.2))

Finally, we have two useful properties of the Topological Palais Smale condition.

Lemma 6.10 *Let F be a topological space and $f : F \to \mathbb{R}$ a continuous function. If $\mu = \inf_{x \in F} f(x) > -\infty$ and f satisfies the Topological Palais Smale Condition for μ then $K_\mu \cap f^{-1}(\mu) \neq \emptyset$.* □

Lemma 6.11 *Let F be a normal topological space. Suppose $f : F \to \mathbb{R}$ is continuous and satisfies the Topological Palais Smale Condition at level c. If $f_{\leq c}$ is disconnected then there exists $b > c$ such that $f_{<a}$ is disconnected for all $a \in (c, b)$.* □

The proof of the results for normal spaces will presented in the next subsection. We are now ready to prove the results for complete metric spaces.

Proof of lemma 4.1. Lemma 4.1 follows from theorem 6.2 in the previous subsection and definition 3.3. □

Proof of lemma 4.2. We show that if $|df|(x_0) > 0$ then x_0 is not a Connection Point. In this case there exist positive σ and δ and a function $\varphi : B_\delta(x_0) \times [0, \delta] \to X$ satisfying (4.1). By the continuity of f there exists $\rho \in (0, \delta)$ such that $x \in B_\rho(x_0) \Rightarrow f(x) < c + \sigma\delta/2$ and by the local connectivity of X at x_0 there exists a connected neighborhood N of x_0 contained in $B_\rho(x_0)$. The continuity of φ implies that $C = \varphi(N, \delta)$ is connected. If $x \in C$ then $x = \varphi(w, \delta)$ for $w \in N$ and (4.1) implies that

$$f(x) = f(\varphi(w, \delta)) \leq f(w) - \sigma\delta \leq c + \sigma\delta/2 - \sigma\delta \leq c - \sigma\delta/2.$$

Thus $C \subset f_{<c}$ and every $x \in N \cap f_{<c}$ can be connected to C through the path $C_x = \{\varphi(x, t), t \in [0, \delta]\}$ and $C_x \subset f_{<c}$ by the second inequality in (4.1). Lemma 6.1 shows that x_0 is not a Connection Point. □

Proof of lemma 4.3. This lemma follows from lemmas 6.9 and 6.7, which are proved in the next section. □

6.4.1 Proofs of the results for normal spaces

Proof of lemma 6.6. It is clear that if B_1 and B_2 are as in the hypothesis then C is disconnected On the other hand, if C is disconnected then there exist open sets A_1 and A_2 such that

$$A_1 \cap A_2 \cap C = \emptyset, \qquad C \subset A_1 \cup A_2, \qquad A_1 \cap C \neq \emptyset \qquad \text{and} \qquad A_2 \cap C \neq \emptyset.$$

We now explain how to build sets B_1 and B_2 as in the hypothesis from A_1 and A_2. Let us denote the complement of A_i by A_i^c. Since $C \subset A_1 \cup A_2$ we have that $C \cap A_1^c \subset A_2 \cap A_1^c$. Analogously $C \cap A_2^c \subset A_1 \cap A_2^c$. Thus

$$(C \cap A_1^c) \cap (C \cap A_2^c) \subset (A_2 \cap A_1^c) \cap (A_1 \cap A_2^c) = \emptyset \tag{6.76}$$

and the closed sets $C \cap A_1^c$ and $C \cap A_2^c$ are disjoint. By the normality of F there exist open sets B_1 and B_2 with

$$B_1 \cap B_2 = \emptyset, \qquad C \cap A_1^c \subset B_2 \qquad \text{and} \qquad C \cap A_2^c \subset B_1. \tag{6.77}$$

The identity $C \cap A_1 \cap A_2 = \emptyset$ leads to

$$C = (C \cap (A_1 \cap A_2)) \cup (C \cap (A_1 \cap A_2)^c) = C \cap (A_1 \cap A_2)^c = C \cap (A_1^c \cup A_2^c) = (C \cap A_1^c) \cup (C \cap A_2^c) \subset B_2 \cup B_1.$$

Moreover, equation (6.77) implies that

$$B_2 \cap C \supset C \cap A_1^c \cap C = C \cap A_1^c \supset C \cap A_2 \cap A_1^c = (C \cap A_2 \cap A_1^c) \cup (C \cap A_2 \cap A_1) = (C \cap A_2) \cap (A_1^c \cup A_1) = C \cap A_2.$$

and

$$B_2 \cap C = B_2 \cap (A_2^c \cup A_2) \cap C = (B_2 \cap A_2^c \cap C) \cup (B_2 \cap A_2 \cap C) \subset (B_2 \cap B_1) \cup (A_2 \cap C) = A_2 \cap C.$$

