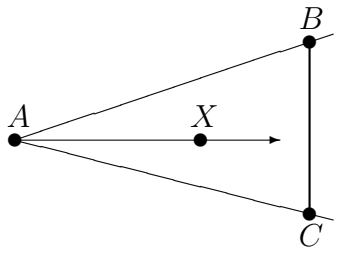
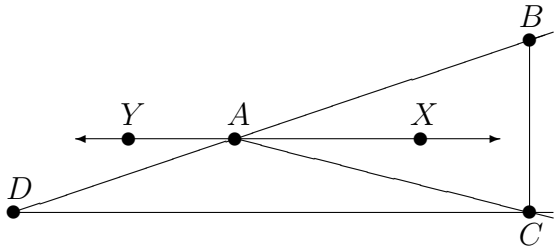
Crossbar Theorem

Theorem 1 *Let X be a point interior to $\angle BAC$. Then the ray $\overrightarrow{AX}$ crosses the segment $\overline{BC}$.*



PROOF Let a point D be chosen so that $B - A - D$. Then we can apply the Pasch axiom to ΔCDB and conclude that $\overleftrightarrow{AX}$ crosses either $\overline{BC}$ or $\overline{DC}$.



Noting that D and B are on opposite sides of $\overleftrightarrow{AC}$, it follows that D and X are on opposite sides of $\overleftrightarrow{AC}$, and hence that all points of $\overline{DC}$ are on the side of $\overleftrightarrow{AC}$ opposite to all points on $\overrightarrow{AX}$. Therefore $\overrightarrow{AX}$ does not cross $\overline{DC}$. We must finally eliminate the possibility that the opposite ray, $\overrightarrow{AY}$, crosses either $\overline{DC}$ or $\overline{BC}$. To do so we have only to notice that both of these segments are on the opposite side of $\overleftrightarrow{DB}$ from $\overrightarrow{AY}$.

The only possibility not eliminated is that $\overrightarrow{AX}$ crosses $\overline{BC}$. ∎