On the Expected Complexity of Random Convex Hulls*

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Abstract

In this paper we present several results on the expected complexity of a convex hull of n points chosen uniformly and independently from a convex shape.

(i) We show that the expected number of vertices of the convex hull of n points, chosen uniformly and independently from a disk, is $O(n^{1/3})$. Applying the same technique to the case where the points are chosen from a convex polygon with k sides, is $O(k \log n)$. Those results are well known (see [RS63, Ray70, PS85]), but, we believe that the elementary proof given here are simpler and more intuitive.

(ii) Let $\mathcal{D}$ be a set of directions in the plane, we define a generalized notion of convexity induced by $\mathcal{D}$, which extends both rectilinear convexity, and standard convexity.

We prove that the expected complexity of the $\mathcal{D}$-convex hull of a set of n points, chosen uniformly and independently from a disk, is $O\left(n^{1/3} + \sqrt{n\alpha(\mathcal{D})}\right)$, where $\alpha(\mathcal{D})$ is the largest angle between two consecutive vectors in $\mathcal{D}$. This result extends the known bounds for the cases of rectilinear and standard convexity.

(iii) Let $\mathcal{B}$ be an axis parallel hypercube in $\mathbb{R}^d$. We prove that the expected number of points on the boundary of the quadrant hull of a set S of n points, chosen uniformly and independently from $\mathcal{B}$ is $O(\log^{d-1} n)$. Quadrant hull of a set of points is an extension of rectilinear convexity to higher dimensions. In particular, this number is larger than the number of maxima in S, and is also larger than the number of points of S that are vertices of the convex hull of S. Implying that this bound holds also for those cases.

Those bounds are known [BKST78], but we believe the new proof is simpler.

1 Introduction

Let C be a fixed compact convex shape, and let X_n be a random sample of n points chosen uniformly and independently from C. Let Z_n denote the number of vertices of the convex hull of X_n. Rényi and Sulanke [RS63] showed that $E[Z_n] = O(k \log n)$, when C is a convex polygon with k vertices in the plane. Raynaud [Ray70] showed that expected number of facets of the convex hull is $O(n^{(d-1)/(d+1)})$, where C is a ball in $\mathbb{R}^d$, so $E[Z_n] = O(n^{1/3})$ when C is a disk in the plane. Raynaud [Ray70] showed that the expected number of facets of $ConvexHull(X_n)$ is

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$O\left((\log(n))^{(d-1)/2}\right)$, where the points are chosen from $\mathbb{R}^d$ by a d-dimensional normal distribution. See [WW93] for a survey of related results.

All these bounds are essentially derived by computing or estimating integrals that quantify the probability of two specific points of X_n to form an edge of the convex hull (multiplying this probability by $\binom{n}{2}$ gives $E[Z_n]$). Those integrals are fairly complicated to analyze, and the resulting proofs are rather long, counter-intuitive and not elementary.

Efron [Efr65] showed that instead of arguing about the expected number of vertices directly, one can argue about the expected area/volume of the convex hull, and this in turn implies a bound on the expected number of vertices of the convex hull. In this paper, we present a new argument on the expected area/volume of the convex hull. The argument goes as follows: Decompose C the into smaller shapes (called tiles). Using the topology of the tiling and the underlining type of convexity, we argue about the expected number of tiles that are exposed by the random convex hull, where a tile is exposed if it does not lie completely in the interior of the random convex hull. Resulting in a lower bound on the area/volume of the random convex hull. We apply this technique to the standard case, and also for more exotic types of convexity.

In Section 2, we give a rather simple and elementary proofs of the aforementioned bounds $E[Z_n] = O(n^{1/3})$ for C a disk, and $E[Z_n] = O(k \log n)$ for C a convex k-gon. We believe that these new elementary proofs are indeed simpler and more intuitive[1] than the previous integral-based proofs.

The question on the expected complexity of the convex hull remains valid, even if we change our type of convexity. In Section 3, we define a generalized notion of convexity induced by $\mathcal{D}$, a given set of directions. This extends both rectilinear convexity, and standard convexity. We prove that the expected complexity of the $\mathcal{D}$-convex hull of a set of n points, chosen uniformly and independently from a disk, is $O\left(n^{1/3} + \sqrt{n\alpha(\mathcal{D})}\right)$, where $\alpha(\mathcal{D})$ is the largest angle between two consecutive vectors in $\mathcal{D}$. This result extends the known bounds for the cases of rectilinear and standard convexity.

Finally, in Section 4, we deal with another type convexity, which is an extension of the generalized convexity mentioned above for the higher dimensions, where the set of the directions is the standard orthonormal basis of $\mathbb{R}^d$. We prove that the expected number of points that line on the boundary of the quadrant hull of n points, chosen uniformly and independently from the axis-parallel unit hypercube in $\mathbb{R}^d$, is $O(\log^{d-1} n)$. This readily implies, imply $O(\log^{d-1} n)$ bounds on the expected number of maxima and expected number of vertices of the convex hull of such a point set. Those bounds are known [BKST78], but we believe the new proof is simpler and more intuitive.

[1]Preparata and Shamos [PS85, pp. 152] comment on the older proof for the case of a disk: "Because the circle has no corners, the expected number of hull vertices is comparatively high, although we know of no elementary explanation of the $n^{1/3}$ phenomenon in the planar case." It is the author's belief that the proof given here remedies this situation.

2 On the Complexity of the Convex Hull of a Random Point Set

In this section, we show that the expected number of vertices of the convex hull of n points, chosen uniformly and independently from a disk, is $O(n^{1/3})$. Applying the same technique to a convex polygon with k sides, we prove that the expected number of vertices of the convex hull is $O(k \log n)$. As already noted, these results are well known ([RS63, Ray70, PS85]), but we believe that the elementary proofs given here are simpler and more intuitive.

Lemma 2.1 *Let C be a bounded convex set in the plane, such that the expected area of the convex hull of n points, chosen uniformly and independently from C, is at least $(1 - f(n))\, Area(C)$, where $1 \geq f(n) \geq 0$, for $n \geq 0$. Then the expected number of vertices of the convex hull is $\leq nf(n/2)$.*

Proof: Let N be a random sample of n points, chosen uniformly and independently from C. Let N_1 (resp. N_2) denote the set of the first (resp. last) $n/2$ points of N. Let V_1 (resp. V_2) denote the number of vertices of $H = CH(N_1 \cup N_2)$ that belong to N_1 (resp. N_2), where $CH(N_1 \cup N_2) = \text{ConvexHull}(N_1 \cup N_2)$.

Clearly, the expected number of vertices of C is $E[V_1] + E[V_2]$. On the other hand,

$$E\left[V_1 \,\middle|\, N_2\right] \leq \frac{n}{2}\left(\frac{Area(C) - Area(CH(N_2))}{Area(C)}\right),$$

since V_1 is bounded by the expected number of points of N_1 falling outside $CH(N_2)$.

We have

$$\begin{aligned} E[V_1] &= E_{N_2}\left[E[V_1|N_2]\right] \leq E\left[\frac{n}{2}\left(\frac{Area(C) - Area(CH(N_2))}{Area(C)}\right)\right] \\ &\leq \frac{n}{2} f(n/2), \end{aligned}$$

since $E[X] = E_Y[E[X|Y]]$ for any two random variables X, Y. Thus, the expected number of vertices of H is $E[V_1] + E[V_2] \leq nf(n/2)$. ■

Remark 2.2 Lemma 2.1 is known as *Efron's Theorem.* See [Efr65].

Theorem 2.3 *The expected number of vertices of the convex hull of n points, chosen uniformly and independently from the unit disk, is $O(n^{1/3})$.*

Proof: We claim that the expected area of the convex hull of n points, chosen uniformly and independently from the unit disk, is at least $\pi - O\left(n^{-2/3}\right)$.