Thus, $B_2 \cap C = A_2 \cap C$ and, analogously, $B_1 \cap C = A_1 \cap C$. These identities and (6.77) prove lemma 6.6. □

Proof of lemma 6.7. Since $f_{<a}$ is open and disconnected there exists an open partition $f_{<a} = A \cup C$. If $\overline{A} \cap \overline{C} \neq \emptyset$ then any $z \in \overline{A} \cap \overline{C}$ is a Connection Point at level a and we are done. To complete this proof we assume that there are no Connection Points in $f^{-1}([a,c))$ and that $\overline{A} \cap \overline{C} = \emptyset$ and show that there exists a bridge in $f^{-1}([a,c))$. We start by noticing that (A,C,a) is a member of the class $\mathscr{T}$ of triples (U,V,b) such that

$$b \in [a,c), \tag{6.78}$$

$$f_{<b} = U \cup V, \tag{6.79}$$

$$\overline{U} \cap \overline{V} = \emptyset, \tag{6.80}$$

$$A \subset U \text{ and } C \subset V \tag{6.81}$$

and U and V are open. Consider the partial order $\preceq$ in $\mathscr{T}$ given by

$$(U_1,V_1,b_1) \preceq (U_2,V_2,b_2) \Leftrightarrow (b_1 \leq b_2,\ U_1 \subset U_2 \text{ and } V_1 \subset V_2). \tag{6.82}$$

We claim that every chain for $\preceq$ has an upper bound. In fact, given a chain $\mathfrak{C} = \{(U_\lambda,V_\lambda,b_\lambda),\ \lambda \in \Lambda\}$ in $(\mathscr{T},\preceq)$ define $b_\infty = \sup_{\lambda\in\Lambda} b_\lambda$ and

$$U_\infty = \bigcup_{\lambda\in\Lambda} U_\lambda \qquad \text{and} \qquad V_\infty = \bigcup_{\lambda\in\Lambda} V_\lambda.$$

It is clear that $f_{<b_\infty} = U_\infty \cup V_\infty$, U_∞ and V_∞ are open and $U_\infty \cap V_\infty = \emptyset$. This implies that $U_\infty \cap \overline{V_\infty} = \overline{U_\infty} \cap V_\infty = \emptyset$. The open partition $f_{<b_\infty} = U_\infty \cup V_\infty$ shows that $b_\infty < c$, because $f_{<c}$ is connected and $f_{<b_\infty}$ is not. If $b_\infty \in \{b_\lambda,\ \lambda \in \Lambda\}$ then $(b_\lambda, U_\lambda, V_\lambda)$ for λ such that $b_\lambda = b_\infty$ is an upper bound for the chain. Thus, we can assume that $b_\infty > b_\lambda$ for all λ. Notice that there is no $z \in \overline{U_\infty} \cap \overline{V_\infty}$ with $f(z) = b_\infty$ (such z would be a Connection Point). We now show that $\overline{U_\infty} \cap \overline{V_\infty} = \emptyset$ by taking $z \in \overline{U_\infty}$ and showing that $z \notin \overline{V_\infty}$. In fact, we may assume that $f(z) < b_\infty$ and then there exist α such that $f(z) < b_\alpha$, because $b_\infty > b_\lambda$ for all λ. Equation (6.79) implies that $z \in U_\alpha \cup V_\alpha$, However, $z \notin V_\alpha$ because $\overline{U_\infty} \cap V_\alpha \subset \overline{U_\infty} \cap V_\infty = \emptyset$. Thus, $z \in U_\alpha$ and $z \notin \overline{V_\infty}$ because U_α is open and $U_\alpha \cap V_\infty \subset U_\infty \cap V_\infty = \emptyset$. Therefore, $\overline{U}_\infty \cap \overline{V}_\infty = \emptyset$ and $(U_\infty, V_\infty, b_\infty)$ is an upper bound for the chain $\mathfrak{C}$.

