An Introduction to the K-theory of Banach Algebras

The K-theory of Banach algebras, and most particularly C^*-algebras, is a branch of mathematics that has caused great excitement in recent years, and played a large part in the development of "non-commutative geometry", a term which will be explained later. This course contains the basic definitions and results of the area, with indications of how it can be used to distinguish between algebras when this would otherwise be difficult. We shall only scratch the surface of the subject, which is taken much further in the books of Blackadar and Wegge-Olsen.

Briefly, we shall construct a sequence of functors $K_0, K_1, K_2, \ldots$ which take Banach algebras A to Abelian groups $K_n(A)$. Given an ideal $I \subset A$, we shall obtain from the short exact sequence

$$0 \to I \to A \to A/I \to 0$$

a long exact sequence

$$... \to K_{n+1}(A/I) \to K_n(I) \to K_n(A) \to K_n(A/I) \to K_{n-1}(I) \to ...$$

The famous Bott periodicity theorem states that the sequence of functors above is periodic with period 2 (or 8 in the real case!). So $K_{n+2} = K_n$ and the long exact sequence reduces to a cyclic exact sequence of size 6, involving only K_0 and K_1. What is beautiful about this result (or at least one of its beautiful aspects) is that it is often possible to make judicious choices of A and I so that one can calculate a few of the terms in the cyclic sequence directly, and then use exactness to calculate the rest, ending up with highly non-obvious results. Thus, what is to follow is *not* mere abstract nonsense, despite the large number of definitions, commutative diagrams, universal constructions, checks that maps are well-defined etc.

Before defining K_0, we need some preliminary definitions and remarks. An *idempotent* in a normed algebra A is an element $x \in A$ such that $x^2 = x$. An obvious example, which is helpful to have in mind in what follows, is when $A = B(H)$ and x is an orthogonal projection. Two idempotents x and y are said to be *orthogonal* if $xy = yx = 0$. If $xy = yx = y$ then we write $y \leqslant x$ (which says in the projection case that the range of y is a subset of the range of x). In that case $x - y$ is also an idempotent. We shall need to consider three equivalence relations on the set of all idempotents in an algebra A. The second makes sense only if A has an identity.

Definition. Two idempotents x and y are (algebraically) *equivalent* if there exist $z, w \in A$ such that $zw = x$ and $wz = y$. They are *similar* if there exists an invertible u such that $ux = yu$. They are *homotopic* if there is a continuous path of idempotents starting at x and ending at y.

The only non-trivial thing to check is that algebraic equivalence is transitive. Even this is a very easy exercise (left to the reader).

Given an algebra A, $M_n(A)$ stands for the set of all $n \times n$-matrices with entries in A (which is also $M_n \otimes A$). Given elements $x \in M_n(A)$ and $y \in M_p(A)$, write $\text{diag}(x, y)$ for the element $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ of $M_{n+p}(A)$. $M_n(A)$ embeds into $M_{n+1}(A)$ in an obvious way via the map $x \mapsto \text{diag}(x, 0)$. The *inductive limit* $M_\infty(A)$ of the $M_n(A)$ is basically the union of the $M_n(A)$, where each is regarded as a subset of the next using the embedding just defined. More formally, it can be defined as the set of sequences $x_1, x_2, x_3, \ldots$ where $x_n \in M_n(A)$, each x_n is the top left hand corner of x_{n+1} and $x_{n+1} = \text{diag}(x_n, 0)$ for all large enough n. We shall be informal in these notes. For example, to see that $M_\infty(A)$ is an algebra, given x and y choose the smallest n such that $M_n(A)$ contains both x and y and perform the obvious operations (matrix addition and multiplication). One can also think of $M_\infty(A)$ as the set of infinite matrices with entries in A, only finitely many of which are non-zero.

The whole of K-theory is about $M_\infty(A)$ rather than A. To put it another way, K-theory is about so called *stable* properties of an algebra A, which means those properties unaffected by changing to $M_\infty(A)$. (Observe that $M_\infty(M_\infty(A))$ is isomorphic to $M_\infty(A)$.) It does not matter too much what norm is taken on $M_\infty(A)$ - we shall take $\|x\|$ to be the sum of the norms of the entries of x, but we only really care about the topology to which the norm gives rise, so that we can talk about homotopies etc. In the algebra, $M_\infty(A)$, which has no identity, we shall say that x and y are similar if they are similar in $M_n(A)$ for some n.

Lemma 1. *The notions of equivalence, similarity and homotopy coincide in $M_\infty(A)$.*

Proof. If x and y are idempotents then, setting $u = xy + (1-x)(1-y)$, we have $xu = uy$ $(= xy)$. If $x = y$ then $u = 1$. It is therefore obvious (or easy to check if you don't find it obvious) that if x is sufficiently close to y then $\|1-u\| < 1$, which implies that u is invertible. This implies (by compactness of $[0,1]$) that two homotopic idempotents in A are similar in A.

The converse is false in general, which is the first place where matrices come in. Let r_θ be the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and let u be any invertible element of A. Then, as θ varies between 0 and $\pi/2$, $v_\theta = r_\theta \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} r_\theta^{-1}$ is a continuous path of invertibles in $M_2(A)$ starting at $\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ and ending at $\begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix}$. Hence, $w_\theta = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} v_\theta$ is a continuous path of invertibles starting at $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and ending at $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$. Finally, if x and y are idempotents in A with $uxu^{-1} = y$, then $w_\theta \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} w_\theta^{-1}$ is a continuous path of idempotents starting at $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ and ending at $\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}$. Thus, similar idempotents in A are homotopic in $M_2(A)$.

It is trivial that similar idempotents are equivalent: if $xu = uy$ then $(xu)u^{-1} = x$ and $u^{-1}(xu) = y$. Again, the converse is not true in general. However, if $ab = x$ and $ba = y$, it can be checked that if $u = \begin{pmatrix} ybx & 1-y \\ 1-x & xay \end{pmatrix}$, then u is invertible in $M_2(A)$ with inverse $\begin{pmatrix} xay & 1-x \\ 1-y & ybx \end{pmatrix}$, and that $u \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} u^{-1} = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}$. Thus, equivalent idempotents in A are similar in $M_2(A)$. $\square$

Notice the trick of replacing a and b above by xay and ybx. It helps one to remember the matrix, and it will be used again below.