Indeed, let D denote the unit disk, and assume without loss of generality, that $n = m^3$, where m is a positive integer. Partition D into m sectors, $\mathcal{S}_1, \ldots, \mathcal{S}_m$, by placing m equally spaced points on the boundary of D and connecting them to the origin. Let $D_1, \ldots, D_{m^2}$ denote the m^2 disks centered at the origin, such that (i) $D_1 = D$, and (ii) $Area(D_{i-1}) - Area(D_i) = \pi/m^2$, for $i = 2, \ldots, m^2$. Let r_i denote the radius of D_i, for $i = 1, \ldots, m^2$.

Let $S_{i,j} = (D_i \setminus D_{i+1}) \cap \mathcal{S}_j$, and $S_{m^2,j} = D_{m^2} \cap \mathcal{S}_j$, for $i = 1, \ldots, m^2 - 1$, $j = 1, \ldots, m$. The set $S_{i,j}$ is called the i-th *tile* of the sector $\mathcal{S}_j$, and its area is π/n, for $i = 1, \ldots, m^2$, $j = 1, \ldots, m$.

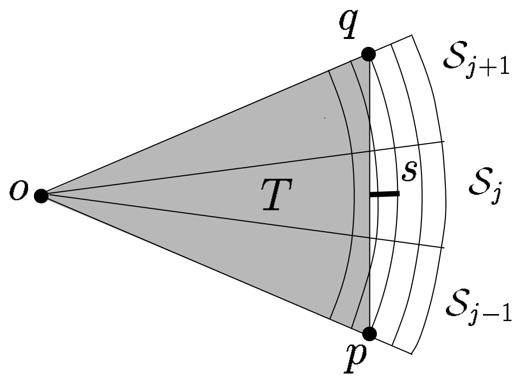

Figure 1: Illustrating the proof that bounds the number of tiles exposed by T inside $\mathcal{S}_j$

Let N be a random sample of n points chosen uniformly and independently from D. Let X_j denote the first index i such that $N \cap S_{i,j} \neq \emptyset$, for $j = 1, \ldots, m$. The probability that $X_j = k$ is upper-bounded by the probability that the tiles $S_{1,j}, \ldots, S_{(k-1),j}$ do not contain any point of N; namely, by $\left(1 - \frac{k-1}{n}\right)^n$. Thus, $P[X_j = k] \leq \left(1 - \frac{k-1}{n}\right)^n \leq e^{-(k-1)}$, since $1 - x \leq e^{-x}$, for $x \geq 0$. Thus,

$$E\left[X_j\right] = \sum_{k=1}^{m^2} kP[X_j = k] \leq \sum_{k=1}^{m^2} ke^{-(k-1)} = O(1),$$

for $j = 1, \ldots, m$.

Let C denote the convex hull of $N \cup \{o\}$, where o is the origin. The tile $S_{i,j}$ is *exposed* by a set K, if $S_{i,j} \setminus K \neq \emptyset$. We claim that at most $X_{j-1} + X_{j+1} + O(1)$ tiles are exposed by C in the sector $\mathcal{S}_j$, for $j = 1, \ldots, m$ (where we put $X_0 = X_m$, $X_{m+1} = X_1$).

Indeed, let $w = max(X_{j-1}, X_{j+1})$, and let p, q be the two points in $S_{j-1,w}, S_{j+1,w}$, respectively, such that the number of sets exposed by the triangle $T = \triangle opq$, in the sector $\mathcal{S}_i$, is maximal. Both p and q lie on ∂D_{w+1} and on the external radii bounding $\mathcal{S}_{j-1}$ and $\mathcal{S}_{j+1}$, as shown in Figure 1. Clearly, any tile which is exposed in $\mathcal{S}_j$ by C is also exposed by T. Let s denote the segment connecting the middle of the base of T to its closest point on ∂D_w. The number of tiles in $\mathcal{S}_j$ exposed by T is bounded by $\max(X_{j-1}, X_{j+1})$, plus the number of tiles intersecting the segment s. The length of s is

$$|oq| - |oq| \cos\left(\frac{3}{2} \cdot \frac{2\pi}{m}\right) \leq 1 - \cos\left(\frac{3}{2} \cdot \frac{2\pi}{m}\right) \leq \frac{1}{2}\left(\frac{3\pi}{m}\right)^2 = \frac{4.5\pi^2}{m^2},$$

since $\cos(x) \geq 1 - x^2/2$, for $x \geq 0$.

On the other hand, $r_{i+1} - r_i \geq r_i - r_{i-1} \geq 1/(2m^2)$, for $i = 2, \ldots, m^2$. Thus, the segment s intersects at most $\lceil ||s||/(1/(2m^2)) \rceil = \lceil 9\pi^2 \rceil = 89$ tiles, and we have that the number of tiles exposed in the sector $\mathcal{S}_i$ by C is at most $\max(X_{j-1}, X_{j+1}) + 89 \leq X_{j-1} + X_{j+1} + 89$, for $j = 1, \ldots, m$.

Thus, the expected number of tiles exposed by C is at most

$$E\left[\sum_{i=1}^{m} (X_{j-1} + X_{j+1} + 89)\right] = O(m).$$

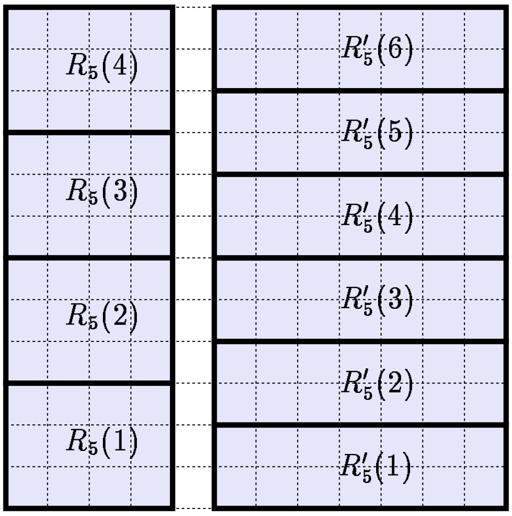

Figure 2: Illustrating the proof that bounds the number of tiles exposed by $CH(N)$ inside the j-th column, by using a non-uniform tiling of the strips to the left and to the right of the j-th column. The area of such a larger tile is at least $1/n$.

The area of $C' = CH(N)$ is bounded from below by the area of tiles which are not exposed by C'. The probability that $C' \neq C$ (namely, the origin is not inside C', or, equivalently, all points of N lie in some semidisk) is at most $2\pi/2^n$, as easily verified. Hence the expected area of C' is at least

$$E[Area(C)] - P\left[C \neq C'\right]\pi = \pi - O(m)\frac{\pi}{n} - \frac{2\pi}{2^n}\pi = \pi - O\left(n^{-2/3}\right).$$

The assertion of the theorem now follows from Lemma 2.1. ∎

Lemma 2.4 *The expected number of vertices of the convex hull of n points, chosen uniformly and independently from the unit square, is $O(\log n)$.*

Proof: We claim that the expected area of the convex hull of n points, chosen uniformly and independently from the unit square, is at least $1 - O\left(\log(n)/n\right)$.

Let S denote the unit square. Partition S into n rows and n columns, such that S is partitioned into n^2 identical squares. Let $S_{i,j} = [(i-1)/n, i/n] \times [(j-1)/n, j/n]$ denote the j-th square in the i-th column, for $1 \leq i, j \leq n$. Let $\mathcal{S}_i = \cup_{j=1}^{n} S_{i,j}$ denote the i-th column of S, for $i = 1, \ldots, n$, and let $\mathcal{S}(l,k) = \cup_{i=l}^{k} \mathcal{S}_i$, for $1 \leq l \leq k \leq n$.

Let N be a random sample of n points chosen uniformly and independently from S. Let X_j denote the first index i such that $N \cap (\cup_{l=1}^{j-1} S_{l,i}) \neq \emptyset$, for $j = 2, \ldots, n-1$; namely, X_j is the index of the first row in $\mathcal{S}(1, j-1)$ that contains a point from N. Symmetrically, let X'_j be the index of the first row in $\mathcal{S}(j+1, n)$ that contains a point of N. Clearly, $E[X_j] = E[X'_{n-j+1}]$, for $j = 2, \ldots, n-1$.

