Continuous deformations of Fredholm operators in B(H)

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Resumo

DIAS, R. L. **Deformações contínuas de operadores de Fredholm em B(H)**. 2021. Dissertação (Mestrado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2021.

Seja X um espaço topológico Hausdorff compacto. O K-grupo de X, denotado por $K(X)$, é o grupo de Grothendieck associado ao monoide comutativo das classes de isomorfismos de fibrados vetoriais complexos sobre X, munido da soma de Whitney.

Sejam H um espaço de Hilbert de dimensão infinita e $F(H)$ o conjunto dos operadores de Fredholm em H. O Teorema de Atiyah-Jänich afirma que o *families-index* é um isomorfismo natural entre o monoide das classes de homotopia das funções de X em $F(H)$ e o grupo $K(X)$. No caso em que X consiste de apenas um ponto, o *families-index* é o clássico índice de Fredholm, e o Teorema de Atiyah-Jänich afirma que as componentes conexas por caminhos de $F(H)$ são caracterizadas pelo índice de Fredholm.

Nesse trabalho, fazemos uma exposição detalhada do Teorema de Atiyah-Jänich, estudando os elementos necessários para entender a construção do K-grupo de um espaço topológico Hausdorff compacto, a definição do *families-index* e a demonstração de que tal índice é o isomorfismo mencionado.

Palavras-chave: K-teoria, operadores de Fredholm, índice de Fredholm, teoria do índice, K-teoria de espaços compactos.

Abstract

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Let X be a compact Hausdorff topological space. The K-group of X, denoted by $K(X)$, is the Grothendieck group associated to the commutative monoid of isomorphism classes of complex vector bundles over X, equipped with the Whitney sum.

Let H be an infinite dimensional Hilbert space and $F(H)$ be the set of Fredholm operators on H. The Atiyah-Jänich Theorem states that the *families-index* is a natural isomorphism between the monoid of homotopy classes of functions from X into $F(H)$ and the group $K(X)$. In case X is a singleton, the *families-index* is the classic Fredholm index, and the Atiyah-Jänich Theorem states that the arcwise connected components of $F(H)$ are characterized by the Fredholm index.

In this work, we give a detailed exposition of the Atiyah-Jänich Theorem, studying the necessary elements to understand the construction of the K-group of a compact Hausdorff topological space, the definition of the *families-index* and giving a proof that such an index is the mentioned isomorphism.

Keywords: K-theory, Fredholm operators, Fredholm index, index theory, K-theory of compact spaces.

Contents

Introduction

It is of great interest to mathematicians and enthusiasts to understand the behavior of linear operators, given that they naturally appear in many problems of mathematics, physics and related areas. A motivating source of examples is the field of linear differential equations. For instance, the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = g(x, y, z)$$

can be written as $\Delta f = g$, where $\Delta \doteq \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a linear differential operator defined over suitable function spaces. Let $D\colon X \to Y$ be a linear differential operator between appropriate function spaces X and Y. Given $g \in Y$, one can ask for existence and uniqueness of solutions for the equation $Df = g$, and the answer for this question is associated to the surjectivity and injectivity of D, respectively. If D is an elliptic differential operator, we can impose a finite number of restrictions to g in order to be able to solve the equation $Df = g$, and the solution is uniquely determined if we give a finite number of parameters. More specifically, that is to say that Ker D and Coker D are finite dimensional. This last property is precisely what defines a Fredholm operator: a linear map $T\colon V \to W$ such that Ker T and Coker T are finite dimensional (if V and W are both topologized, we also require T to be continuous and $T(V)$ to be closed). For a Fredholm operator T, the integer number

$$\text{ind } T \doteq \dim \text{Ker } T - \dim \text{Coker } T,$$

called Fredholm index, is a measurement for the injectivity and surjectivity of T. Surprisingly, this algebraic object turns out to be a topological invariant, as one can see in the following classical result.

For a Hilbert space H and Fredholm operators $T_1, T_2\colon H \to H$ it is true that $\text{ind}(T_1) = \text{ind}(T_2)$ if and only if there exists a continuous path of Fredholm operators joining T_1 and T_2.