We are now ready to define K_0. Given an algebra A with a unit, let $V(A)$ be the set of equivalence classes of idempotents in $M_\infty(A)$. We can define addition on $V(A)$ as follows. Write $[x]$ for the equivalence class of an idempotent x and 1_n and 0_n for the identity and zero matrices in $M_n(A)$. Then, setting $u = \begin{pmatrix} 0_n & 1_n \\ 1_n & 0_n \end{pmatrix} \in M_{2n}(A)$, we have, for any idempotent $x \in M_n(A)$, that $u \operatorname{diag}(x, 0_n) u^{-1} = \operatorname{diag}(0_n, x)$, so that $\operatorname{diag}(x, 0_n)$ and $\operatorname{diag}(0_n, x)$ are similar in $M_{2n}(A)$ and hence equivalent. It follows that, given two idempotents x and y in $M_\infty(A)$, it is possible to choose orthogonal representatives x' and y' from the equivalence classes $[x]$ and $[y]$. One then defines $[x] + [y]$ to be $[x' + y']$.

To show that this operation is well defined, we need to show that if $x \sim y$, $x' \sim y'$, $xy = yx = 0$ and $x'y' = y'x' = 0$, then $x + x' \sim y + y'$. If $ab = x$, $ba = y$, $a'b' = x'$, $b'a' = y'$, then one can check that $(xay+x'a'y')(ybx+y'b'x') = x+x'$ and $(ybx+y'b'x')(xay+x'a'y') = y+y'$. It is obvious that this addition is commutative, so $V(A)$ is an Abelian semigroup. To make it into the group $K_0(A)$ is straightforward, and similar to constructing $\mathbb{Z}$ from $\mathbb{N}$, with the small difference that $V(A)$ may not satisfy the cancellation law. Anyhow, $K_0(A)$ is defined to be the set of pairs $([x],[y])$, where two such pairs $([x],[y])$ and $([x'],[y'])$ are equivalent if there exists $[z]$ such that $[x]+[y']+[z] = [x']+[y]+[z]$. (This is called the *Grothendieck group* of $V(A)$ and has the obvious universal property that any homomorphism from $V(A)$ into an Abelian group factors through it. This construction was not what made Grothendieck famous.) It is easy to check that K_0 is a functor, i.e., that a homomorphism ϕ of unital algebras A and B gives rise to a homomorphism from $K_0(A)$ to $K_0(B)$ well defined by $\phi^*([x],[y]) = ([\phi x],[\phi y])$.

It is important to define $K_0(A)$ for algebras A without units. To any such algebra one can adjoin a unit to give a new algebra A^+. It is defined as the set of ordered pairs $(x,\lambda) \in A \times \mathbb{C}$ with pointwise addition and $(x,\lambda)(y,\mu) = (xy+\lambda y+\mu x, \lambda\mu)$. The obvious thing to do now would be to define $K_0(A)$ to be $K_0(A^+)$. Unfortunately, this does not work: sequences one needs to be exact are not exact with this definition. Instead, one proceeds as follows. If K_0 is to be a (covariant) functor, then it takes left and right inverses to left and right inverses. Now there is a short exact sequence

$$0 \to A \to A^+ \to \mathbb{C} \to 0$$

(with the obvious maps), and the map π from A^+ to $\mathbb{C}$ has a right inverse. (N.B. this is saying more than just that the map from A^+ to $\mathbb{C}$ is surjective, as the right inverse is an algebra homomorphism - this stronger property is called *split exactness* of the sequence.) A is an ideal in A^+, so if we are to obtain the cyclic exact sequence mentioned at the beginning, then the sequence

$$K_1(A^+) \to K_1(\mathbb{C}) \to K_0(A) \to K_0(A^+) \to K_0(\mathbb{C}) \to 0$$

must be exact. (The final zero follows from the fact that π^* has a right inverse and is thus surjective.) It will turn out (by the definition of K_1 later) that $K_1(\mathbb{C}) = 0$, but the functoriality of K_1 implies that the map from $K_1(A^+)$ to $K_1(\mathbb{C})$ is surjective again, so for either reason we obtain the split exact sequence

$$0 \to K_0(A) \to K_0(A^+) \to K_0(\mathbb{C}) \to 0\ .$$

This leaves no choice for the definition of $K_0(A)$ - it must be the kernel of π^*. (We shall see soon that $K_0(\mathbb{C}) = \mathbb{Z}$.)

The following point should be emphasized. Just because a functor preserves right and left inverses, it does not follow that it preserves exactness. Indeed, it is very important that K_0 should *not* preserve short exact sequences, as this would mess up the cyclic exact sequence we are eventually trying to construct. Given the short exact sequence

$$0 \to I \to A \to A/I \to 0$$

metioned at the beginning, it will not in general be possible to find an algebra homomorphism lifting A/I to A.

A natural reaction at this point is one of anxiety. If A is a non-unital algebra, then there are several steps to the construction of $K_0(A)$, involving many equivalence relations. It is therefore a good idea to consider what a typical element of $K_0(A)$ looks like, which we shall do by showing that every equivalence class in K_0 has a representative of a certain nice form. First, we shall prove a simple lemma calculating $K_0(\mathbb{C})$ (as promised earlier).

Lemma 2. *$K_0(\mathbb{C})$ is isomorphic to $\mathbb{Z}$.*

Proof. An idempotent in $M_n(\mathbb{C})$ is characterized by its kernel and image in $\mathbb{C}^n$. It is therefore easy to see that two such idempotents are similar (in $M_n(\mathbb{C})$) if and only if they have the same rank. Consequently $V(\mathbb{C}) = \mathbb{N} \cup \{0\}$. Since the cancellation law applies here, $K_0(\mathbb{C}) = \mathbb{Z}$ as claimed. $\square$