Let Z_j denote the number of squares $S_{i,j}$ in the bottom of the j-th column that are exposed by $CH(N)$, for $j = 2, \ldots, n-1$. Arguing as in the proof of Theorem 2.3, we have that $Z_j \leq \max(X_j, X'_j) \leq X_j + X'_j$. Thus, in order to bound $E[Z_j]$, we first bound $E[X_j]$ by covering the strips $\mathcal{S}(1, j-1), \mathcal{S}(j+1, n)$ by tiles of area $\geq 1/n$. In particular, let $h(j) = \lceil n/(j-1) \rceil$, and let

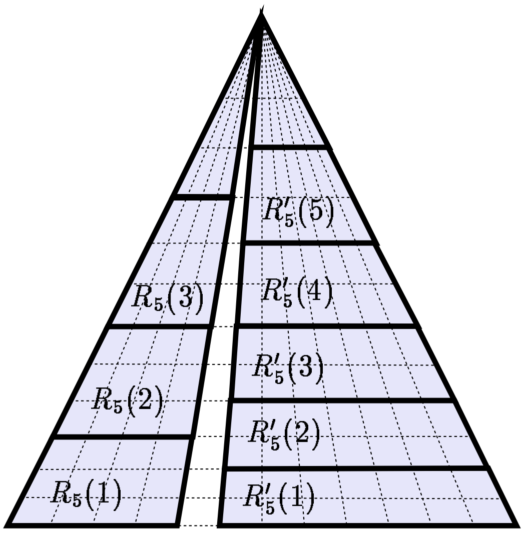

Figure 3: Illustrating the proof of Lemma 2.4 for the case of a triangle.

$R_j(m) = [0, (j-1)/n] \times [h(j)(m-1)/n, h(j)m/n]$, and let $R'_j(m) = [(j+1)/n, 1] \times [h(j)(m-1)/n, h(j)m/n]$, for $j = 2, \ldots, n-1$. See Figure 2.

Let Y_j denote the minimal index i such that $R_j(i) \cap N \neq \emptyset$. The area of $R_j(i)$ is at least $1/n$, for any i and j. Arguing as in the proof of Theorem 2.3, it follows that $E[Y_j] = O(1)$. On the other hand, $E[X_j] \leq h(j)E[Y_j] = O(n/(j-1))$. Symmetrically, $E[X'_j] = O(n/(n-j))$.

Thus, by applying the above argument to the four directions (top, bottom, left, right), we have that the expected number of squares $S_{i,j}$ exposed by $CH(N)$ is bounded by

$$4n - 4 + 4\sum_{j=2}^{n-1} E[Z_j] < 4n + 4\sum_{j=2}^{n-1}(E[X_j] + E[X'_j]) = 4n + 8\sum_{j=2}^{n-1} O\left(\frac{n}{j-1}\right) = O(n \log n),$$

where $4n - 4$ is the number of squares adjacent to the boundary of S.

Since the area of each square is $1/n^2$, it follows that the expected area of $CH(N)$ is at least $1 - O(\log(n)/n)$.

By Lemma 2.1, the expected number of vertices of the convex hull is $O(\log n)$. ∎

Lemma 2.5 *The expected number of vertices of the convex hull of n points, chosen uniformly and independently from a triangle, is $O(\log n)$.*

Proof: We claim that the expected area of the convex hull of n points, chosen uniformly and independently from a triangle T, is at least $(1 - O(\log(n)/n))Area(T)$. We adapt the tiling used in Lemma 2.4 to a triangle. Namely, we partition T into n equal-area triangles, by segments emanating from a fixed vertex, each of which is then partitioned into n equal-area trapezoids by segments parallel to the opposite side, such that each resulting trapezoid has area $1/n^2$. See Figure 3.

Notice that this tiling has identical topology to the tiling used in Lemma 2.4. Thus, the proof of Lemma 2.4 can be applied directly to this case, repeating the tiling process three times, once for each vertex of T. This readily implies the asserted bound. ∎

Theorem 2.6 *The expected number of vertices of the convex hull of n points, chosen uniformly and independently from a convex polygon P having k sides, is $O(k \log n)$.*

Proof: We triangulate P into k triangles $T_1, \ldots, T_k$. Let N be a random sample of n points, chosen uniformly and independently from P. Let $Y_i = |T_i \cap N|$, $N_i = T_i \cap N$, and $Z_i = |CH(N_i)|$, for $i = 1, \ldots, k$. Notice that the distribution of the points of N_i inside T_i is identical to the distribution of Y_i points chosen uniformly and independently from T_i. In particular, $E[Z_i|Y_i] = O(\log Y_i)$, by Lemma 2.5, and $E[Z_i] = E_{Y_i}[E[Z_i|Y_i]] = O(\log n)$, for $i = 1, \ldots, k$.

Thus, $E[|CH(N)|] \leq E\left[\sum_{i=1}^{k} |CH(N_i)|\right] \leq \sum_{i=1}^{k} E[Z_i] = O(k \log n)$. ∎

3 On the Expected Complexity of a Generalized Convex Hull Inside a Disk

In this section, we derive a bound on the expected complexity on a generalized convex hull of a set of points, chosen uniformly and independently for the unit disk. The new bound matches the known bounds, for the case of standard convexity and maxima. The bound follows by extending the proof of Theorem 2.3.

We begin with some terminology and some initial observations, most of them taken or adapted from [MP97]. A set $\mathcal{D}$ of vectors in the plane is a *set of directions*, if the length of all the vectors in $\mathcal{D}$ is 1, and if $v \in \mathcal{D}$ then $-v \in \mathcal{D}$. Let $\mathcal{D}_{\mathbb{R}}$ denote the set of all possible directions. A set C is *$\mathcal{D}$-convex* if the intersection of C with any line with a direction in $\mathcal{D}$ is connected. By definition, a set C is convex (in the standard sense), if and only if it is $\mathcal{D}_{\mathbb{R}}$-convex.

For a set C in the plane, we denote by $\mathcal{CH}_D(C)$ the *$\mathcal{D}$-convex hull* of C; that is, the smallest $\mathcal{D}$-convex set that contains C. While this seems like a reasonable extension of the regular notion of convexity, its behavior is counterintuitive. For example, let $\mathcal{D}_Q$ denote the set of all rational directions (the slopes of the directions are rational numbers). Since $\mathcal{D}_Q$ is dense in $\mathcal{D}_{\mathbb{R}}$, one would expect that $\mathcal{CH}_{\mathcal{D}_Q}(C) = \mathcal{CH}_{\mathcal{D}_{\mathbb{R}}}(C) = ConvexHull(C)$. However, if C is a set of points such that the slope of any line connecting a pair of points of C is irrational, then $\mathcal{CH}_{D_Q}(C) = C$. See [OSSW85, RW88, RW87] for further discussion of this type of convexity.

Definition 3.1 *Let f be a real function defined on a $\mathcal{D}$-convex set C. We say that f is* $\mathcal{D}$-convex *if, for any $x \in C$ and any $v \in \mathcal{D}$, the function $g(t) = f(x + tv)$ is a convex function of the real variable t. (The domain of g is an interval in $\mathbb{R}$, as C is assumed to be $\mathcal{D}$-convex.)*

Clearly, any convex function, in the standard sense, defined over the whole plane satisfies this condition.