The last paragraph and Zorn's lemma yield a maximum element (U_*, V_*, b_*) for $\preceq$. If $B = f_{\leq b_*} - \left(\overline{U}_* \cup \overline{V}_*\right)$ could be decomposed as $B = B_u \cup B_v$ in such way that $\left(\overline{U}_* \cup \overline{B}_u\right) \cap \left(\overline{V}_* \cup \overline{B}_v\right) = \emptyset$ then applying f's Expansion Property for $U = \overline{U}_* \cup \overline{B}_u$ and $V = \overline{V}_* \cup \overline{B}_v$ would yield $d \in (b,c)$ and open sets U_d and V_d such that (U_d, V_d, d) would contradict the maximality of (U_*, V_*, b_*). Therefore, there are no B_u and B_v as above and B is a Bridge. □

Proof of lemma 6.8. This proof uses lemmas 6.12 and 6.13, which are proved at the end of this subsection. Let A_1 and A_2 be open sets with $f_{\leq c} \subset A_1 \cup A_2$ and $\overline{A}_1 \cap \overline{A}_2 = \emptyset$ and K_c the fc-compact set definition 6.4. The set $R = K_c - A_1 \cup A_2$ is also fc-compact and $R \cap f_{\leq c} = \emptyset$. Therefore, there exists $s > c$ such that $R \subset f_{>s}$. The open set $A = f_{>s} \cup A_1 \cup A_2$ contains K_c. Thus, the Topological Palais Smale Condition yields $d \in (c,s)$ and a continuous function $\psi : f_{<d} \times [0,1] \to X$ which satisfies (6.73) – (6.75). To complete this proof we show that $\varphi(x) = \psi(x,1)$ and, for $i = 1,2$,

$$B_i = \{x \in X \text{ with } \psi(x,t) \in A_i \text{ for } t \in [0,1]\} \cap f_{<d}, \tag{6.83}$$

satisfy (6.87)–(6.89). Lemma 6.13 shows that B_1 and B_2 are open. By taking $t = 0$ in the definitions of B_1 and B_2 and using (6.73) we conclude that $B_1 \subset A_1$, $B_2 \subset A_2$. Taking $t = 1$ in these definitions we get (6.87).

To verify (6.88) notice that if $x \in A_1 \cap f_{\leq c}$ then (6.74) implies that $\psi(x,t) \in f_{\leq c} \subset A_1 \cup A_2$ for all t. Since $\psi(x,0) = x \in A_1$, $C_x = \{\psi(x,t), t \in [0,1]\}$ is connected and $A_1 \cap A_2 = \emptyset$ we must have $C_x \subset A_1$. In particular, $\varphi(x) = \psi(x,1) \in A_1$. Therefore, (6.88) holds.

Finally, to prove (6.89), let us take $x \in f_{<d}$ with $f(\varphi(x)) \geq c$. In this case (6.75) shows that $\psi(x,t) \in A = f_{>c} \cup A_1 \cup A_2$ for all t. Moreover, (6.74) shows that $f(\psi(x,t)) < d < s$ for all t and, by definition of s, $\psi(x,t) \notin f_{>c}$ for all t. Therefore, $\psi(x,t) \in A_1 \cup A_2$ for all t. By the same connectivity argument of the last paragraph we conclude that either (i) $\psi(x,t) \in A_1$ for all t or (ii) $\psi(x,t) \in A_2$ for all t. Equation (6.83) shows that in case (i) $x \in B_1$ and in case (ii) $x \in B_2$. This proves (6.89). □

Proof of lemma 6.9. Our proof uses theorem (2.14) in [9]. The fc-compact set K_c is defined as

$$K_c = \left\{ x \in f^{-1}(c) \text{ with } |df|(x) = 0 \right\}.$$

It is compact and contained in $f^{-1}(c)$ and $\overline{f_c^{\min}} \subset K_c$. Theorem (2.14) in [9] can then be stated as:

Theorem 6.3 *(by Corvellec, Degiovanni and Marzocchi) Let X be a complete metric space, $f : X \to \mathbb{R}$ a continuous function and $c \in \mathbb{R}$. If f satisfies the Metric Palais Smale Condition at level c then given $\overline{\varepsilon} > 0$ and a neighborhood B of K_c and $\lambda > 0$ there exists $\varepsilon > 0$ and $\phi : X \times [0,1] \to X$ continuous with*

$$\operatorname{dist}(\phi(x,t),x) \leq \lambda t, \tag{6.84}$$
$$f(\phi(x,t)) \leq f(x), \tag{6.85}$$
$$|f(u) - c| \geq \overline{\varepsilon} \Rightarrow \phi(u,t) = u,$$
$$u \in f_{\leq c+\varepsilon} - B \Rightarrow f(\phi(x,1)) \leq c - \varepsilon. \tag{6.86}$$