We are now ready for the "standard portrait" of $K_0(A)$. By definition an element of $K_0(A)$ is of the form $[x] - [y]$, where x and y are idempotents in $M_\infty(A^+)$ and $\pi^*([x] - [y]) = 0$. ($[x] - [y]$ is of course shorthand for $([x], [y])$, or more accurately its equivalence class in the Grothendieck group of $V(A^+)$.) Note that $x - y$ may not be an idempotent so $[x - y]$ is not in general defined.) Let us write p_k for the projection given by the infinite matrix with the unit of A^+ in the first k places in the diagonal, and zero everywhere else. If x and y are both in $M_n(A^+)$, then $x < p_n$ and $y < p_n$. Let $x' \in M_{2n}(A^+)$ be similar to x but orthogonal to p_n (obtained from x by "moving down the diagonal"). Then $x' + p_n - y$ and p_n are both idempotents, and

$$\begin{aligned}[x' + p_n - y] - [p_n] &= [x'] + [p_n - y] - ([p_n - y] + [y]) \\ &= [x] + [y] \ .\end{aligned}$$

Thus, all elements of $K_0(A)$ are of the form $[x] - [p_n]$ for some n, with $\pi^*([x] - [p_n]) = 0$. An element x in $M_n(A^+)$ can be written (uniquely) as an ordered pair (x_1, x_2), where x_1 is an element of $M_n(A)$ and x_2 is an element of $M_n(\mathbb{C})$. The latter is called the *scalar part* of x. Notice that if x is an idempotent, then so is x_2. To say that $\pi^*([x] - [p_n]) = 0$ is to say that the scalar part of x is equivalent to $[p_n]$. Now $x \in M_{2n}(A^+)$, so we can find an invertible $u \in M_{2n}(\mathbb{C})$ such that the scalar part of $u^{-1}xu$ is equal to p_n. We have proved the following result.

Proposition 3. *Every element of $K_0(A)$ can be written in the form $[x + p_n] - [p_n]$ for some n, with x an element of $M_{2n}(A)$.*

It is also useful to rewrite the above element when it turns out to be zero. Then we know that there exists an idempotent $z \in M_k(A^+)$ for some k such that $[x + p_n] + [z] = [p_n] + [z]$ in $V(A^+)$. Adding $[p_k - z]$ to both sides tells us that $\mathrm{diag}(x + p_n, p_k)$ is similar to $\mathrm{diag}(p_n, 0_n, p_k)$. It is now easy to construct a scalar matrix u such that, writing $x' = uxu^{-1}$, we have $[x + p_n] - [p_n] = [x' + p_{n+k}] - [p_{n+k}]$ with $x' + p_{n+k}$ similar to p_{n+k}. It will be important later that x' is similar to x via a *scalar* matrix u.

The next lemma but one shows that $K_0(A)$ can be constructed in the same way whether or not A is unital. Note first that one can adjoin a unit to A, obtaining A^+, even if A has a unit already. (It then ceases to be one, having been usurped by the new unit.)

Lemma 4. *If A has a unit, then $A^+ \sim A \oplus \mathbb{C}$.*

Proof. $A \oplus \mathbb{C}$ stands for the algebra obtained using pointwise operations in the two components, so it is not the same as A^+. However, the map $(x, \lambda) \mapsto (x + \lambda, \lambda)$ gives the desired algebra isomorphism. $\square$

Lemma 5. *Let A and B be unital algebras. Then $K_0(A \oplus B) = K_0(A) \oplus K_0(B)$.*

Proof. If (x, y) is similar to (x', y') in $A \oplus B$, then x and y are similar to x' and y' in A and B. This shows that $V(A \oplus B) = V(A) \oplus V(B)$. The result follows easily from the definition of the Grothendieck group (or from the universal property if you are a true algebraist). $\square$

Now we unify the definitions of K_0 for unital and non-unital algebras.

Lemma 6. *Let A be a unital algebra, and consider the short exact sequence $0 \to A \to A^+ \to \mathbb{C} \to 0$. Then $K_0(A)$ is equal to the kernel of π^*, the map from $K_0(A^+)$ to $K_0(\mathbb{C})$ given by the projection from A^+ to $\mathbb{C}$.*

Proof. The projection $\pi : A^+ \to \mathbb{C}$ is defined by $(a, \lambda) \mapsto \lambda$. Since (a, λ) corresponds to the element $(a - \lambda, \lambda)$ in $A \oplus \mathbb{C}$ (via the isomorphism above), the corresponding projection on $A \oplus \mathbb{C}$ is given by the same formula. But then the map $\pi : A \oplus \mathbb{C} \to \mathbb{C}$ induces a map $\pi^* : K_0(A \oplus \mathbb{C}) = K_0(A) \oplus K_0(\mathbb{C}) \to K_0(\mathbb{C})$ with the obvious formula, so its kernel is $K_0(A)$. $\square$

It is not hard to show that K_0, extended to non-unital algebras, is still a functor. We content ourselves with the definition of $\phi^* : K_0(A) \to K_0(B)$ when $\phi : A \to B$. First, ϕ extends in an obvious way to a homomorphism $\phi^+ : A^+ \to B^+$. Writing π_A and π_B for the projections to $\mathbb{C}$ of A^+ and B^+, we have $1_{\mathbb{C}} \circ \pi_A = \pi_B \circ \phi^+$, which implies (by functoriality of K_0 for unital algebras) that $1_{\mathbb{Z}} \circ \pi_A^* = \pi_B^* \circ \phi^{+*}$, and therefore that ϕ^{+*} maps the kernel of π_A^* into the kernel of π_B^*, giving us our desired map from $K_0(A)$ to $K_0(B)$. (Notice that we have used Lemma 6 and therefore avoided splitting into cases according to whether A and B are unital or not.) It is easy to see that a typical element $[x + p_n] - [p_n]$ of $K_0(A)$ maps under ϕ^{+*} to the element $[\phi(x) + p_n] - [p_n]$ of $K_0(B)$.

Before we define K_1, we shall need two more lemmas. Given a unital algebra A, let $GL_n(A)$ stand for the group of invertible $n \times n$-matrices with entries in A. Define $GL_\infty(A)$ to be the limit of the $GL_n(A)$, where this time the embedding from $GL_n(A)$ to $GL_{n+1}(A)$ is given by $x \mapsto \mathrm{diag}(x, 1_A)$. This can be thought of as infinite invertible matrices over A with only finitely many entries differing from the infinite identity matrix.

Given an element x of an algebra A, we define $\exp(x)$ using the power series for exp. This of course converges. It is not in general true that $\exp(x + y) = \exp(x)\exp(y)$ though this is clearly true when x and y commute. For $\|x\| < 1$ we can also define $\log(1 + x)$ using a power series, and it can be checked quite easily that $\exp\log(1 + x) = 1 + x$.