Another way to state this result is: *the Fredholm index is a bijection from the (arcwise) connected components of the space of Fredholm operators in H and the set of integer numbers.*

Topological K-theory, a tool created by Atiyah and Hirzebruch in the 60's, inspired in the previous work of Grothendieck, provides a successful way to generalize the above result. The K group of a compact Hausdorff topological space X, denoted by $\mathsf{K}(X)$, is constructed using isomorphism classes of vector bundles over X. It is possible to associate an element $\text{ind}(T) \in \mathsf{K}(X)$ to every continuous family T of Fredholm operators parameterized by X, in a natural manner, in such a way that $\text{ind}(T)$ coincides with the classical Fredholm index whenever the space X consists of one point only (and T is therefore a single Fredholm operator). This is the reason why we can call $\text{ind}(T)$ the *index bundle* of T. The Atiyah-Jänich Theorem is the mentioned generalization and it asserts that the *families-index* is an isomorphism from the monoid of homotopy classes of continuous families of Fredholm operators parameterized by X onto the abelian group $\mathsf{K}(X)$.

The discussion above could help one to understand the following Atiyah's quote: "Abstract Functional Analysis provides the natural meeting ground of Algebraic Topology and Partial Differetial Equations" (see [Ati70]).

This Master's Thesis is primarily written with the intention of presenting a systematic and comprehensive account of the Atiyah-Jänich Theorem for beginners. The prerequisites and requirements necessary for understanding this text are essentially basic functional analysis and topology, so that any senior undergraduate student of mathematics should be able to read this thesis with no trouble.

The content treated here is not original, but rather a detailed exposition of the work of K. Jänich, F. Hirzebruch, M. Atiyah, among other great mathematicians. One can find a discussion of Atiyah-Jänich Theorem in the Appendix of [Ati67], where one approach for the *families-index* is given. Another approach is given in [Bre16], whose description of the families-index has some similarity with the index of the paper Atiyah-Singer IV [AS71]. I have discussed these two approaches in the slightly more general setting that the continuous families of Fredholm operators can be considered to be defined between distinct Hilbert spaces. To prove the equivalence of these approaches, it was needed to suppose that our Hilbert spaces are complexes. Using Kuiper's Theorem [Kui65] and a generalization of it, that was shown in [Ill65], I gave a proof for the Atiyah-Jänich Theorem dropping the assumption of separability for the Hilbert space, and assuming only that its dimension are infinite. At last, a digression was made to give a category theoretical interpretation for Atiyah-Jänich Theorem, concluding that the constructed families-index is a natural isomorphism between the K-theory functor and the functor that, to each compact Hausdorff space X, associate the monoid of homotopy classes of functions from X into the space of Fredholm operators in an infinite dimensional Hilbert space.

Chapter 1

Background and Prerequisites

In this section we discuss some preliminary concepts. Most of them shall be used throughout the text, even if we do not mention them explicitly sometimes.

1.1 Some Algebraic Requisites

Definition 1.1. A *monoid* (or *semigroup*) is a pair $(M, \star)$ where M is a set and $\star\colon M \times M \to M$ is a binary operation in M such that

(i) $(a \star b) \star c = a \star (b \star c)$ for every $a, b, c \in M$;

(ii) there exists $e \in M$ such that $a \star e = e \star a = a$ for every $a \in M$.

The element e is called *neutral element*. If $\star$ also satisfies

(iii) $a \star b = b \star a$ for every $a, b \in M$,

we shall call $(M, \star)$ a *commutative monoid*. In this case, we will use the additive notation for operation $\star$, writing simply $+$, and the neutral element will be denoted by 0.

An element a in a monoid M is said to be *invertible* if there exists $b \in M$ such that $ab = ba = e$, and such b is called an *inverse* for a. Differently from the groups, and by the very definition, elements in monoids need not to be invertible, but when it is, its inverse is unique. For if $a \in M$ and there exist $x, y \in M$ such that $ax = xa = e = ay = ya$, then $x = xe = x(ay) = (xa)y = ey = y$. If $a \in M$ is invertible, we denote its inverse by a^{-1} (and by $-a$ in the commutative case).

Example 1.2.

1. Every group (resp. abelian group) is a monoid (resp. commutative monoid).
2. $(\mathbb{N}, +)$ is a commutative monoid with neutral element $0 \in \mathbb{N}$.
3. The set of non zero integers $\mathbb{Z}^\times$ equipped with integer multiplication is a commutative monoid with neutral element $1 \in \mathbb{Z}^\times$.
4. Given a set X, the set of all functions from X to itself equipped with the composition operation, $(X^X, \circ)$, is a monoid with neutral element $id_X \in X^X$.
5. The set $\{e, a\}$, $e \neq a$, equipped with multiplication table given by

	e	a
e	e	a
a	a	a

is a commutative monoid.

Definition 1.3. Let M and N be monoids. We say that a map $f\colon M \to N$ is a *monoid morphism* if it satisfies

(*i*) $f(ab) = f(a)f(b)$ for every $a, b \in M$,

(*ii*) $f(e) = e$.