Definition 3.2 *Let $C \subseteq \mathbb{R}^2$. The set $\mathcal{CH}^{\mathcal{D}}(C)$, called the* functional $\mathcal{D}$-convex hull of C, *is defined as*

$$\mathcal{CH}^{\mathcal{D}}(C) = \left\{ x \in \mathbb{R}^2 \;\middle|\; f(x) \leq \sup_{y \in C} f(y) \text{ for all } \mathcal{D}\text{-convex } f : \mathbb{R}^2 \to \mathbb{R} \right\}$$

A set C is functionally $\mathcal{D}$-convex *if $C = \mathcal{CH}^{\mathcal{D}}(C)$.*

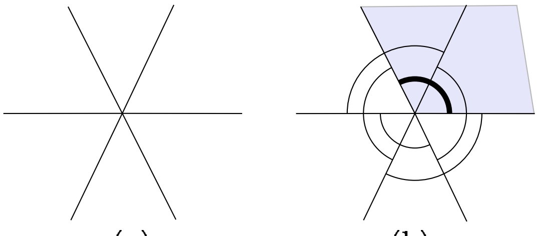

Figure 4: (a) A set of directions $\mathcal{D}$, (b) the set of quadrants $\mathcal{Q}(\mathcal{D})$ induced by $\mathcal{D}$, and (c) the $\mathcal{D}$-$\mathcal{CH}$ of three points.

Definition 3.3 *Let $\mathcal{D}$ be a set of directions. A pair of vectors $v_1, v_2 \in \mathcal{D}$, is a $\mathcal{D}$-*pair*, if v_2 is counterclockwise from v_1, and there is no vector in $\mathcal{D}$ between v_1 and v_2. Let $\mathcal{D}_{pairs}$ denote the set of all $\mathcal{D}$-pairs. Let $pspan(u_1, u_2)$ denote the portion of the plane that can be represented as a* positive *linear combination of $u_1, u_2 \in \mathcal{D}$. Thus $pspan(u_1, u_2)$ is the* open *wedge bounded by the rays emanating from the origin in directions u_1, u_2. We define by $(v_1, v_2)_L = pspan(-v_1, v_2)$ and $(v_1, v_2)_R = pspan(v_1, -v_2)$: these are two of the four quadrants of the plane induced by the lines containing v_1 and v_2. Similarly, for $v \in \mathcal{D}$ we denote by v_L and v_R the two open half-planes defined by the line passing through v. Let*

$$\mathcal{Q}(\mathcal{D}) = \left\{ v_L, v_R \;\middle|\; v \in \mathcal{D} \right\} \cup \left\{ (v_1, v_2)_R, (v_1, v_2)_L \;\middle|\; (v_1, v_2) \in D_{pairs} \right\}.$$

Definition 3.4 *For a set $S \subseteq \mathbb{R}^2$ we denote by $T(S)$ the set of translations of S in the plane, that is $T(S) = \left\{ S + p \;\middle|\; p \in \mathbb{R}^2 \right\}$. Given a set of directions $\mathcal{D}$, let $\mathcal{T}(\mathcal{D}) = \bigcup_{Q \in \mathcal{Q}(\mathcal{D})} T(Q)$.*

For $\mathcal{D}_{\mathbb{R}}$, the set $\mathcal{T}(\mathcal{D}_{\mathbb{R}})$ is the set of all open half-planes. The standard convex hull of a planar point set S can be defined as follows: start from the whole plane, and remove from it all the open half-planes H^+ such that $H^+ \cap S = \emptyset$. We extend this definition to handle $\mathcal{D}$-convexity for an arbitrary set of directions $\mathcal{D}$, as follows:

$$\mathcal{D}\text{-}\mathcal{CH}(S) = \mathbb{R}^2 \setminus \left(\bigcup_{I \in \mathcal{T}(\mathcal{D}), I \cap S = \emptyset} I \right);$$

that is, we remove from the plane all the translations of quadrants and halfplanes in $\mathcal{Q}(\mathcal{D})$ that do not contain a point of S. See Figures 4, 5.

For the case $\mathcal{D}_{xy} = \{(0,1), (1,0), (0,-1), (-1,0)\}$, Matoušek and Plecháč [MP97] showed that if $\mathcal{D}_{xy}$-$\mathcal{CH}(S)$ is connected, then $\mathcal{CH}^{\mathcal{D}_{xy}}(S) = \mathcal{D}_{xy}$-$\mathcal{CH}(S)$.

Definition 3.5 *For a set of directions $\mathcal{D}$, we define the* density *of $\mathcal{D}$ to be*

$$\alpha(\mathcal{D}) = \max_{(v_1, v_2) \in \mathcal{D}_{pairs}} \alpha(v_1, v_2),$$

where $\alpha(v_1, v_2)$ denotes the counterclockwise angle from v_1 to v_2.

See Figure 5, for an example of a set of directions with density larger than $\pi/2$.

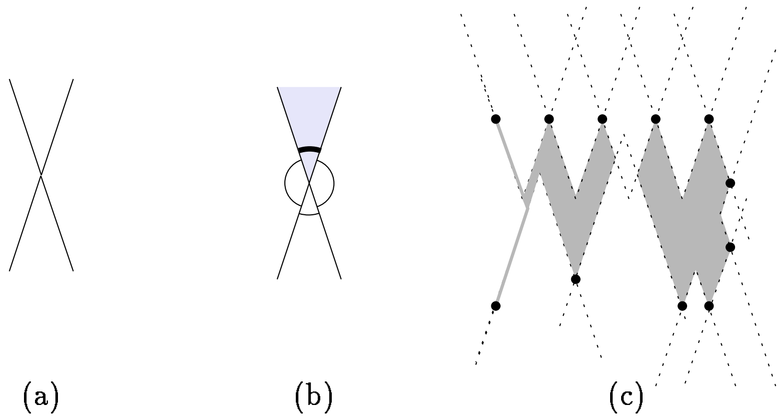

Figure 5: (a) A set of directions $\mathcal{D}$, such that $\alpha(\mathcal{D}) > \pi/2$, (b) the set of quadrants $\mathcal{Q}(\mathcal{D})$ induced by $\mathcal{D}$, and (c) the $\mathcal{D}$-$\mathcal{CH}$ of a set of points which is not connected.

Corollary 3.6 *Let $\mathcal{D}$ be a set of directions in the plane. Then:*

- *The set $\mathcal{D}$-$\mathcal{CH}(A)$ is $\mathcal{D}$-convex, for any $A \subseteq \mathbb{R}^2$.*
- *For any $A \subseteq B \subseteq \mathbb{R}^2$, one has $\mathcal{D}$-$\mathcal{CH}(A) \subseteq \mathcal{D}$-$\mathcal{CH}(B)$.*
- *For two sets of directions $\mathcal{D}_1 \subseteq \mathcal{D}_2$ we have $\mathcal{D}_1$-$\mathcal{CH}(S) \subseteq \mathcal{D}_2$-$\mathcal{CH}(S)$, for any $S \subseteq \mathbb{R}^2$.*
- *Let S be a bounded set in the plane, and let $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \mathcal{D}_3 \cdots$ be a sequence of sets of directions, such that $\lim_{i\to\infty} \alpha(D_i) = 0$. Then, $int\, ConvexHull(S) \subseteq \lim_{i\to\infty} \mathcal{D}_i$-$\mathcal{CH}(S) \subseteq ConvexHull(S)$.*
- *Let S be a finite set of points in general position in the plane, and let $\mathcal{D}_1 \supseteq \mathcal{D}_2 \supseteq \mathcal{D}_3 \cdots$ be a sequence of sets of directions, such that $\lim_{i\to\infty} \alpha(D_i) = \pi$. Then, $\lim_{i\to\infty} \mathcal{D}_i$-$\mathcal{CH}(S) = S$.*

Lemma 3.7 *Let $\mathcal{D}$ a set of directions, and let S be a finite set of points in the plane. Then $C = \mathcal{D}$-$\mathcal{CH}(S)$ is a polygonal set whose complexity is $O(|S \cap \partial C|)$.*

Proof: It is easy to show that C is polygonal. We charge each vertex of C to some point of $S' = S \cap \partial C$. Let C' be a connected component of C. If C' is a single point, then this is a point of S'. Otherwise, let e be an edge of C', and let I be a set in $\mathcal{T}(\mathcal{D})$ such that $e \subseteq \partial I$, and $I \cap S = \emptyset$.