Lemma 7. *Let A be a unital Banach algebra. The component of the identity in A is the group generated by elements of the form $\exp(x)$ with $x \in A$.*

Proof. Suppose there is a path of invertibles starting at 1 and ending at u. An easy compactness argument implies that there is a sequence $1 = u_0, u_1, \ldots, u_r = u$ of invertible elements such that $\left\|1 - u_{i-1}^{-1} u_i\right\| < 1$ for every i. Hence we can define x_i to be $\log(u_{i-1}^{-1} u_i)$, and we have expressed u as $\exp(x_1) \dots \exp(x_r)$. □

Notice that since $GL_n(A)$ is a Banach algebra, the conclusion holds there as well. Let us write $GL_n^0(A)$ for the component of the identity in $GL_n(A)$.

Lemma 8. *Let ϕ be a continuous surjective homomorphism between unital Banach algebras A and B. Then every invertible in $GL_n^0(B)$ lifts to an invertible in $GL_n^0(A)$.*

Proof. Let v be an invertible in $GL_n^0(B)$. Write v as a product of elements of the form $\exp(y_i)$, and lift each y_i to some $x_i \in GL_n(A)$. Then the product of the $\exp(x_i)$ maps to v and is in $GL_n^0(A)$ by Lemma 7. □

The above lemma will be particularly useful when B is A/I for some ideal I.

The definition of K_1 is easier than that of K_0. It is simply the group quotient of $GL_\infty(A)$ by $GL_\infty^0(A)$. (For those who are worried, two elements of $GL_\infty(A)$ are in the same component if they are in the same component of $GL_n(A)$ for some n.) The above definition relies on A having a unit. However, in this case we simply define $K_1(A)$ to be $K_1(A^+)$. This is not laziness: as in the K_0 case the definition is forced upon us, but it is simpler here because $K_1(\mathbb{C}) = 0$. This last fact follows from (or rather, *is*) the connectedness of $GL_n(\mathbb{C})$, which is an easy exercise.

It turns out that $K_1(A)$ is Abelian. To see this, notice that $[x][y] = [xy] = [\mathrm{diag}(xy, 1)] = [\mathrm{diag}(x, 1)\mathrm{diag}(y, 1)]$. In the proof of Lemma 1 we constructed a continuous path from $\mathrm{diag}(y, 1)$ to $\mathrm{diag}(1, y)$ which shows that $[xy] = [\mathrm{diag}(x, y)]$. In a similar way we can show that this equals $[yx] = [y][x]$.

The next step in the theory is to define the *index map*, which takes $K_1(A/I)$ to $K_0(I)$, when I is an ideal in the algebra A. This is a generalization of the Fredholm index of an operator. Consider the algebra $A = B(H)$ and the ideal I of compact operators. We have seen that every invertible in A/I (the *Calkin* algebra) is Fredholm with a well-defined index, and that the component of the identity is the operators of index zero. But $K_0(I) = \mathbb{Z}$ (this is because I is the closure of the finite-rank operators, so from the point of view of K_0 is basically the same as $\mathbb{C}$), so there ought to be some way of regarding the index as a map from $K_1(A/I)$ to $K_0(I)$. With this example in mind, the definition below ought to seem more natural than it otherwise would.

Let $[x]$ be an element of $K_1(A/I)$ with $x \in GL_n((A/I)^+)$. We would like to lift x to an element of $GL_n(A^+)$, but in general cannot (including in the example above). Instead, we take any element of $GL^0_{n+m}((A/I)^+)$ of the form $\mathrm{diag}(x, y)$ (such as $\mathrm{diag}(x, x^{-1})$ for example) and lift that (which we can do by Lemma 8) to an element w of $GL^0_{2n}(A^+)$. Let $\pi : GL_\infty(A^+) \to GL_\infty((A/I)^+)$ be the quotient map. (Observe that $(A/I)^+$ is isomorphic to A^+/I.) Then $\pi(wp_nw^{-1}) = p_n$, so wp_nw^{-1} is an element of I^+. Moreover, wp_nw^{-1} is an idempotent. We now define $\partial[x]$ to be $[wp_nw^{-1}] - [p_n] \in K_0(I)$.

This definition is of course ludicrously unwell-seeming. We must check that it does not depend on the representative of $[x]$, the choice of y and the lift of $\mathrm{diag}(x, y)$. We must also check that ∂ is a group homomorphism. Then all the definitions will be in place, and our task will be to show the exact sequence results outlined at the beginning.

To see that ∂ does not depend on the lift w, suppose that v is another lift of $\mathrm{diag}(x, y)$. Then $vp_nv^{-1} = (vw^{-1})wp_nw^{-1}(vw^{-1})^{-1}$, which is similar in $GL_\infty(I^+)$ to wp_nw^{-1}, since vw^{-1} is an invertible in I^+.

To see that ∂ does not depend on the representative x of $[x]$ in $GL_n((A/I)^+)$ or the choice of y, let x' and y' be different choices. Without loss of generality $y' \in M_{m+k}((A/I)^+)$ with $k \geqslant 0$. Then $x^{-1}x'$ and $\mathrm{diag}(y^{-1}, 1_k)y'$ are in $GL^0_n((A/I)^+)$ and $GL^0_{m+k}((A/I)^+)$, so they have lifts a and b in $GL_n(A^+)$ and $GL_{m+k}(A^+)$. But then $u = w\mathrm{diag}(a, b)$ is a lift of $\mathrm{diag}(x', y')$. Since $\mathrm{diag}(a, b)$ commutes with p_n, we have $up_nu^{-1} = wp_nw^{-1}$. We are now done, since we were free to choose any lift.

Finally, to show that ∂ is a group homomorphism is easy if one uses the fact that $[xy] = [\mathrm{diag}(x, y)]$ in $K_1((A/I)^+)$ and the fact that lifts can be chosen however one likes.

We shall now obtain the long exact sequence

$$K_1(I) \to K_1(A) \to K_1(A/I) \to K_0(I) \to K_0(A) \to K_0(A/I)$$

over the course of a few lemmas. At a first reading, there is something to be said for skipping the proofs of the next three lemmas, since the three after that are easier and provide considerable motivation for them. As usual we shall let ι and π be the inclusion from I into A and the projection from A onto A/I respectively.