□

Given an open set $A \supset K_c$ we apply theorem 6.3 with $B = \{x \in X \text{ with } \operatorname{dist}(x,K_c) < \delta\}$, $\delta = \operatorname{dist}(K_c, X - A)/3$, $\overline{\varepsilon} = 1$ and $\lambda = \delta$ and obtain ε and ϕ as in (6.84) – (6.86). Take $d = c + \varepsilon$ and ψ as the restriction of ϕ to $f_{<d} \times [0,1]$. Equation (6.84) shows that ψ satisfies condition (6.73) in definition 6.4 and equation (6.85) yields (6.74). Finally, (6.86) shows that if $f(\psi(x,1)) \geq c$ then $x \in B$. The definition of B and equations (6.84) imply that, for $t \in [0,1]$ and $x \in A$,

$$\operatorname{dist}(\psi(x,t),K_c) \leq 2\delta < 3\delta = \operatorname{dist}(X - A, K_c).$$

Since $K_c \subset A$, the last inequalities imply that $\psi(x,t) \in A$ for $t \in [0,1]$. In resume, we have shown that $f(\psi(x,1)) \geq c \Rightarrow \psi(x,t) \in A$ for $t \in [0,1]$. This is the last requirement in definition 6.4 and this proof is complete. □

Proof of lemma 6.10. We argue by contradiction to show that $a = \inf_{x \in K_\mu} f(x) = \mu$ (To handle the degenerate case $K_\mu = \emptyset$ take $a = \mu + 1$). In fact, if $a > \mu$ then the fc-compactness of K_μ yields $s \in (\mu, a)$ such that $K_\mu \subset f_{>s}$. Taking the open set $A = f_{>s}$, the Topological Palais Smale Condition yields d and ψ satisfying (6.73)–(6.75). However, taking $b = \min\{d,s\}$ we have that $b > \mu = \inf_{x \in F} f(x)$ and there exists $x \in F$ with $f(x) \in (\mu, b)$. By (6.74) the point $z = \psi(x,1)$ satisfies $f(z) < b \leq s$. Therefore, $z \notin A = f_{>s}$ and (6.75) implies that $f(z) = f(\psi(x,1)) < \mu$. This contradicts the minimality of μ and our argument is complete. □

Proof of lemma 6.11 This proof uses lemma 6.6 and lemma 6.13, which is stated and proved at the end of this subsection. Since F is normal and $f_{\leq c}$ is disconnected, lemma 6.6 yields disjoint open sets U_1 and U_2 and $f_{\leq c} \subset U_1 \cup U_2$. The set $C = K_c - (U_1 \cup U_2)$ is fc-compact and $f(x) > c$ for $x \in C$. Therefore, there exists $e > c$ such that $C \subset f_{>e}$ and the open set $A = U_1 \cup U_2 \cup f_{>e}$ contains K_c. Let ψ and d be the corresponding function and number given by definition 6.4. Take $b = \min\{d,e\}$. For each $a \in (c,b)$, $x \in f_{<a}$ and $t \in [0,1]$, equation (6.74) shows that $\psi(x,t) \notin f_{<d}$. As a consequence, equation (6.75) implies that if $x \in f_{<a}$ and $\psi(x,1) \geq c$ then $x \in B_1 \cup B_2$, where $B_i = \{x \in f_{<a} \text{ with } \psi(x,t) \in U_i \text{ for } t \in [0,1]\}$. The sets B_i are open by lemma 6.13 applied to $X = f_{<d}$ and $Y = [0,1]$. Defining $\varphi(x) = \psi(x,1)$, the sets $C_i = \varphi^{-1}(U_i \cap f_{<c})$ are also open and so are $D_i = B_i \cup C_i$. It is clear that $f_{<a} = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$, because $B_1 \cap B_2 = B_1 \cap C_2 = C_1 \cap B_2 = C_1 \cap C_2$ is a direct consequence of the way these sets were defined. Therefore, $f_{<a}$ is disconnected. □