The *kernel* of a monoid morphism $f\colon M \to N$ is $\operatorname{Ker} f \doteq f^{-1}(e)$.

Example 1.4. Let X be a set and fix $\phi \in X^X$. Define, for $n \geq 0$,

$$\phi^{\circ n} = \begin{cases} id_X & \text{if } n = 0, \\ \phi^{\circ(n-1)} \circ \phi & \text{if } n > 0. \end{cases}$$

The map $\mathbb{N} \to X^X$, $n \longmapsto \phi^{\circ n}$, is a monoid morphism.

It is known that a group morphism is injective if and only if its kernel is trivial. This is not true for monoids.

Example 1.5. Let M be the commutative monoid described in Example 1.2.5. Define $f\colon \mathbb{N} \to M$ by $f(n) = a^n$, that is,

$$f(n) = \begin{cases} e & \text{if } n = 0, \\ a & \text{if } n > 0. \end{cases}$$

Then f is a non injective monoid morphism with $\operatorname{Ker} f = \{0\}$.

The following is a result will be needed later in the text.

Proposition 1.6. *Let M be a monoid, G be a group and $f\colon M \to G$ be a surjective monoid morphism. If* $\operatorname{Ker} f = \{e\}$*, then f is injective.*

Proof. Let $a, b \in M$ be such that $f(a) = f(b)$. Since f is surjective and G is a group, we can consider $x \in M$ such that $f(x) = f(a)^{-1} = f(b)^{-1} \in G$. Then $f(ax) = f(a)f(x) = e = f(x)f(a) = f(xa)$ and we have $ax, xa \in \operatorname{Ker} f = \{e\}$, from where we conclude that $ax = e = xa$. In a similar way, $bx = e = xb$. Then $a = x^{-1} = b$, and therefore f is injective. □

1.2 Category Theory

In this section, it will be discussed basics concepts of Category Theory. For further discussion, see [Mac98].

Definition 1.7. A *category* C consists of

- a class $\mathsf{ob}(C)$ of *objects*,

- a class $\mathsf{hom}(C)$ of *morphisms* (or *arrows*, or *maps*) between objects,
- a *domain* (or *source object*) class function $\mathsf{dom}\colon \mathsf{hom}(C) \to \mathsf{ob}(C)$,
- a *codomain* (or *target object*) class function $\mathsf{cod}\colon \mathsf{hom}(C) \to \mathsf{ob}(C)$,
- for every $a, b, c \in \mathsf{ob}(C)$, a binary operation[1] $C(a,b) \times C(b,c) \to C(a,c)$ called *composition of morphisms*: the composition of $(f,g) \in C(a,b) \times C(b,c)$ is denoted by $g \circ f$ or simply gf. The composition must satisfy the following axioms:
  - ⋆ (Associativity) for every $a_1, a_2, a_3, a_4 \in \mathsf{ob}(C)$ and every $f_i \in C(a_i, a_{i+1})$, $i = 1,2,3$, the equality $f_3 \circ (f_2 \circ f_1) = (f_3 \circ f_2) \circ f_1$ holds,
  - ⋆ (Identity) for every $a \in \mathsf{ob}(C)$ there exists $id_a \in C(a,a)$, called *identity morphism for* a, such that for every $b \in \mathsf{ob}(C)$ and every $f \in C(a,b)$ and $g \in C(b,a)$ the equalities $f \circ id_a = f$ and $id_a \circ g = g$ hold.

Given $a, b \in \mathsf{ob}(C)$, a morphism $f \in C(a,b)$ is an *isomorphism* if there exists $g \in C(b,a)$ such that $g \circ f = id_a$ and $f \circ g = id_b$.

Remark 1.8. Let C be a category and fix $a \in \mathsf{ob}(C)$. Observe that the identity morphism for a is unique, for if $id'_a \in C(a,a)$ is a morphism satisfying the same property we would have

$$id_a = id_a \circ id'_a = id'_a \ .$$

Example 1.9. 1. The category of sets, Set, consists of the class of all sets $\mathsf{ob}(\mathsf{Set})$ and the class of all maps between all sets, $\mathsf{hom}(\mathsf{Set})$. Domain and codomain coincides with the usual notion of domain and codomain of maps. Composition coincides with the usual notion of composition of maps.