Lemma 10. *The sequence is exact at $K_0(A)$.*

Proof. A typical element of $K_0(I)$ can be written $[x] - [p_n]$ with $x - p_n \in M_{2n}(I)$. Its image in A is denoted by the same expression, and then

$$\pi^*([x]-[p_n]) = [\pi^+(x)] - [p_n] = [p_n] - [p_n] = 0 \ .$$

Conversely, suppose $[y]-[p_k]$ is an element of $K_0(A)$, $y \in M_n(A^+)$ and that $[\pi(y)] - [p_k] = 0$. Then there exists m such that $[\pi(y)] + [p_m] = [p_{k+m}]$. Hence, there exists an invertible $u \in M_{n+m}((A/I)^+)$ such that $u\mathrm{diag}(\pi(y), p_m)u^{-1} = p_{k+m}$. Then we also have $\mathrm{diag}(u, u^{-1})\mathrm{diag}(\pi(y), p_m)\mathrm{diag}(u^{-1}, u) = p_{k+m}$. We can lift $\mathrm{diag}(u, u^{-1})$ to an invertible w in $GL^0_{2n+2m}(A^+)$. Then $\pi(w\mathrm{diag}(y, p_m)w^{-1}) = p_{k+m}$, so $[w\mathrm{diag}(y, p_m)w^{-1}] - [p_{k+m}]$ is an element of $K_0(I)$ equivalent to $[y]-[p_k]$. □

Lemma 11. *The sequence is exact at $K_0(I)$.*

Proof. Let u be an invertible in $(A/I)^+$ and let us write $\partial([u]) = [wp_nw^{-1}] - [p_n]$ for some lift w of a suitable $\mathrm{diag}(u, v)$. This is rather obviously zero when considered as an element of $K_0(A)$ rather than $K_0(I)$.

Let $[x+p_n]-[p_n]$ be an element of $K_0(I)$ written in the usual way and suppose that it is zero in $K_0(A)$. Then by the remark following Proposition 3 we can find m and x' similar to x in $M_{m+n}(I^+)$ such that $x'+p_m$ is similar to p_m in $M_{n+m}(A^+)$, and rewrite the element as $[x'+p_m]-[p_m]$. Let w be an invertible in $M_{2n+m}(A^+)$ such that $wp_mw^{-1} = x'+p_m$. Now x' is an element of $M_{n+m}(I)$, so $\pi(x') = 0$ and the above equation tells us that $\pi(w)$ commutes with p_m. It follows that $\pi(w)$ has a matrix of the form $\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$.

Then $[z_{11}]$ is an inverse image under ∂ for $[x+p_n]-[p_n]$. This is because $\pi(w) = \mathrm{diag}(z_{11}, z_{22})$ and is in $GL^0_{2n+m}((A/I)^+)$, so we have

$$\partial[z_{11}] = [wp_mw^{-1}] - [p_m] = [x'+p_m] - [p_m] = [x+p_n] - [p_n] \ .$$

Lemma 12. *The sequence is exact at $K_1(A/I)$.*

Proof. Let $x \in GL_n(A^+)$ and let $y = \pi(x)$. Then $\mathrm{diag}(y, y^{-1})$ lifts to $\mathrm{diag}(x, x^{-1})$ which commutes with p_n, so $\partial[y] = 0$.

Conversely, if $y \in GL_n((A/I)^+)$ and $\partial[y] = 0$, then let w be a lift of $\mathrm{diag}(y, y^{-1})$. We know that $[wp_nw^{-1}] - [p_n] = 0$, so that $\mathrm{diag}(wp_nw^{-1}, p_m)$ is similar to $\mathrm{diag}(p_n, p_m)$ in $M_{2n+m}(I^+)$ for some m. Pick $u \in M_{4n+2m}(I^+)$ such that $u\mathrm{diag}(wp_nw^{-1}, p_m)u^{-1} = \mathrm{diag}(p_n, p_m)$. This equation tells us that $u\mathrm{diag}(w, p_m)$ commutes with $\mathrm{diag}(p_n, 0_n, p_m)$

and hence has a 4×4 block matrix of the form $z = (z_{ij})$ with $z_{ij} = 0$ whenever $i + j$ is odd. That implies that $\pi(u)\mathrm{diag}(y, y^{-1}, p_m)$ commutes with $\mathrm{diag}(p_n, p_m)$, or that $\pi(u)$ commutes with $\mathrm{diag}(p_n, p_m)$, so $\pi(u)$ has a 4×4 scalar block matrix (λ_{ij}) of the same form. But then $\begin{pmatrix} \lambda_{11} & \lambda_{13} \\ \lambda_{31} & \lambda_{33} \end{pmatrix}$ is invertible and $GL_{n+m}(\mathbb{C})$ is connected, which implies that the equivalence class of $\begin{pmatrix} z_{11} & z_{13} \\ z_{31} & z_{33} \end{pmatrix}$ is a lift of $[\mathrm{diag}(x, p_m)] = [x]$. $\square$

Instead of proving directly that the sequence is exact at $K_1(A)$ we shall now show that $K_1(A)$ is in fact K_0 of a different algebra. The *suspension* SA of the algebra A is defined to be the algebra of continuous functions f from $[0, 1]$ into A such that $f(0) = f(1) = 0$. The *cone* of A is the same except that $f(1)$ does not have to be 0. All operations are pointwise, and the norm is the uniform norm.

Lemma 13. *Let A be any algebra. Then $K_0(CA) = K_1(CA) = 0$.*

Proof. Let $(x(t), \lambda)$ be an idempotent in $M_n(CA^+)$. The non-scalar part is easily seen to be homotopic to the zero map via the homotopy $x_\theta(t) = x(\theta t)$. Hence $K_0(CA^+) = K_0(\mathbb{C}) = \mathbb{Z}$ and $K_0(CA)$, the kernel of π^*, is 0. A similar but easier argument does K_1. $\square$

Lemma 14. *Let A be any algebra. Then $K_1(A) \cong K_0(SA)$.*

Proof. There is an obvious short exact sequence

$$0 \to SA \to CA \to A \to 0 \; .$$

SA is certainly an ideal in CA, so by Lemmas 10,11 and 12 we obtain an exact sequence

$$K_1(CA) \to K_1(A) \to K_0(SA) \to K_0(CA) \to K_0(A) \; .$$

By Lemma 13 this gives us the sequence

$$0 \to K_1(A) \to K_0(SA) \to 0$$

and therefore shows that ∂ is the required isomorphism. $\square$

The isomorphism above is a *natural* one. This means that a homomorphism from A to B yields an obvious commutative diagram taking the isomorphism from $K_1(A)$ to $K_0(SA)$ to the corresponding isomorphism for B. This property is left as an exercise.