Since e is an edge of C', there is no $q \in \mathbb{R}^2$ such that $e \subseteq q + I$, and $(q + I) \cap S = \emptyset$. This implies that there must be a point p of S on $\partial I \cap l_e$, where l_e is the line passing through e. However, C is a $\mathcal{D}$-convex set, and the direction of e belongs to $\mathcal{D}$. It follows that l_e intersects C along a connected set (i.e., the segment e), and $p \in l_e \cap C = e$. We charge the edge e to p. We claim that a point p of S' can be charged at most 4 times. Indeed, for each edge e' of C incident to p, there is a supporting set in $\mathcal{T}(\mathcal{D})$, such that p and e' lie on its boundary. Only two of those sets can have angle less than $\pi/2$ at p (because such a set corresponds to a $\mathcal{D}$-pair(v_1, v_2) with $\alpha(v_1, v_2) > \pi/2$). Thus, a point of S' is charged at most $\max(2\pi/(\pi/2), \pi/(\pi/2) + 2) = 4$ times. ■

Lemma 3.8 *Let $\mathcal{D}$ be a set of directions, and let K be a bounded convex body in the plane, such that the expected area of $\mathcal{D}$-$\mathcal{CH}(N)$ of a set N of n points, chosen uniformly and independently from K, is at least $(1 - f(n))\,Area(K)$, where $1 \geq f(n) \geq 0$, for $n \geq 1$. Then, the expected number of vertices of $C = \mathcal{D}$-$\mathcal{CH}(N)$ is $O(nf(n/2))$.*

Proof: By Lemma 3.7, the complexity of C is proportional to the number of points of N on the boundary of C. Using this observation, it is easy to verify that the proof of Lemma 2.1 can be extended to this case. ∎

We will first show that, with high probability, the $\mathcal{D}$-$\mathcal{CH}$ of a random point set inside a disk, contains a "large" disk in its interior. Next, we argue that this implies that the random $\mathcal{D}$-$\mathcal{CH}$ covers almost the whole disk, and the desired bound will readily follows from the above Lemma.

Definition 3.9 *For $r \geq 0$, let B_r denote the disk of radius of r centered at the origin.*

Lemma 3.10 *Let $\mathcal{D}$ be a set of directions, such that $0 \leq \alpha(\mathcal{D}) \leq \pi/2$. Let N be a set of n points chosen uniformly and independently from the unit disk. Then, with probability $1 - n^{-10}$ the set $\mathcal{D}$-$\mathcal{CH}(N)$ contains B_r in its interior, where $r = 1 - c\sqrt{\log n/n}$, for an appropriate constant c.*

Proof: Let $r' = 1 - c\sqrt{(\log n)/n}$, where c is a constant to be specified shortly. Let q be any point of $B_{r'}$. We bound the probability that q lies outside $C = \mathcal{D}$-$\mathcal{CH}(N)$ as follows: Draw 8 rays around q, such that the angle between any two consecutive rays is $\pi/4$. This partitions $q + B_{r''}$, where $r'' = c\sqrt{(\log n)/n}$, into eight portions $R_1, \ldots, R_8$, each having area $\pi c^2 \log n/(8n)$. Moreover, $R_i \subseteq q + B_{r''} \subseteq B_1$, for $i = 1, \ldots, 8$. The probability of a point of N to lie outside R_i is $1 - c^2 \log n/(8n)$. Thus, the probability that all the points of N lie outside R_i is

$$P\left[N \cap R_i = \emptyset\right] \leq \left(1 - \frac{c^2 \log n}{8n}\right)^n \leq e^{-(c^2 \log n)/8} = n^{-c^2/8},$$

since $1 - x \leq e^{-x}$, for $x \geq 0$. Thus, the probability that one of the R_i's does not contain a point of N is bounded by $8n^{-c^2/8}$. We claim that if $R_i \cap N \neq \emptyset$, for every $i = 1, \ldots, 8$, then $q \in C$. Indeed, if $q \notin C$ then there exists a set $Q \in \mathcal{Q}(\mathcal{D})$, such that $(q + Q) \cap N = \emptyset$. Since $\alpha(\mathcal{D}) \leq \pi/2$ there exists an i, $1 \leq i \leq 8$, such that $R_i \subseteq q + Q$; see Figure 6. This is a contradiction, since $R_i \cap N \neq \emptyset$. Thus, the probability that q lies outside C is $\leq 8n^{-c^2/8}$.

Let N' denote a set of n^{10} points spread uniformly on the boundary of $B_{r'}$. By the above analysis, all the points of N' lie inside C with probability at least $1 - 8n^{10-c^2/8}$. Furthermore, arguing as above, we conclude that $B_r \subseteq \mathcal{D}$-$\mathcal{CH}(N')$, where $r = 1 - 2c\sqrt{(\log n)/n}$. Hence, with probability at least $1 - 8n^{10-c^2/8}$, $\mathcal{D}$-$\mathcal{CH}(C)$ contains B_r. The lemma now follows by setting $c = 20$, say. ∎

Since the set of directions may contain large gaps, there are points in $B_1 \setminus B_r$ that are "unsafe", in the following sense:

Definition 3.11 *Let $\mathcal{D}$ be a set of directions, and let $0 \leq r \leq 1$ be a prescribed constant, such that $0 \leq \alpha(\mathcal{D}) \leq \pi/2$. A point p in B_1 is* safe, *relative to B_r, if $op \subseteq \mathcal{D}$-$\mathcal{CH}(B_r \cup \{p\})$.*

See Figure 7 for an example how the unsafe area looks like. The behavior of the $\mathcal{D}$-$\mathcal{CH}$ inside the unsafe areas is somewhat unpredictable. Fortunately, those areas are relatively small. start

`start`

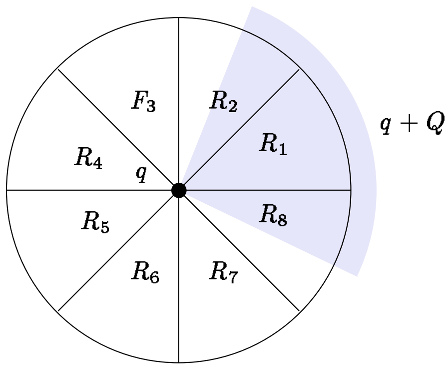

Figure 6: Since $\alpha(\mathcal{D}) \leq \pi/2$, any quadrant $Q \in \mathcal{Q}(\mathcal{D})$, when translated by q, must contain one of the R_i's.

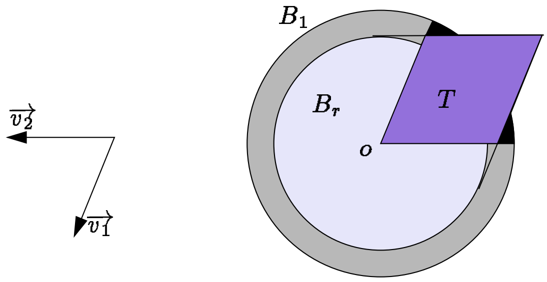

Figure 7: The dark areas are the unsafe areas for a consecutive pairs of directions $v_1, v_2 \in \mathcal{D}$.

Lemma 3.12 *Let $\mathcal{D}$ be a set of directions, such that $0 \leq \alpha(\mathcal{D}) \leq \pi/2$, and let $r = 1 - O\left(\sqrt{(\log n)/n}\right)$. The unsafe area in B_1, relative to B_r, can be covered by a union of $O(1)$ caps. Furthermore, the length of the base of such a cap is $O(((\log n)/n)^{1/4})$, and its height is $O\left(\sqrt{(\log n)/n}\right)$.*

Proof: Let p be an unsafe point of B_1. Let $\overrightarrow{v_1}, \overrightarrow{v_2}$ be the consecutive pair of vectors in $\mathcal{D}$, such that the vector $\overrightarrow{po}$ lies between them. If $ray(p, \overrightarrow{v_1}) \cap B_r \neq \emptyset$, and $ray(p, \overrightarrow{v_2}) \cap B_r \neq \emptyset$ then $po \subseteq ConvexHull(\{p, o, p_1, p_2\}) \subseteq \mathcal{D}\text{-}\mathcal{CH}(B_r \cup \{p\})$, for any pair of points $p_1 \in B_r \cap ray(p, \overrightarrow{v_1}), p_2 \in B_r \cap ray(p, \overrightarrow{v_2})$. Thus, p is unsafe only if one of those two rays miss B_r. Since p is close to B_r, the angle between the two tangents to B_r emanating from p is close to π. This implies that the angle between $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ is at least $\pi/4$ (provided n is a at least some sufficiently large constant), and the number of such pairs is at most 8.