2. Let $\mathbb{K}$ be a field. The category of vector spaces over $\mathbb{K}$, $\mathsf{Vect}_{\mathbb{K}}$, consists of the class of all vector spaces $\mathsf{ob}(\mathsf{Vect}_{\mathbb{K}})$ and the class of all linear transformations between them, $\mathsf{hom}(\mathsf{Vect}_{\mathbb{K}})$. Again, domain, codomain and composition coincide with the usual notion.

3. The category of topological spaces, Top, consists of the class of all topological spaces $\mathsf{ob}(\mathsf{Top})$ and the class of all continuous maps between them, $\mathsf{hom}(\mathsf{Top})$. Domain, codomain and composition are defined as usual.

The morphisms of a category C can often be thought of as maps between objects in C that preserves the structure of interest (linear maps preserve linear structures of the spaces, continuous maps preserve topological structures, etc). There is also a notion of special maps between categories, that preserve the structure of the categories themselves.

Definition 1.10. Let C and D be two categories. A *covariant functor* F between C and D, denoted by $F\colon C \to D$, is a pair of class functions $F_1\colon \mathsf{ob}(C) \to \mathsf{ob}(D)$ and $F_2\colon \mathsf{hom}(C) \to \mathsf{hom}(D)$ satisfying

- for every $a, b \in \mathsf{ob}(C)$ and every $f \in C(a,b)$, it holds that $F_2(f) \in D(F_1(a), F_1(b))$,
- $F_2(id_a) = id_{F_1(a)}$ for every $a \in \mathsf{ob}(C)$,

[1]Given $a, b \in \mathsf{ob}(C)$ the symbol $C(a,b)$ stands for the subclass consisting of morphisms $f \in \mathsf{hom}(C)$ such that $\mathsf{dom}(f) = a$ and $\mathsf{cod}(f) = b$.

- for every $a, b, c \in \mathsf{ob}(C)$, $f \in C(a,b)$ and $g \in C(b,c)$, it holds that $F_2(g \circ f) = F_2(g) \circ F_2(f)$.

These properties are called *functorial properties.*

Similarly, a *contravariant functor* G between C and D, also written $G\colon C \to D$, is a pair of class functions $G_1\colon \mathsf{ob}(C) \to \mathsf{ob}(D)$ and $G_2\colon \mathsf{hom}(C) \to \mathsf{hom}(D)$ satisfying

- for every $a, b \in \mathsf{ob}(C)$ and every $f \in C(a,b)$, it holds that $G_2(f) \in D(G_1(b), G_1(a))$,
- $G_2(id_a) = id_{G_1(a)}$ for every $a \in \mathsf{ob}(C)$,
- for every $a, b, c \in \mathsf{ob}(C)$, $f \in C(a,b)$ and $g \in C(b,c)$, it holds that $G_2(g \circ f) = G_2(f) \circ G_2(g)$.

Let F be a functor. For simplicity, it is usual to write simply F instead of F_1 and F_2, so that the functorial properties are written as

$$F(id_a) = id_{F(a)} \qquad \text{and} \qquad F(g \circ f) = F(g) \circ F(f)$$

if F is covariant, and as

$$F(id_a) = id_{F(a)} \qquad \text{and} \qquad F(g \circ f) = F(f) \circ F(g)$$

if F is contravariant.

We shall see examples of functors in Chapter 2.

Definition 1.11. Let C and D be two categories and let F and G be covariant functors between C and D. A *natural transformation* between F and G is a class function $\eta\colon \mathsf{ob}(C) \to \mathsf{hom}(D)$ that satisfies

- for every $a \in \mathsf{ob}(C)$, $\eta(a)$ is a morphism from $F(a)$ to $G(a)$,
- for every $a, b \in \mathsf{ob}(C)$ and every $f \in C(a,b)$, we have that $\eta(b) \circ F(f) = G(f) \circ \eta(a)$, making commutative the diagram

$$\begin{array}{ccc} F(a) & \xrightarrow{\eta(a)} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(b) & \xrightarrow[\eta(b)]{} & G(b) \end{array} \tag{1.1}$$

Remark 1.12. If F and G are both contravariant functors, we require η to make the following diagram commutative, instead of diagram (1.1):

$$\begin{array}{ccc} F(a) & \xrightarrow{\eta(a)} & G(a) \\ {\scriptstyle F(f)}\uparrow & & \uparrow{\scriptstyle G(f)} \\ F(b) & \xrightarrow[\eta(b)]{} & G(b) \end{array}$$

We say that a natural transformation η is a *natural isomorphism* (or a *natural equivalence*, or a *isomorphism of functors*) if $\eta(a)$ is an isomorphism for every $a \in \mathsf{ob}(C)$.