Lemma 6.12 *Let F be a normal topological space, $f : F \to \mathbb{R}$ a continuous function and $c \in \mathbb{R}$. If for every pair A_1, A_2 of open sets with $f_{\leq c} \subset A_1 \cup A_2$ and $\overline{A}_1 \cap \overline{A}_2 = \emptyset$ there exist $d > c$ and open sets $B_1 \subset A_1 \cap f_{<d}$ and $B_2 \subset A_2 \cap f_{<d}$ and a continuous function $\varphi : f_{<d} \to F$ such that*

$$\varphi(B_1) \subset A_1 \quad \text{and} \quad \varphi(B_2) \subset A_2, \tag{6.87}$$
$$\varphi(A_1 \cap f_{<c}) \subset A_1 \quad \text{and} \quad \varphi(A_2 \cap f_{<c}) \subset A_2, \tag{6.88}$$

and, for $x \in f_{<d}$,

$$f(\varphi(x)) \geq c \quad \Rightarrow \quad x \in B_1 \cup B_2, \tag{6.89}$$

then f has the Extension Property at level c. □

Proof of lemma 6.12. Let $f_{\leq c} = U_1 \cup U_2$ be a closed partition of $f_{\leq c}$. The normality of X yields disjoint open sets $F_1 \supset U_1$ and $F_2 \supset U_2$. Normality also implies that there exist disjoint open sets $A_1 \supset U_1$ and $G_1 \supset X - F_1$. This implies that $\overline{A}_1 \cap (X - F_1) = \emptyset$ and then $\overline{A}_1 \subset F_1$. For the same reason there exists an open set $A_2 \supset U_2$ with $\overline{A}_2 \subset F_2$. Therefore, $\overline{A}_1 \cap \overline{A}_2 \subset F_1 \cap F_2 = \emptyset$ and the hypothesis yields $d > c$, B_1, B_2 and φ as in (6.87) – (6.89).

The sets

$$C_1 = A_1 \cap f_{<c} \qquad \text{and} \qquad C_2 = A_2 \cap f_{<c} \tag{6.90}$$

are open and disjoint. By the continuity of φ, the sets

$$D_1 = \varphi^{-1}(C_1) \qquad \text{and} \qquad D_2 = \varphi^{-1}(C_2) \tag{6.91}$$

are also open and disjoint. Equations (6.89) and (6.90) and the fact that $f_{<c} \subset A_1 \cup A_2$ lead to

$$f_{<d} \subset E_1 \cup E_2, \tag{6.92}$$

for

$$E_1 = B_1 \cup D_1 \qquad \text{and} \qquad E_2 = B_2 \cup D_2. \tag{6.93}$$

According to (6.87), $\varphi(B_1) \subset A_1$. This implies that $B_1 \cap D_2 = \emptyset$, because $\varphi(D_2) \subset C_2 \subset A_2$ and $A_2 \cap \varphi(B_1) = \emptyset$. For the same reason $B_2 \cap D_1 = \emptyset$. Therefore, $E_1 \cap E_2 = \emptyset$ and since these sets are open

$$\overline{E}_1 \cap E_2 = E_1 \cap \overline{E}_2 = \emptyset. \tag{6.94}$$

The sequence $c_n = ((n+1)c + d)/(n+2)$ is decreasing, $c_n \in (c,d)$ and $\lim_{n\to\infty} c_n = c$. We now show that

$$A_{1n} = E_1 \cap f_{<c_n} \qquad \text{and} \qquad A_{2n} = E_2 \cap f_{<c_n} \tag{6.95}$$

satisfy the requirements of definition 6.2. We start by noticing that (6.92) and $c_n < d$ yield

$$A_{1n} \cup A_{2n} = (E_1 \cup E_2) \cap f_{<c_n} \supset f_{<d} \cap f_{<c_n} = f_{<c_n}.$$

On the other hand, (6.95) shows that $A_{1n} \cup A_{2n} \subset f_{<c_n}$. Therefore, the requirement $A_{1n} \cup A_{2n} = f_{<c_n}$ is satisfied. To prove that $U_1 \subset A_{1n}$, consider $x \in U_1 \subset f_{\leq c}$. If $f(\varphi(x)) \geq c$ then (6.89) leads to

$$x \in (B_1 \cup B_2) \cap U_1 \subset B_1 \cap f_{\leq c} \subset A_{1n}.$$

If $f(\varphi(x)) < c$ then (6.90) implies that $\varphi(x) \in C_1$, because (6.88) shows that $\varphi(x) \in A_1$. Therefore $x \in D_1 = \varphi^{-1}(C_1)$ and $x \in (B_1 \cup D_1) \cap f_{\leq c} \subset (B_1 \cup D_1) \cap f_{<c_n} = A_{1n}$. This shows that the requirements $U_1 \subset A_{1n}$ and $U_2 \subset A_{2n}$ are satisfied too. Now, (6.95) shows that $\overline{A}_{1n} \subset \overline{E}_1$ and (6.94) implies that $\overline{A}_{1n} \cap E_2 = \emptyset$. Equation (6.95) also shows that $\overline{A}_{1n} \subset \overline{f_{<c_n}} \subset f_{<d}$ and (6.92) combined with $\overline{A}_{1n} \cap E_2 = \emptyset$ yield $\overline{A}_{1n} \subset E_1$. For the same reasons, $\overline{A}_{2n} \subset E_2$. Finally, $E_1 \cap E_2 = \emptyset$ implies the last requirement: $\overline{A}_{1n} \cap \overline{A}_{2n} = \emptyset$. □