Corollary 15. *Defining $K_n(A)$ to be $K_0(S^nA)$, we have the long exact sequence*

$$\to K_{n+1}(A/I) \to K_n(I) \to K_n(A) \to K_n(A/I) \to K_{n-1}(A) \to \ldots \to K_0(A/I)$$

mentioned right at the beginning. □

Note that we also obtain such a long exact sequence from a more general looking short exact sequence $0 \to A \to B \to C \to 0$. However we can think of A in this case as an ideal in B (as it is isomorphic to its image in B, which is the kernel of the map from A to C) so this is not really a generalization.

Corollary 16. *K_0 and K_1 preserve split exactness.*

Proof. By Lemma 14 it is enough to prove this for K_0. Consider a split exact sequence $0 \to A \to B \to C \to 0$. The map σ from B to C has a right inverse which is an algebra homomorphism, and hence the corresponding maps σ^* from $K_1(B)$ to $K_1(C)$ and from $K_0(B)$ to $K_0(C)$ have right inverses and are therefore surjections. It follows by exactness that ∂ is the zero map in this case, and therefore that the long exact sequence

$$K_1(B) \to K_1(C) \to K_0(A) \to K_0(B) \to K_0(C)$$

gives us the short exact sequence

$$0 \to K_0(A) \to K_0(B) \to K_0(C) \to 0 \ ,$$

where the last zero comes from σ^* being surjective. Since σ^* has a right inverse, this sequence is split exact. □

To complete the picture, all that is left is to show that $K_2 = K_0$, which, by Lemma 14, amounts to showing that $K_0(A)$ is isomorphic to $K_1(SA)$. This is the Bott periodicity theorem. We shall give a detailed sketch rather than a complete proof (which can be found in Wegge-Olsen). Those details left out will all be simple exercises. Why has the result any chance of being true? We must somehow find a way of associating matrices in $GL_\infty(SA^+)$ to idempotents in $M_\infty(A^+)$. Now an element of $GL_n(SA^+)$ corresponds in an obvious way to a loop in $GL_n(A^+)$. The beginning (and end) of the corresponding loop will be a scalar matrix, since a typical element of SA^+ is the sum of a scalar and a loop in A starting and ending at zero. Since $GL_n(\mathbb{C})$ is connected, every element of $GL_n(SA^+)$ is equivalent to a loop in $GL_n(A^+)$ starting and ending at 1. But such a loop lies in $GL_n^0(A^+)$.

It turns out that the so-called *Bott map* β from $K_0(A)$ to $K_1(SA)$ is extremely simple to define. Given an idempotent $x \in M_n(A)$, let $\beta[x]$ be the (equivalence class of the) loop $f_x : z \mapsto zx + (1-x)$. Notice that f_x is a homomorphism from $\mathbb{T}$ to its image. This shows that f_x really is invertible. More generally, define $\beta([x]-[y])$ to be $f_x f_y^{-1}$. It is clear that a homotopy between two idempotents results in a homotopy of loops, so β is well defined. Moreover it is a homomorphism since $\beta([x]+[y]) = f_{\mathrm{diag}(x,y)}$ and so without loss of generality x and y are orthogonal. This then gives $f_{x+y} = f_x f_y$ as required. (The extension from VA to $K_0(A)$ is automatic.)

What is much less obvious is that β is an isomorphism. Our task will be to show that every element of $K_1(SA)$, or loop in $GL_n(A^+)$ starting and ending at 1, is equivalent to some loop of the very special form $f_x f_y^{-1}$. An initial simplification is that we may assume that A is unital. To see this, bear in mind that by Corollary 16 we have the following pair of split exact sequences:

$$0 \to K_0(A) \to K_0(A^+) \to K_0(\mathbb{C}) \to 0$$

$$0 \to K_1(SA) \to K_1(SA^+) \to K_1(S\mathbb{C}) \to 0 \ .$$

If we know the result for unital algebras, then β_{A^+} and $\beta_{\mathbb{C}}$ are isomorphisms from $K_0(A^+)$ to $K_1(SA^+)$ and from $K_0(\mathbb{C})$ to $K_1(S\mathbb{C})$ respectively. By the five lemma, β_A is also an isomorphism.

For the remainder of the proof we shall therefore assume that A is unital. Given a loop in $GL_n(A)$ starting and ending at 1, we wish to find an equivalent loop of the form $zx + (1-x)$ for some idempotent x. Let us define a *trigonometric loop* to be one of the form $z \mapsto \sum_{i=-N}^{N} a_i z^i$, where the coefficients a_i are elements of $GL_n(A)$, and a *polynomial loop* to be a trigonometric loop with no non-zero coefficients a_i for negative i.

Lemma 17. *Every loop f can be uniformly approximated by trigonometric loops, and is therefore equivalent to one.*

Proof. For $0 \leqslant k \leqslant m$ let I_k be the set of elements of $\mathbb{T}$ with argument in the interval $\frac{2\pi}{m}(k \pm 1)$, let $z_k = \exp(2\pi i k/m)$ and let $a_k = f(z_k)$. If m is large enough, then the diameter of any $f(I_k)$ is at most ϵ. Let $g_0, g_1, \ldots, g_m$ be a partition of unity with g_k supported in I_k. It is easy to check that $g(z) = \sum_{k=0}^{m} a_k g_k(z)$ approximates $f(z)$ to within ϵ. By the Stone-Weierstrass theorem (or even just the Weierstrass bit) each g_k can be approximated uniformly by a trigonometric polynomial, and therefore $g(z)$ can be approximated uniformly by a trigonometric loop, and finally so can $f(z)$.