1.3 Topology

Definition 1.13. Let X be a topological space.

An *open cover* of X is a family $\mathcal{F}$ of open subsets of X such that $X = \bigcup \mathcal{F}$. If $\mathcal{F}$ is an open cover of X and $\mathcal{F}' \subseteq \mathcal{F}$ satisfies $X = \bigcup \mathcal{F}'$, we say that $\mathcal{F}'$ is a *subcover* of $\mathcal{F}$. We say that X is *compact* if every open cover of X admits a finite subcover. The space X is said to be *locally compact* if every $x \in X$ admits a compact neighborhood, i.e., if for every $x \in X$ there exist an open set $U \subseteq X$ and a compact set $K \subseteq X$ such that $x \in U \subseteq K$.

We say that X is *normal* if it is Hausdorff and if for every closed sets $C_1, C_2 \subseteq X$ one can find open sets $U_1, U_2 \subseteq X$ such that $C_i \subseteq U_i$, $i = 1, 2$, and $U_1 \cap U_2 = \emptyset$.

Theorem 1.14 (Tietze Extension Theorem)**.** *Let X be a normal topological space, Y be a closed subset of X and $f \colon Y \to \mathbb{C}^N$ be a continuous function. There exists continuous function $\tilde{f} \colon X \to \mathbb{C}^N$ such that $\tilde{f}(x) = f(x)$ for every $x \in Y$.*

Lemma 1.15 (Tube Lemma)**.** *Let X be compact, Y be an arbitrary topological space and $y \in Y$. If U is an open subset of $X \times Y$ such that $X \times \{y\} \subseteq U$, then there exists an open set $V \subseteq Y$ such that $X \times \{y\} \subseteq X \times V \subseteq U$.*

Proof. For each $x \in X$ there exist open sets $W_x \subseteq X$ and $V_x \subseteq Y$ such that $(x, y) \in W_x \times V_x \subseteq U$. By compactness, there exists a finite set $\{x_1, \cdots, x_n\}$ such that $X = \bigcup_{i=1}^n W_{x_i}$. Consider the open set $V \doteq \bigcap_{i=1}^n V_{x_i}$. We clearly have $y \in V$ so that $X \times \{y\} \subseteq X \times V$. If

$$(x, z) \in X \times V = \left(\bigcup_{i=1}^n W_{x_i} \right) \times \left(\bigcap_{i=1}^n V_{x_i} \right)$$

there exists $j \in \{1, \cdots, n\}$ such that $x \in W_{x_j}$ and one has $(x, z) \in W_{x_j} \times V_{x_j} \subseteq U$. This concludes the proof. □

Corollary 1.16. *Let X be compact, Y be an arbitrary topological space and $S \subseteq Y$. If U is an open subset of $X \times Y$ such that $X \times S \subseteq U$, then there exists an open set $V \subseteq Y$ such that $X \times S \subseteq X \times V \subseteq U$.*

Proof. For each $y \in S$ we have $X \times \{y\} \subseteq U$. By Tube Lemma, there exists an open set $V_y \subseteq Y$ such that $X \times \{y\} \subseteq X \times V_y \subseteq U$. The set $V \doteq \bigcup_{y \in S} V_y$ satisfies the desired conditions. □

Lemma 1.17. *Let X and Y be topological spaces, and denote by $\mathcal{B}_x$ a local basis at a point $x \in X$. Let $f \colon X \to Y$ be a continuous bijection. If $f(B)$ is a neighborhood of $f(x)$ in Y for every $B \in \mathcal{B}_x$ and for every $x \in X$, then f is a homeomorphism.*

Proof. Fix $x \in X$. It suffices to prove that f^{-1} is continuous at $f(x)$. Let V be a neighborhood of x in X and consider $B \in \mathcal{B}_x$ such that $B \subseteq V$. By hypothesis, there exists an open set $U \subseteq Y$ such that $f(x) \in U \subseteq f(B)$. Applying f^{-1}, we obtain $x \in f(U) \subseteq B \subseteq V$, and it follows that f^{-1} is continuous at $f(x)$, as desired. □

1.3.1 Quotient Topology

Let X be a topological space and $\sim$ be an equivalence relation in X. Denote by $[x]$ the equivalence class of an element $x \in X$, and let $\pi \colon X \to X/{\sim}$ be the quotient projection, $\pi(x) = [x]$. The *quotient topology* in $X/{\sim}$ is defined by

$$A \text{ is open in } X/\!\sim \iff \pi^{-1}(A) \text{ is open in } X\,,$$

and it is the finest topology that makes continuous the map π.