The area in the plane that sees o in a direction between $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$, is a quadrant Q of the plane. The area in Q which is is safe, is a parallelogram T. Thus, the unsafe area in B_1 that induced by the pair $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ is $(B_1 \cap Q) \setminus T$. Since $\alpha(\mathcal{D}) \leq \pi/2$, this set can covered with two caps of B_1 with their base lying on the boundary of B_r. See Figure 7.

The height of such a cap is $1 - r = O\left(\sqrt{\frac{\log n}{n(\pi - \alpha)}}\right)$, and the length of the base of such a cap is $2\sqrt{1 - r^2} = O\left(\left(\frac{\log n}{n(\pi-\alpha)}\right)^{1/4}\right)$. ■

The proof of Lemma 3.12 is where our assumption that $\alpha(\mathcal{D}) \leq \pi/2$ plays a critical role.

Indeed, if $\alpha(\mathcal{D}) > \pi/2$, then the unsafe areas in $B_1 \setminus B_r$ becomes much larger, as indicated by the proof. $\boxed{end}$ end

Theorem 3.13 *Let $\mathcal{D}$ be a set of directions, such that $0 \le \alpha(\mathcal{D}) \le \pi/2$. The expected number of vertices of $\mathcal{D}$-$\mathcal{CH}(N)$, where N is a set of n points, chosen uniformly and independently from the unit disk, is $O\left(n^{1/3} + \sqrt{n\alpha(\mathcal{D})}\right)$.*

Proof: We claim that the expected area $\mathcal{D}$-$\mathcal{CH}(N)$ is at least $\pi - O\left(n^{-2/3} + \sqrt{\alpha/n}\right)$, where $\alpha = \alpha(\mathcal{D})$. The theorem will then follow from Lemma 3.8.

Indeed, let m be an integer to be specified later, and assume, without loss of generality, that m divides n. Partition B into m congruent sectors, $\mathcal{S}_1, \dots, \mathcal{S}_m$. Let $B^1, \dots, B^\mu$ denote the $\mu = n/m$ disks centered at the origin, such that (i) $B^1 = B_1$, and (ii) $Area(B^{i-1}) - Area(B^i) = \pi/\mu$, for $i = 2, \dots, \mu$. Let r_i denote the radius of B^i, for $i = 1, \dots, \mu$. Note[2], that $r_i - r_{i+1} \ge r_{i-1} - r_i \ge 1/(2\mu)$, for $i = 2, \dots, \mu - 1$.

Let $r = 1 - O\left(\sqrt{(\log n)/n}\right)$, and let U be the set of sectors that either intersect an unsafe area of B relative to B_r, or their neighboring sectors intersect the unsafe area of B. By Lemma 3.12, the number of sectors in U is $O(1) \cdot O\left(\frac{((\log n)/n)^{1/4}}{(2\pi/m)}\right) = O(m((\log n)/n)^{1/4})$.

Let $S_{i,j} = (B^i \setminus B^{i+1}) \cap \mathcal{S}_j$, and $S_{\mu,j} = B^\mu \cap \mathcal{S}_j$, for $i = 1, \dots, \mu - 1$, and $j = 1, \dots, m$. The set $S_{i,j}$ is called the i-th *tile* of the sector $\mathcal{S}_j$, and its area is π/n, for $i = 1, \dots, \mu$, and $j = 1, \dots, m$.

Let X_j denote the first index i such that $N \cap S_{i,j} \ne \emptyset$, for $j = 1, \dots, m$. The probability that $X_j = k$ is upper-bounded by the probability that the tiles $S_{1,j}, \dots, S_{(k-1),j}$ do not contain any point of N; namely, by $\left(1 - \frac{k-1}{n}\right)^n$. Thus, $P[X_j = k] \le \left(1 - \frac{k-1}{n}\right)^n \le e^{-(k-1)}$. Thus,

$$E\left[X_j\right] = \sum_{k=1}^{\mu} kP[X_j = k] \le \sum_{k=1}^{\mu} ke^{-(k-1)} = O(1),$$

for $j = 1, \dots, m$.

Let C denote the set $\mathcal{D}$-$\mathcal{CH}(N \cup B_r)$. The tile $S_{i,j}$ is *exposed* by a set K, if $S_{i,j} \setminus K \ne \emptyset$.

We claim that the expected number of tiles exposed by C in a section $S_j \notin U$ is at most $X_{j-1} + X_{j+1} + O(\mu/m^2 + \alpha\mu/m)$, for $j = 1, \dots, m$ (where we put $X_0 = X_m$, $X_{m+1} = X_1$).

Indeed, let $w = max(X_{j-1}, X_{j+1})$, and let p, q be the two points in $S_{j-1,w}, S_{j+1,w}$, respectively, such that the number of sets exposed by the triangle $T = \triangle opq$, in the sector $\mathcal{S}_j$, is maximal. Both p and q lie on ∂B^{w+1} and on the external radii bounding $\mathcal{S}_{j-1}$ and $\mathcal{S}_{j+1}$, as shown in Figure 1. Let s denote the segment connecting the midpoint ρ of the base of T to its closest point on ∂B^w. The number of tiles in $\mathcal{S}_j$ exposed by T is bounded by w, plus the number of tiles intersecting the segment s. The length of s is

$$|oq| - |oq| \cos\left(\frac{3}{2} \cdot \frac{2\pi}{m}\right) \le 1 - \cos\left(\frac{3\pi}{m}\right) \le \frac{1}{2}\left(\frac{3\pi}{m}\right)^2 = \frac{4.5\pi^2}{m^2},$$

since $\cos x \ge 1 - x^2/2$, for $x \ge 0$.

[2] $Area(B^1) - Area(B^2) = \pi(1 - r_2^2) = \pi/\mu$, thus $r_2^2 = 1 - 1/\mu$. We have $r_2 \le 1 - 1/(2\mu)$, and $r_1 - r_2 \ge 1 - (1 - 1/(2\mu)) = 1/(2\mu)$.

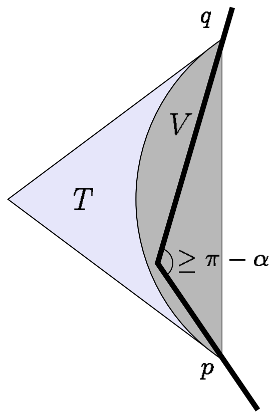

Figure 8: The portion of T that can be removed by a quadrant Q of $\mathcal{T}(\mathcal{D})$, is covered by the darkly-shaded circular cap, such that any point on its bounding arc creates an angle $\pi - \alpha$ with p and q.

On the other hand, the segment s intersects at most $\lceil ||s||/(1/(2\mu)) \rceil = O(\mu/m^2)$ tiles, and we have that the number of tiles exposed in the sector $\mathcal{S}_i$ by T is at most $w + O(\mu/m^2)$, for $j = 1, \ldots, m$.

Since $\mathcal{S}_j \notin U$, the points p, q are safe, and $op, oq \subseteq C$. This implies that the only additional tiles that might be exposed in $\mathcal{S}_j$ by C, are exposed by the portion of the boundary of C between p and q that lie inside T. Let V be the circular cap consisting of the points in T lying between pq and a circular arc $\gamma \subseteq T$, connecting p to q, such that for any point $p' \in \gamma$ one has $\angle pp'q = \pi - \alpha$. See Figure 8.