For the last part, observe that any two loops in $GL_n(A)$ starting and ending at 1 that are sufficiently uniformly close are homotopic via loops of the same form. □

Lemma 18. *Every polynomial loop is equivalent to a linear loop.*

Proof. Let $P(z)$ be a polynomial of degree m with values in $GL_N(A)$. Write $P(z) = Q(z) + az^m$ with $Q(z)$ of degree at most $m-1$. Then

$$\begin{pmatrix} 1 & \lambda a z^{m-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P(z) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda z & 1 \end{pmatrix} = \begin{pmatrix} P(z) - \lambda^2 a z^m & \lambda a z^{m-1} \\ -\lambda z & 1 \end{pmatrix}$$

so that letting λ vary from 0 to 1 gives a homotopy in $GL_{2N}(A)$ from $\mathrm{diag}(P(z), 1)$ to a polynomial of smaller degree. Repeating the process, we can get the degree down to 1. □

We now wish to show that a loop of the form $z \mapsto az+b$ is equivalent to an idempotent loop. Setting $z = 1$ we see that $a + b$ is an invertible scalar matrix, and hence homotopic to the identity. Therefore the loop $az + b$ is homotopic to the loop $(a + b)^{-1}(az + b)$, so without loss of generality $a + b = 1$ and our linear loop is of the form $z \mapsto az + (1 - a)$. This at least looks reasonably close to what we want - the rest of the proof consists in building an idempotent out of a using holomorphic functional calculus, taking the obvious path from a to that idempotent and checking that everything works.

Lemma 19. *Suppose $az + (1 - a)$ is invertible for every $z \in \mathbb{T}$. Then the spectrum of a is disjoint from the line* $\mathrm{Re}(z) = 1/2$.

Proof. When $z \neq 1$, $az + (1 - a) = 1 + a(z - 1) = (z - 1)(\frac{1}{z-1} + a)$ is invertible, so $-(z - 1)^{-1}$ is not in the spectrum of a. But the map $z \mapsto -(z - 1)^{-1}$ maps $\mathbb{T} \setminus \{1\}$ onto the line $\mathrm{Re}(z) = 1/2$. □

Lemma 20. *Let a be an element of a Banach algebra, and let C be a closed contour in $\mathbb{C}$ disjoint from the spectrum of a. Then*

$$x = \frac{1}{2\pi i} \int_C (z - a)^{-1} dz$$

is an idempotent which commutes with a.

Proof. Let C' be a contour outside C still disjoint from the spectrum of a.

$$\begin{aligned}
x^2 &= \frac{1}{(2\pi i)^2}\int_C (z-a)^{-1}dz \int_{C'}(z'-a)^{-1}dz' \\
&= \frac{1}{(2\pi i)^2}\int_C\int_{C'}(z-a)^{-1}(z'-a)^{-1}dz'dz \\
&= \frac{1}{(2\pi i)^2}\int_C\int_{C'}\frac{(z-a)^{-1}-(z'-a)^{-1}}{z'-z}dz'dz \\
&= \frac{1}{(2\pi i)^2}\int_C(z-a)^{-1}\int_{C'}\frac{dz'}{z'-z}dz + \frac{1}{(2\pi i)^2}\int_{C'}(z'-a)^{-1}\int_C\frac{dz}{z'-z}dz' \\
&= \frac{1}{2\pi i}\int_C (z-a)^{-1}dz
\end{aligned}$$

by Cauchy's integral formula and Cauchy's theorem. The commutativity with a follows from the fact that x is in the closed subalgebra generated by 1 and a. $\square$

The argument of Lemma 20 is a special case of a more general method of defining analytic functions on Banach algebras. If f is a complex function analytic on some domain D containing the spectrum $\sigma = \sigma(x)$ of an element $x \in A$, and if ϕ is a cycle in $D \setminus \sigma$ that winds exactly once round each point in σ, then $f(x)$ is defined to be $\int_\phi f(z)(z-x)^{-1}dz$. It can be shown that $\sigma(f(x)) = f(\sigma(x))$ (the *spectral mapping theorem*), which we shall use below. If σ has two components σ_1 and σ_2, then it is possible to find $D = D_1 \cup D_2$ disjoint open sets disconnecting σ. The function f that is 1 on D_1 and 0 on D_2 is analytic on $D_1 \cup D_2$. The resulting function $f(x)$ is basically the idempotent constructed above. For more details, look up "holomorphic (or analytic) functional calculus" in a typical functional analysis textbook. E.g. Functional Analysis by H. G. Heuser is quite good on the topic (H3/HEU in the library). There may well be better accounts.

Lemma 21. *Let a and x be as above and let $0 \leqslant t \leqslant 1$. The spectrum of $(1-t)a + tx$ is disjoint from the line* $\mathrm{Re}(z) = 1/2$.

Proof. Let f be defined on the spectrum of a by $f(z) = 1$ when $\mathrm{Re}(z) > 1/2$ and 0 when $\mathrm{Re}(z) < 1/2$. As we commented above, the element x is $f(a)$, where this has its functional calculus meaning. Moreover, $(1-t)a + tx = (1-t)a + tf(a)$. The spectral mapping theorem implies that

$$\sigma\Big((1-t)a + tf(a)\Big) = \{(1-t)z + tf(z) : z \in \sigma(a)\}\ .$$

From the definition of the function f, it is clear that this set is disjoint from the line $\mathrm{Re}(z) = 1/2$, as required. $\square$

Now let b be any element of $M_n(A)$ of the form $(1-t)a + tx$ with $t \in [0,1]$. What we have just proved about the spectrum of b implies that $bz + (1-b)$ is invertible for every $z \in \mathbb{T}$. It follows that the straight line from a to x gives rise to a homotopy from the loop $az + (1-a)$ to the idempotent loop $xz + (1-x)$. This completes the proof that β is surjective.

To show that β is injective is easier. Given a loop f, we construct a sequence of equivalent loops as above, ending up with an idempotent loop x. Then $\beta^{-1}[f]$ is defined to be $[x] - [p_N]$, where N was the power of z used to turn the trigonometric loop into a polynomial one. It must be checked that this map β^{-1} is well-defined - in other words, that different choices lead to an element $[x'] - [p_M]$ equal to $[x] - [p_N]$ in $K_0(A)$. This is a straightforward exercise. (It appears in great detail in Wegge-Olsen, or as a very short remark in Blackadar.) For example, two homotopic polynomial loops result in two homotopic linear loops, and thereby to homotopic choices for a, and therefore (small estimate needed) to homotopic choices for x.