Proposition 1.18. *Let X and Y be topological spaces, $\sim$ be an equivalence relation in X and π be as above. Consider a continuous map $f\colon X \to Y$. If $f(x) = f(x')$ whenever $x \sim x'$, then there exists a continuous map $\tilde{f}\colon X/\!\sim \to Y$ such that $\tilde{f}([x]) = f(x)$ for every $x \in X$, making commutative the following diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\pi}\downarrow & \nearrow_{\tilde{f}} & \\ X/\!\sim & & \end{array}$$

Proposition 1.19. *Let X and Y be topological spaces, $\sim$ be an equivalence relation in X and π be as above. Suppose we have maps $f\colon X \to Y$ and $g\colon X/\!\sim \to Y$ such that $g([x]) = f(x)$ for every $x \in X$, that is, such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\pi}\downarrow & \nearrow_{g} & \\ X/\!\sim & & \end{array}$$

is commutative. Then f is continuous if and only if g is continuous.

1.4 Functional Analysis

Let V_1 and V_2 be normed vector spaces over $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.20. A linear operator $T\colon V_1 \to V_2$ is said to be *bounded* if there exists $C > 0$ such that $\|T(v)\| \leq C\,\|v\|$ for every $v \in V$. The set of bounded linear operators from V_1 to V_2 is denoted by $\mathcal{B}(V_1, V_2)$ and it is itself a normed space under the operator norm

$$\begin{aligned} \|T\| &\doteq \inf\{c > 0 : \|T(v)\| \leq c\,\|v\| \text{ for every } v \in V_1\} \\ &= \sup\{\|T(v)\| : v \in V_1 \text{ and } \|v\| = 1\}\,. \end{aligned}$$

Theorem 1.21 (Open Mapping Theorem)**.** *If V_1 and V_2 are Banach spaces and $T \in \mathcal{B}(V_1, V_2)$ is surjective, then T is an open map.*

Recalling that boundedness of a linear operator between normed spaces is equivalent to continuity, we have the following

Corollary 1.22. *Let V_1 and V_2 be Banach spaces. If $T \in \mathcal{B}(V_1, V_2)$ is a bijection, then T^{-1} belongs to $\mathcal{B}(V_2, V_1)$.*

Definition 1.23. A *normed algebra* is an algebra A over $\mathbb{R}$ or $\mathbb{C}$ equipped with a map $\|\cdot\|\colon A \to [0, \infty)$ that turns $(A, \|\cdot\|)$ into a normed vector space and that satisfies

$$\|ab\| \leq \|a\|\,\|b\| \qquad \text{for every } a, b \in A\,.$$

If the algebra A is unital, we also require that $\|1\| = 1$.

Besides, if $(A, \|\cdot\|)$ is a Banach space, we say that A is a *Banach algebra*.

Lemma 1.24. *Let A be a unital Banach algebra. If $a \in A$ is such that $\|1-a\| < 1$, then a is invertible and $a^{-1} = \sum_{n=0}^{\infty}(1-a)^n$.*

Proof. First note that $\sum_{n=0}^{\infty}(1-a)^n$ is absolutely convergent since $\|1-a\| < 1$. Completeness of A provides convergence of this series. Now we have

$$\begin{aligned} a\sum_{n=0}^{N}(1-a)^n &= (1-(1-a))\sum_{n=0}^{N}(1-a)^n \\ &= \sum_{n=0}^{N}(1-a)^n - \sum_{n=0}^{N}(1-a)^{n+1} \\ &= 1-(1-a)^{N+1} \longrightarrow 1\,, \end{aligned}$$

implying the equality $a\big(\sum_{n=0}^{\infty}(1-a)^n\big) = 1$. Similarly, $\big(\sum_{n=0}^{\infty}(1-a)^n\big)a = 1$. □

Proposition 1.25. *Let V_1 be a Banach space and V_2 be a normed vector space. The subset of linear isomorphisms in $\mathcal{B}(V_1, V_2)$ is open.*

Proof. Let $T \in \mathcal{B}(V_1, V_2)$ be an isomorphism. Let $S \in \mathcal{B}(V_1, V_2)$ be such that $\|T-S\| < 1/\|T^{-1}\|$. Then $\|id_{V_1} - T^{-1}S\| = \|T^{-1}(T-S)\| \leq \|T^{-1}\|\|T-S\| < 1$ and Lemma 1.24 gives that $T^{-1}S$ is invertible in the Banach algebra $\mathcal{B}(V_1, V_1)$ (since V_1 is Banach, so is $\mathcal{B}(V_1, V_1)$). Therefore $S = T(T^{-1}S)$ is an isomorphism in $\mathcal{B}(V_1, V_2)$. The arbitrariness of T implies the desired result. □

1.4.1 Spectral Theory

Let us deal with some spectral results. For references, see [Sch12].