Lemma 6.13 *Let X be a topological space and Y a compact set. If $\psi : X \times Y \to X$ is continuous and $A \subset X$ is open then $B = \{x \in X \text{ with } \psi(x,y) \in A \text{ for all } y \in Y\}$ is also open.* □

Proof of lemma 6.13. We show that the complement of B, $B^c = \{x \in X \text{ such that } \psi(x,y) \in A^c \text{ for some } y \in Y\}$, is closed. To do that we take a net $\{x_\gamma, \gamma \in \Gamma\} \subset B^c$ converging to $x_\infty \in X$ and show that $x_\infty \in B^c$. For every $\gamma \in \Gamma$ there exists $y_\gamma \in Y$ such that $\psi(x_\gamma, y_\gamma) \in A^c$. Since Y is compact $\{y_\gamma, \gamma \in \Gamma\}$ has a subnet converging to some $y_\infty \in Y$. Without loss of generality we may assume that $\{y_\gamma, \gamma \in \Gamma\}$ itself is this subnet. Since ψ is continuous and A^c is close, $\{\psi(x_\gamma, y_\gamma), \gamma \in \Gamma\} \subset A^c$ converges to $\psi(x_\infty, y_\infty) \in A^c$. Therefore $x_\infty \in B^c$ and we are done. □

6.5 Proofs for section 5.

As for the proofs about metric spaces, here we develop the theory in the more abstract frame of normal topological spaces, which may not be Hausdorff. At the end of this subsection we prove the following lemmas:

Lemma 6.14 *Let F be a topological space and $x \in C \subset F$. If $\overline{\mathscr{C}(x,C)} \subset C$ then $\mathscr{C}(x,C)$ is closed. In particular, if C is closed then $\mathscr{C}(x,C)$ is closed.* □

Lemma 6.15 *Let F be a topological space. If $A \subset B \subset F$ and $x \in A$ then $\mathscr{C}(x,A) \subset \mathscr{C}(x,B)$.* □

Lemma 6.16 *Let F be a topological space. If $A \subset B \subset F$ and $\mathscr{C}(x,B) \subset A$ then $\mathscr{C}(x,A) = \mathscr{C}(x,B)$.* □

Lemma 6.17 *Let F be a normal topological space and let $D \subset F$ be closed and compact. If $x, y \in D$ and $y \notin \mathscr{C}(x,D)$ then there exist disjoint open sets A and B with $D \subset A \cup B$, $\mathscr{C}(x,D) \subset A$ and $y \in B$.* □

Lemma 6.18 *Let F be a normal topological space, $f : F \to \mathbb{R}$ a continuous function and $c > \inf_{x \in F} f(x)$. If $\overline{f_c^{\min}}$ is compact and there exist $x \in f_c^{\min}$ and disjoint open sets A and C such that $\overline{f_{<c}} \subset A$ and $\mathscr{C}\left(x, f_c^{\min}\right) \subset C$ then $f_{\leq c}$ is disconnected.* $\square$

Proof of theorem 5.1 for normal spaces. Lemma 6.10 shows that the set $f_\mu^{\min}$ in definition 5.2 is not empty for $\mu = \inf_{x \in F} f(x)$. Therefore, f has at least one global minimizer. If f has only one local minimizer then it satisfies item (i) in the connectivity alternative and we are done. Let us then assume that f has multiple local minimizers. Lemma 6.11 shows that the sets $f_{\leq c}$ are connected for all $c \in \mathbb{R}$. Therefore $f_\mu^{\min} = f_{\leq \mu}$ is connected. If $f_\mu^{\min}$ has more than one element then it is a Plateau according to definition 2.4 and item (iv).(b) in the Connectivity alternative is satisfied. If w is another local minimizer with $c = f(w) > \mu$ then $\overline{\mathscr{C}(x, f_c^{\min})} \cap \overline{f_{<c}} \neq \emptyset$ because $x \notin D_c = \emptyset$. The set $\mathscr{C}\left(x, f_c^{\min}\right)$ is a Terrace and w satisfies item (iv).(a) in the Connectivity alternative. $\square$