A similarly easy exercise (which would nevertheless be quite long to write out in full) is that β is a *natural* isomorphism between $K_0(A)$ and $K_2(A)$. That is, if $\phi : A \to B$ is a homomorphism, then the diagram

$$\begin{array}{ccc} K_0(A) & \to & K_0(B) \\ \\ K_2(A) & \to & K_2(B) \end{array}$$

commutes.

We have thus reached the culmination of these notes:

Theorem 23. *There is a natural isomorphism between $K_0(A)$ and $K_1(SA)$. Therefore a short exact sequence $0 \to A \to B \to C \to 0$ of Banach algebras gives a cyclic six-term exact sequence*

$$\begin{array}{ccccc} K_0(A) & \to & K_0(B) & \to & K_0(C) \\ \\ K_1(C) & \leftarrow & K_1(B) & \leftarrow & K_1(A) \end{array}$$

of Abelian groups. $\square$

Now that we have proved the Bott periodicity theorem, let us briefly see what Banach algebra K-theory has to do with the version familiar to topologists. Let $X = S_n$ for some

n. Then $A = C[X]$ is a Banach algebra (even a C^*-algebra) and we can therefore look at $K_0(A)$ and $K_1(A)$ of this (unital) algebra. $V(A)$ then consists of equivalence classes of idempotents, and an idempotent in $M_n(A)$ is basically a continuous map from X to $M_n(\mathbb{C})$, where every $x \in X$ maps to an idempotent. Since the map is continuous, the ranks of these idempotents are all the same, k say, and the images form a k-dimensional vector bundle, which is a k-dimensional section of the trivial vector bundle of dimension n. Moreover, two such vector bundles are equivalent if and only if the corresponding idempotents are equivalent.

The original version of K-theory started with equivalence classes of vector bundles, with addition defined by direct sum. This gave a semigroup and K_0 was the corresponding Grothendieck group. Given a manifold X, the vector bundle $K_0(X)$ is equal to the Banach algebra K_0 of $C[X]$. However, the algebra version works for non-commutative algebras, which obviously do not arise as $C[X]$ for any X: hence the term "non-commutative geometry".

More generally, if X is not compact, then $C_0[X]$ stands for the algebra of continuous functions f on X such that $|f| \geqslant \epsilon$ on a compact subset of X for every $\epsilon > 0$. (E.g. $C_0[\mathbb{R}]$ is continuous functions tending to zero at infinity.) If $Y \subset X$ and Y and X are both compact, then the ideal in $C[X]$ of functions vanishing on Y is isomorphic to $C_0[X \setminus Y]$. We therefore have an exact sequence

$$0 \to C_0[X \setminus Y] \to C[X] \to C[Y] \to 0 \ ,$$

and this may enable us to calculate the K-theory of X from that of Y and how X is put together from Y.

This does not yet explain how to work out K_1 in the topological case. The *suspension* of a space X is defined to be $X \times [0,1]$ with $X \times \{0\}$ and $X \times \{1\}$ identified to two points. (For the *cone* just do this to $X \times \{0\}$.) For example, the suspension of S_n is S_{n+1}. The definition of $K_{n+1}(X)$ is now of course $K_n(SX)$, and the Bott periodicity theorem in its original form states that $K_2(X) = K_0(X)$.

The following glossary translates concepts from topological K-theory into their Banach-algebra counterparts.

Topological space X	$C_0(X)$
Vector bundle on X	Idempotent in $C_0(X)$

Direct sum of bundles	Orthogonal sum of idempotents in $M_\infty(C_0(X))$
One-point compactifying X	Adjoining a unit to $C_0(X)$
Suspension and cone of X	Suspension and cone of $C_0(X)$
$K_n(X)$	$K_n(C_0(X))$

I strongly recommend browsing through the books of Blackadar and Wegge-Olsen to get some sort of feel for what K-theory is good for, where the theory goes next and so on. It turns out that there are many other cyclic six-term exact sequences one can define (e.g. there is one - the Pimsner-Voiculescu exact sequence - which can help to determine the K-groups of the tensor product of two C^*-algebras). In other words, the Bott periodicity theorem is not an isolated fact, but more like an example of a phenomenon that appears all over the place.

A very nice chapter in Wegge-Olsen explains that K-theory can be done axiomatically, just like homology and cohomology. That is, there is a small list of properties that characterize the functors K_0 and K_1, and it is possible to deduce the Bott periodicity theorem very cleanly from these properties. However, as far as I know this has been done only for C^*-algebras.

Finally, a bluffer's guide to K-theory would not be complete without a mention of Kasparov's KK-theory. This is a functor $KK(A, B)$ defined on *pairs* of algebras. Various restrictions of it specialize to known other theories, including K-theory. One can define a product of some sort, and it turns out that $KK(A, B)KK(B, C) = KK(A, C)$. This is difficult to prove and, for some reason, frightfully important.

Exercises.

1. Show that K_0 is a functor.
2. Let A be an algebra, and let γ be the obvious embedding of A into A^+. Under what circumstances will γ have a left inverse which is also an algebra homomorphism?
3. Show that $K_0(A \oplus B) = K_0(A) \oplus K_0(B)$ for arbitrary algebras A and B.
4. Show that Lemma 7 is false if one defines $K_0(A)$ to be $K_0(A^+)$ for non-unital algebras.
5. Let $A = L(H)$ for H a separable Hilbert space and let $I = K(H)$ (the ideal of compact operators). Then the index map from $K_1(A/I)$ to $K_0(A)$ takes a Fredholm operator (i.e., an invertible in A/I) to an element of $K_0(I)$. How does this element of $K_0(I)$ relate to the index of the Fredholm operator as usually understood?

6. Prove that the index map from $K_1(A)$ to $K_0(SA)$ is not just an isomorphism (which you may assume) but a natural one.

7. What are $K_0(S_n)$ and $K_1(S_n)$? What about $K_0(\mathbb{T}^n)$ and $K_1(\mathbb{T}^n)$?

8. Prove that $GL_n(\mathbb{C})$ is connected.

9. Give an example of a non-zero element of $K_1(C[\mathbb{T}])$.

10. Let X be a Banach space isomorphic to its square. Show that $K_0(L(X)) = \{0\}$.

11. (For the keen - I have not done this.) What should the definition of the index map be in the topological version of K-theory?