Definition 1.26. Let H be a Hilbert space. The *spectrum* of a bounded operator $T \in \mathcal{B}(H, H)$ is the set

$$\sigma(T) \doteq \{\lambda \in \mathbb{C} : \lambda - T \in \mathcal{B}(H, H) \text{ is not invertible}\}\,.$$

It can be shown that $\sigma(T)$ is a nonempty compact subset of $\mathbb{C}$ (in fact, one shows that $\sigma(T)$ is contained in $\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$). Moreover, if T is a self-adjoint operator, we have that $\sigma(T)$ is entirely contained in $\mathbb{R}$.

Definition 1.27. Let H be a Hilbert space and $(\Omega, \mathcal{M})$ be a measurable space. A *spectral measure* on $\mathcal{M}$ is a map

$$E\colon \mathcal{M} \to \{P \in \mathcal{B}(H, H) : P^2 = P = P^*\}\,,$$

from the σ-algebra $\mathcal{M}$ into the set of orthogonal projections on H, satisfying

(i) $E(\Omega) = id_H$

(ii) $E\big(\bigcup_{n=1}^{\infty} M_n\big) = \sum_{n=1}^{\infty} E(M_n)$ for any sequence $\{M_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ such that $M_j \cap M_k = \emptyset$ whenever $j \neq k$ (this infinite sum is taken on the strong operator topology).

We can state the Spectral Theorem. For a clear proof of this theorem, see Chapter 5 of [Sch12].

Theorem 1.28 (Spectral Theorem for Bounded Self-Adjoint Operators)**.** *Let H be a Hilbert space and $T \in \mathcal{B}(H,H)$ be a self-adjoint operator. Let $a,b \in \mathbb{R}$ be such that $\sigma(T) \subseteq [a,b]$. There exists a unique spectral measure E_T on the Borel σ-algebra $B([a,b])$ such that*

$$T = \int_{[a,b]} \lambda \; \mathrm{d}E_T(\lambda)\,.$$

Let $T \in \mathcal{B}(H,H)$ be self-adjoint, $a,b \in \mathbb{R}$ be such that $\sigma(T) \subseteq [a,b]$ and E_T be the measure given in the Spectral Theorem. For a continuous function $f\colon [a,b] \to \mathbb{C}$ we define

$$f(T) \doteq \int_{[a,b]} f(\lambda) \; \mathrm{d}E_T(\lambda)\,.$$

Proposition 1.29 (Properties of the Continuous Functional Calculus)**.** *Let $T \in \mathcal{B}(H,H)$ be a self-adjoint operator, $a,b \in \mathbb{R}$ be such that $\sigma(T) \subseteq [a,b]$ and E_T be given by the Spectral Theorem. For continuous functions $f,g\colon [a,b] \to \mathbb{C}$, we have*

(*i*) $\|f(T)\| \leq \|f\|_\infty \doteq \sup\{\ |f(\lambda)|\ : \lambda \in [a,b]\}\,.$

(*ii*) $\overline{f}(T) = f(T)^*$. *In particular, $f(T)$ is self-adjoint if and only if $f(\lambda) = \overline{f(\lambda)}$ for every $\lambda \in \sigma(T)$.*

(*iii*) $(fg)(T) = f(T)g(T)\,.$

(*iv*) $p(T) = \sum_{k=0}^{d} \alpha_k T^k$ *for every polynomial function* $p(\lambda) = \sum_{k=0}^{d} \alpha_k \lambda^k\,.$

(*v*) *If $f(\lambda) \geq 0$ for every $\lambda \in \sigma(T)$, then $f(T)$ is positive*[2].