Proof of lemma 5.1. If $x \in D_c$ then $\overline{f_{<c}} \cap \overline{\mathscr{C}(x, f_c^{\min})} = \emptyset$ and $\overline{\mathscr{C}(x, f_c^{\min})} \subset f_c^{\min}$ because $\overline{f_c^{\min}} \subset f_c^{\min} \cup \overline{f_{<c}}$. and lemma 6.14 shows that $\mathscr{C}\left(x, f_c^{\min}\right)$ is closed. The normality of F yields disjoint open sets $A \supset \overline{f_{<c}}$ and $C \supset \mathscr{C}\left(x, f_c^{\min}\right)$. and lemma 5.1 follows from lemma 6.18. $\square$

Proof of lemma 6.14. We claim that if $y \in \overline{\mathscr{C}(x,C)}$ then $y \in \mathscr{C}(x,C)$ because $U = \{y\} \cup \mathscr{C}(x,C) \subset C$ is connected and contains y. In fact, let A and B be open sets such that $U \subset A \cup B$, $A \cap B \cap U = \emptyset$ and $x \in A$. By the connectivity of $\mathscr{C}(x,C)$ we have $\mathscr{C}(x,C) \subset A$. This shows that $B \cap \mathscr{C}(x,C) = \emptyset$ and $y \notin B$ because $y \in \overline{\mathscr{C}(x,C)}$. Therefore, $y \in A$, $B \cap U = \emptyset$ and U is indeed connected. Finally, $\mathscr{C}(x,C) \subset C$ implies that $\overline{\mathscr{C}(x,C)} \subset \overline{C}$ and if C is closed then $\overline{C} = C$ and the first part of this proof implies that $\mathscr{C}(x,C)$ is closed. $\square$

Proof of lemma 6.15. If $x \in \mathscr{C}(x,A)$ then there exists a connected set $C \subset A$ with $x, y \in C$. Since $A \subset B$ we have that $C \subset B$ and $y \in \mathscr{C}(x,B)$. $\square$

Proof of lemma 6.16. Lemma 6.15 shows that $\mathscr{C}(x,A) \subset \mathscr{C}(x,B)$. On the other hand, if $y \in \mathscr{C}(x,B)$ then there exists $C \subset B$ connected with $x, y \in C$. This implies that $C \subset \mathscr{C}(x,B) \subset A$, $y \in \mathscr{C}(x,A)$ and $\mathscr{C}(x,B) \subset \mathscr{C}(x,A)$. $\square$

Proof of lemma 6.17. Consider the set

$$\mathscr{D}(x,D) = \{y \in D \text{ such that if } A \text{ and } B \text{ are open and disjoint, } D \subset A \cup B \text{ and } \mathscr{C}(x,D) \subset A \text{ then } y \in A\}.$$

This set is closed in D because if $y \in D - \mathscr{D}(x,D)$ then there exist disjoint open sets A and B with $\mathscr{C}(x,D) \subset A$ and $y \in B$ and $B \cap D$ is a neighborhood of y in D contained in $D - \mathscr{D}(x,D)$. Since D is closed in F, $\mathscr{D}(x,D)$ is also closed in F. It is also clear from the definition of $\mathscr{D}(x,D)$ that $\mathscr{C}(x,D) \subset \mathscr{D}(x,D)$.

Lemma 6.17 is logically equivalent to the statement $\mathscr{D}(x,D) \subset \mathscr{C}(x,D)$. Since $x \in \mathscr{D}(x,D) \subset D$, to prove this statement we show that $\mathscr{D}(x,D)$ is connected. The normality of F, the fact that $\mathscr{D}(x,D)$ is closed and lemma 6.6 show that to verify this connectivity it is enough to assume that R and S are disjoint open sets with $x \in R$ and $\mathscr{D}(x,D) \subset R \cup S$ and prove that $\mathscr{D}(x,D) \cap S = \emptyset$. Since $\mathscr{C}(x,D) \subset \mathscr{D}(x,D)$ we have $\mathscr{C}(x,D) \subset R \cup S$ and the connectivity of $\mathscr{C}(x,D)$ and $x \in R$ imply that $\mathscr{C}(x,D) \subset R$.