Let $Q \in \mathcal{T}(\mathcal{D})$ be any quadrant of the plane induce by $\mathcal{D}$, such that $Q \cap N = \emptyset$ (i.e. $C \cap Q = \emptyset$), and $Q \cap T \neq \emptyset$. Then, $Q \cap op = \emptyset, Q \cap oq = \emptyset$ since p and q are safe. Moreover, the angle of Q is at least $\pi - \alpha$, which implies that $Q \cap T \subseteq V$. See Figure 8.

Let s' be the segment $o\rho \cap V$, where ρ is as above, the midpoint of pq. The length of s'' is

$$|s'| \leq \sin\left(\frac{3}{2} \cdot \frac{2\pi}{m}\right) \tan\frac{\alpha}{2} \leq \frac{3\pi}{m}\frac{\sqrt{2}\alpha}{2} \leq \frac{3\pi\alpha}{m},$$

since $\sin x \leq x$, for $x \geq 0$, and $1/\sqrt{2} \leq \cos(\alpha/2)$ (because $0 \leq \alpha \leq \pi/2$).

Thus, the expected number of tiles exposed by C, in a sector $\mathcal{S}_j \notin U$, is bounded by

$$X_{j-1} + X_{j+1} + O\left(\frac{\mu}{m^2}\right) + O\left(\frac{3\pi\alpha/m}{1/(2\mu)}\right) = X_{j-1} + X_{j+1} + O\left(\frac{\mu}{m^2}\right) + O\left(\frac{\alpha\mu}{m}\right).$$

Thus, the expected number of tiles exposed by C, in sectors that do not belong to U, is at most

$$E\left[\sum_{j=1}^{m}\left(X_{j-1} + X_{j+1} + O\left(\frac{\mu}{m^2}\right) + O\left(\frac{\alpha\mu}{m}\right)\right)\right] = O\left(m + \frac{\mu}{m} + \alpha\mu\right).$$

Adding all the tiles that lie outside B_r in the sectors that belong to U, it follows that the expected number of tiles exposed by C is at most

$$O\left(m+\frac{\mu}{m}+\alpha\mu+|U|\cdot\frac{1-r}{1/2\mu}\right)=O\left(m+\frac{\mu}{m}+\alpha\mu+m\left(\frac{\log n}{n}\right)^{1/4}\cdot\mu\sqrt{\left(\frac{\log n}{n}\right)}\right)$$

$$=O\left(m+\frac{n}{m^2}+\frac{\alpha n}{m}+n\left(\frac{\log n}{n}\right)^{3/4}\right)=O\left(m+\frac{n}{m^2}+\frac{\alpha n}{m}+n^{1/4}\log^{3/4}n\right).$$

Setting $m=\max\left(n^{1/3},\sqrt{n\alpha}\right)$, we conclude that the expected number of tiles exposed by C is $O\left(n^{1/3}+\sqrt{n\alpha}\right)$.

The area of $C'=\mathcal{D}\text{-}\mathcal{CH}(N)$ is bounded from below by the area of the tiles which are not exposed by C'. The probability that $C'\neq C$ (namely, that the disk B_r is not inside C') is at most n^{-10}, by Lemma 3.10. Hence the expected area of C' is at least

$$E[Area(C)]-Prob\left[C\neq C'\right]\pi=\pi-O\left(n^{1/3}+\sqrt{n\alpha}\right)\frac{\pi}{n}-n^{-10}\pi=\pi-O\left(n^{-2/3}+\sqrt{\frac{\alpha}{n}}\right).$$

The assertion of the theorem now follows from Lemma 3.8. ■

The expected complexity of the $\mathcal{D}_{xy}\text{-}\mathcal{CH}$ of n points, chosen uniformly and independently from the unit square, is $O(\log n)$ (Lemma 2.4). Unfortunately, this is a degenerate case for a set of directions with $\alpha(\mathcal{D})=\pi/2$, as the following corollary testifies:

Corollary 3.14 *Let $\mathcal{D}'_{xy}$ be the set of directions resulting from rotating $\mathcal{D}_{xy}$ by 45 degrees. Let N be a set of n points, chosen independently and uniformly from the unit square S'. The expected complexity of $\mathcal{D}'_{xy}\text{-}\mathcal{CH}(N)$ is $\Omega\left(\sqrt{n}\right)$.*

Proof: Without loss of generality, assume that $n=m^2$ for some integer m. Tile S' with n translated copies of a square of area $1/n$. Let $\mathcal{S}_1,\ldots,\mathcal{S}_m$ denote the squares in the top raw of this tiling, from left to right. Let A_j denote the event that $\mathcal{S}_j$ contains a point of N, and neither of the two adjacent squares $\mathcal{S}_{j-1},\mathcal{S}_{j+1}$ contains a point of N, for $j=2,\ldots,m-1$.

We have

$$Prob\left[A_j\right]=Prob\left[\mathcal{S}_{j+1}\cap N=\emptyset\text{ and }\mathcal{S}_{j-1}\cap N=\emptyset\right]-Prob\left[(\mathcal{S}_{j-1}\cup\mathcal{S}_j\cup\mathcal{S}_{j+1})\cap N=\emptyset\right],$$

for $j=2,\ldots,m-1$. Hence,

$$\lim_{n\to\infty}Prob\left[A_j\right]=\lim_{n\to\infty}\left(\left(1-\frac{2}{n}\right)^n-\left(1-\frac{3}{n}\right)^n\right)=e^{-2}-e^{-3}\approx0.0855$$

This implies, that for n large enough, $Prob\left[A_j\right]\geq0.01$. Thus, the expected value of Y is $\Omega(m)=\Omega\left(\sqrt{n}\right)$, where Y is the number of A_j's that have occurred, for $j=2,\ldots,m-1$. However, if A_j occurs, then $C=\mathcal{D}'_{xy}\text{-}\mathcal{CH}(N)$ must have a vertex at $\mathcal{S}_j$. Indeed, let Q_{top} denote the quadrant of $\mathcal{Q}(\mathcal{D}'_{xy})$ that contains the positive y-axis. If we translate Q_{top} to the highest point in $S_j\cap N$, then it does not contain a point of N in its interior, implying that this point is a vertex of C, see Figure 9.

This implies that the expected complexity of $\mathcal{D}'_{xy}\text{-}\mathcal{CH}(N)$ is $\Omega\left(\sqrt{n}\right)$ ■

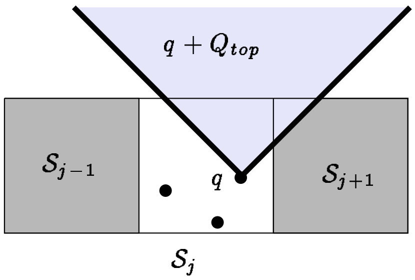

Figure 9: If A_j happens, then the squares $\mathcal{S}_{j-1}, \mathcal{S}_{j+1}$ do not contain a point of N. Thus, if q is the highest point in $\mathcal{S}_j$, then $q + Q_{top}$ can not contain a point of N, and q is a vertex of $\mathcal{D}'_{xy}$-$\mathcal{CH}(N)$.

4 On the Expected Number of Points on the Boundary of the Quadrant Hull Inside a Hypercube

In this section, we show that the expected number of points on the boundary of the quadrant hull of a set S of n points, chosen uniformly and independently from the unit cube is $O(\log^{d-1} n)$. Those bounds are known [BKST78], but we believe the new proof is simpler.