We now show that $\mathscr{D}(x,D) \cap S = \emptyset$. The set $K = D - (R \cup S)$ is closed. Since D is compact K is also compact. Since $K \subset D - \mathscr{D}(x,D)$, if $y \in K$ then $y \in D - \mathscr{D}(x,D)$ and there exist open sets A_y and B_y such that

$$\mathscr{C}(x,D) \subset A_y, \qquad y \in B_y, \qquad D \subset A_y \cup B_y, \qquad \text{and} \qquad A_y \cap B_y = \emptyset. \tag{6.96}$$

The B_y cover K. Let $B_{y_1}, \ldots, B_{y_n}$ be a finite sub covering and define $B = \bigcup_{i=1,\ldots,n} B_{y_i}$ and $A = \bigcap_{i=1,\ldots,n} A_{y_i}$. The equations in (6.96) lead to

$$\mathscr{C}(x,D) \subset A, \qquad K \subset B, \qquad D \subset A \cup B, \qquad \text{and} \qquad A \cap B = \emptyset. \tag{6.97}$$

By definition of K we have $D \subset K \cup R \cup S$. Combining this with $D \subset A \cup B$ and $A \cap K \subset A \cap B = \emptyset$ we obtain

$$D \subset (B \cup (A \cap S)) \quad \cup \quad (A \cap R). \tag{6.98}$$

We have seen above that $\mathscr{C}(x,D) \subset R$ and (6.97) shows that $\mathscr{C}(x,D) \subset A$. Therefore, $\mathscr{C}(x,D) \subset A \cap R$. This identity combined with the open partition (6.98) and the definition of $\mathscr{D}(x,D)$ imply that $\mathscr{D}(x,D) \subset A \cap R$. Since $R \cap S = \emptyset$ we conclude that $\mathscr{D}(x,D) \cap S = \emptyset$ and this proof is complete. $\square$

Proof of lemma 6.18. The set $B = (F - A - C) \cap \overline{f_c^{\min}}$ is closed and compact. Using the identity $f_{\leq c} = f_c^{\min} \cup \overline{f_{<c}}$ the reader can verify that $\overline{f_c^{\min}} = \left(A \cap \overline{f_c^{\min}}\right) \cup D$ for $D = B \cup \left(C \cap f_c^{\min}\right) \subset f_c^{\min}$ and $\mathscr{C}\left(x, f_c^{\min}\right) \subset D$.

Moreover, $D = \overline{f_c^{\min}} - A$ is closed and it follows from lemma 6.16 that $\mathscr{C}(x,D) = \mathscr{C}(x, f_c^{\min}) \subset C$. Therefore, if $y \in B$ then $y \notin \mathscr{C}(x,D)$ and lemma 6.17 yields disjoint open sets $U_y \supset \mathscr{C}(x,D)$ and $V_y \ni y$ such that $D \subset U_y \cup V_y$. The open sets $\{V_y, y \in B\}$ cover the compact set B. Let $V_{y_1}, \ldots, V_{y_n}$ be a finite sub covering. The sets

$$U = \bigcap_{i=1,\ldots,n} U_{y_i} \qquad \text{and} \qquad V = \bigcup_{i=1,\ldots,n} V_{y_i}$$

are open and disjoint, $D \subset U \cup V$, $\mathscr{C}(x,D) \subset U$ and $B \subset V$. To complete this proof we show that the open sets $G = A \cup V$ and $H = U \cap C$ are disjoint, $G \cap f_{\leq c} \neq \emptyset$ and $H \cap f_{\leq c} \neq \emptyset$ and $f_{\leq c} \subset G \cup H$. In fact, $G \cap f_{\leq c} \neq \emptyset$ since $f_{<c} \subset G$ and $f_{<c} \neq \emptyset$ because $c > \inf_{x \in F} f(x)$. The intersection $H \cap f_{\leq c}$ is not empty because $\mathscr{C}(x,D) \subset H$. The set $G \cap H$ is empty because $A \cap C = \emptyset$, $V \cap U = \emptyset$ and

$$G \cap H = (A \cup V) \cap (U \cap C) = (A \cap C \cap U) \cup (V \cap U \cap C) = \emptyset \cup \emptyset = \emptyset.$$

Finally, $f_{\leq c} \subset A \cup B \cup \left(C \cap f_c^{\min}\right) = A \cup D$, $B \subset V$ and $D \subset U \cup V$ and (i) If $x \in A$ then $x \in G$, (ii) If $x \in B$ then $x \in V$ and $x \in G$ and (iii) If $x \in C \cap f_c^{\min}$ then $x \in D \subset U \cup V$. If $x \in V$ then $x \in G$. If $x \notin V$ then $x \in U \cap C = H$. The items (i),(ii) and (iii) show that $f_{\leq c} \subset G \cup H$ and we are done . □

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