Definition 4.1 ([MP97]) *Let $\mathcal{Q}$ be a family of subsets of $\mathbb{R}^d$. For a set $A \subseteq \mathbb{R}^d$, we define the $\mathcal{Q}$-hull of A as*

$$\mathcal{Q}\text{-co}(A) = \bigcap \left\{ Q \in \mathcal{Q} \;\middle|\; A \subseteq Q \right\}.$$

Definition 4.2 ([MP97]) *For a sign vector $s \in \{-1, +1\}^d$, define*

$$q_s = \left\{ x \in \mathbb{R}^d \;\middle|\; \text{sign}(x_i) = s_i, \text{ for } i = 1, \ldots, d \right\},$$

and for $a \in \mathbb{R}^d$, let $q_s(a) = q_s + a$. We set $\mathcal{Q}_{sc} = \left\{ \mathbb{R}^d \setminus q_s(a) \;\middle|\; a \in \mathbb{R}^d, s \in \{-1,+1\}^d \right\}$. We shall refer to $\mathcal{Q}_{\text{sc}}$-co(A) as the quadrant hull *of A. These are all points which cannot be separated from A by any open orthant in space (i.e., quadrant in the plane).*

Definition 4.3 *Given a set of points $S \subseteq \mathbb{R}^d$, a point $p \in \mathbb{R}^d$ is* exposed *by q_s, if $q_s(p) \cap S = \emptyset$, for $s \in \{-1, +1\}^d$. A set C is* exposed *by q_s, if there exists a point $p \in C$ which is exposed by q_s; namely, $q_s(p) \cap S = \emptyset$.*

Definition 4.4 *For a set $S \subseteq \mathbb{R}^d$, let $n_{sc}(S)$ denote the number of points of S on the boundary of $\mathcal{Q}_{\text{sc}}$-co(S).*

Theorem 4.5 *Let C be a unit axis parallel hypercube in $\mathbb{R}^d$, and let S be a set of n points chosen uniformly and independently from C. Then, the expected number of points of S on the boundary of $H = \mathcal{Q}_{\text{sc}}$-co$(S)$ is $O(\log^{d-1}(n))$.*

Proof: We partition $\mathcal{C}$ into equal size tiles, of volume $1/n^d$; that is $C(i_1, i_2, \ldots, i_d) = [(i_1 - 1)/n, i_1/n] \times \cdots \times [(i_d - 1)/n, i_d/n]$, for $1 \le i_1, i_2, \ldots, i_d \le n$.

We claim that the expect number of tiles in our partition of $\mathcal{C}$ which are exposed by S is $O(n^{d-1}\log^{d-1} n)$.

Indeed, let $q = q_{(-1,-1,\ldots,-1)}$ be the "negative" quadrant of $\mathbb{R}^d$. Let $X(i_2,\ldots,i_d)$ be the maximal integer k, for which $C(k,i_2,\ldots,i_d)$ is exposed by q. The probability that $X(i_2,\ldots,i_d) \geq k$ is bounded by the probability that the cubes $C(l_1,l_2,\ldots,l_d)$ does not contain a point of S, where $l_1 < k, l_2 < i_2, \ldots, l_d < i_d$. Thus,

$$\begin{aligned}
\Pr\left[X(i_2,\ldots,i_d) \geq k\right] &\leq \left(1 - \frac{(k-1)(i_2-1)\cdots(i_d-1)}{n^d}\right)^n \\
&\leq \exp\left(-\frac{(k-1)(i_2-1)\cdots(i_d-1)}{n^{d-1}}\right),
\end{aligned}$$

since $1 - x \leq e^{-x}$, for $x \geq 0$.

Hence, the probability that $\Pr\left[X(i_2,\ldots,i_d) \geq i\cdot m + 1\right] \leq e^{-i}$, where $m = \left\lceil \frac{n^{d-1}}{(i_2-1)\cdots(i_d-1)} \right\rceil$. Thus,

$$\begin{aligned}
E\left[X(i_2,\ldots,i_d)\right] &= \sum_{i=1}^{\infty} i \Pr\left[X(i_2,\ldots,i_d) = i\right] = \sum_{i=0}^{\infty}\sum_{j=im+1}^{(i+1)m} j \Pr\left[X(i_2,\ldots,i_d) = j\right] \\
&\leq \sum_{i=0}^{\infty}(i+1)m \Pr\left[X(i_2,\ldots,i_d) \geq im+1\right] \leq \sum_{i=0}^{\infty}(i+1)me^{-i} = O(m).
\end{aligned}$$

Let r denote the expected number of tiles exposed by q in $\mathcal{C}$. If $C(i_1,\ldots,i_d)$ is exposed by q, then $X(i_2,\ldots,i_d) \geq i_1$. Thus, one can bound r by the number of tiles on the boundary of $\mathcal{C}$, plus the sum of the expectations of the variables $X(i_2,\ldots,i_d)$. We have

$$\begin{aligned}
r &= O(n^{d-1}) + \sum_{i_2=2}^{n-1}\sum_{i_3=2}^{n-1}\cdots\sum_{i_d=2}^{n-1} O\left(\frac{n^{d-1}}{(i_2-1)(i_3-1)\cdots(i_d-1)}\right) \\
&= O\left(n^{d-1}\right)\sum_{i_2=2}^{n-1}\frac{1}{i_2-1}\sum_{i_3=2}^{n-1}\frac{1}{i_3-1}\cdots\sum_{i_d=2}^{n-1}\frac{1}{i_d-1} = O\left(n^{d-1}\log^{d-1} n\right).
\end{aligned}$$

The set $\mathcal{Q}_{sc}$ contains translation of 2^d different quadrants. This implies, by symmetry, that the expected number of tiles exposed in $\mathcal{C}$ by S is $O\left(2^d n^{d-1}\log^{d-1} n\right) = O\left(n^{d-1}\log^{d-1} n\right)$. However, if a tile is not exposed by any q_s, for $s \in \{-1,+1\}^d$, then it lies in the interior of H. Implying that the expected volume of H is at least

$$\frac{n^d - O\left(n^{d-1}\log^{d-1} n\right)}{n^d} = 1 - O\left(\frac{\log^{d-1} n}{n}\right).$$

We are now in position to apply an argument similar to the one used in Lemma 2.1: Partition S into two equal size sets S_1, S_2. We have that $n_{sc}(S)$ is bounded by the number of points of S_1 falling outside $H_2 = \mathcal{Q}_{\mathrm{sc}}\text{-co}(S_2)$, plus the number of points of S_2 falling outside $H_1 = \mathcal{Q}_{\mathrm{sc}}\text{-co}(S_1)$, where $n_{sc}(S)$ is the number of points of S on the boundary of $\mathcal{Q}_{\mathrm{sc}}$-co$(S)$. We conclude,

$$\begin{aligned} E[n_{sc}(S)] &\le E\left[|S_1 \setminus H_2|\right] + E\left[|S_2 \setminus H_1|\right] = E\left[E\left[|S_1 \setminus H_2| \;\middle|\; S_2\right]\right] + E\left[E\left[|S_2 \setminus H_1| \;\middle|\; S_1\right]\right] \\ &= E\left[\frac{n}{2}(1 - \mathrm{Vol}(H_2)) + \frac{n}{2}(1 - \mathrm{Vol}(H_1))\right] = O\left(n \cdot \frac{\log^{d-1} n}{n}\right) = O(\log^{d-1} n), \end{aligned}$$

and the theorem follows. ∎

Remark 4.6 A point p of S is a *maxima*, if there is no point p' in S, such that $p_i \le p'_i$, for $i = 1, \ldots, d$. Clearly, a point which is a maxima, is also on the boundary of $\mathcal{Q}_{\mathrm{sc}}$-co$(S)$. By Theorem 4.5, the expected number of maxima in a set of n points chosen independently and uniformly from the unit hypercube in $\mathbb{R}^d$ is $O(\log^{d-1} n)$. This was also proved in [BKST78], but we believe that our new proof is simpler.

Also, as noted in [BKST78], a vertex of the convex hull of S is a point of S lying on the boundary of the $\mathcal{Q}_{\mathrm{sc}}$-co$(S)$. Hence, the expected number of vertices of the convex hull of a set of n points chosen uniformly and independently from a hypercube in $\mathbb{R}^d$ is $O(\log^{d-1} n)$.

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