Let $\{T_n : n \geq 1\}$ be a sequence of bounded self-adjoint operators in H such that $T_n \longrightarrow T$ in $\mathcal{B}(H,H)$. Since $\|T_n\| \longrightarrow \|T\|$, the set $\{\|T_n\| : n \geq 1\} \cup \{\|T\|\}$ is bounded in $\mathbb{R}$, which allows us to choose $a,b \in \mathbb{R}$ such that $\sigma(T) \cup \bigcup_{n\geq 1} \sigma(T_n) \subseteq [a,b]$. Let $f\colon [a,b] \to \mathbb{C}$ be continuous. Let us prove that $f(T_n) \longrightarrow f(T)$ in $\mathcal{B}(H,H)$. Fix $\varepsilon > 0$. Weierstrass Approximation Theorem gives a polynomial function $p\colon [a,b] \to \mathbb{C}$ such that $\|f - p\|_\infty \leq \varepsilon/3$. Since $T_n \to T$, we have $T_n^k \to T^k$ for every $k \geq 0$, so that $p(T_n) \longrightarrow p(T)$. This gives $n_0 \in \mathbb{N}$ such that $\|p(T_n) - p(T)\| \leq \varepsilon/3$ whenever $n \geq n_0$. Thus, for $n \geq n_0$,

$$\begin{aligned} \|f(T_n) - f(T)\| &\leq \|f(T_n) - p(T_n)\| + \|p(T_n) - p(T)\| + \|p(T) - f(T)\| \\ &\leq \|f - p\|_\infty + \|p(T_n) - p(T)\| + \|p - f\|_\infty \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3\,, \end{aligned}$$

and we have established the following

Proposition 1.30. *Let $\{T_n : n \geq 1\}$ is a sequence of bounded self-adjoint operators in H such that $T_n \to T$ in $\mathcal{B}(H,H)$. Consider $a,b \in \mathbb{R}$ such that $\sigma(T) \cup \bigcup_{n\geq 1} \sigma(T_n) \subseteq [a,b]$. If $f\colon [a,b] \to \mathbb{C}$ is continuous, then $f(T_n) \to f(T)$ in $\mathcal{B}(H,H)$.*

Let T is a positive bounded self-adjoint operator in H. Letting f and g be the map $\lambda \longmapsto \lambda^{1/2}$

[2] A self-adjoint operator T in a Hilbert space H is called *positive* if $\langle T(v), v\rangle \geq 0$ for every $v \in H$. It can be shown that a positive operator T satisfies $\sigma(T) \subseteq [0,\infty)$.

and applying items *(iii)* and *(v)* of Proposition 1.29, we see that

$$T^{1/2} \doteq \int_{[0,b]} \lambda^{1/2} \, \mathrm{d}E_T(\lambda)$$

is a positive operator that satisfies $(T^{1/2})^2 = T$. It can be shown that this is the unique positive bounded operator satisfying such condition.

Corollary 1.31. *Let X be the set of positive bounded self-adjoint operators in H. The map $X \to X$, $T \longmapsto T^{1/2}$, is continuous.*

Proof. Let $T_n, T \in X$ with $T_n \longrightarrow T$. We can choose $b > 0$ such that $\sigma(T) \cup \bigcup_{n\geq 1} \sigma(T_n) \subseteq [0, b]$. Then $f\colon [0, b] \to \mathbb{R}$, given by $f(\lambda) = \lambda^{1/2}$, is a well defined continuous function. By Proposition 1.30, it follows that $T_n^{1/2} = f(T_n) \longrightarrow f(T) = T^{1/2}$, as desired. $\square$

Chapter 2

Topological K-Theory

K-theory is a *generalized cohomology theory* (in the sense of [Whi62]), for example) introduced by Grothendieck in his work on Algebraic Geometry. It was then transposed to topology by Hirzebruch and Atiyah as expounded in [Ati67], which is the approach we are concerned with.

In this chapter, we will see basic properties about *topological* K*-theory*, a functor from the category of compact Hausdorff topological spaces to the category of abelian groups. The elements needed to construct the K-group of a compact Hausdorff space are discussed in details.

2.1 Vector Bundles

The concept of a vector bundle formalizes the idea of a collection of vector spaces parameterized by a topological space in a continuous manner. We will focus on collections of finite dimensional complex vector spaces, although many of these results can be generalized to the case of real vector spaces or to the case of infinite dimensional spaces.

In this section, X denotes a topological space and our vector spaces are finite dimensional and complex.

2.1.1 Basics

Definition 2.1. A *family of vector spaces over* X is a pair (E, p), where E is a topological space and $p\colon E \to X$ is a surjective continuous map, such that

1. $E_x \doteq p^{-1}(x)$ has the structure of a vector space for every $x \in X$.
2. the vector space structures vary continuously with x, meaning that the scalar multiplication map

$$\mathbb{C} \times E \longrightarrow E, \qquad (\lambda, e) \longmapsto \lambda \cdot e$$

and the addition map

$$\big\{(e_1, e_2) \in E \times E : p(e_1) = p(e_2)\big\} \longrightarrow E, \qquad (e_1, e_2) \longmapsto e_1 + e_2$$

are continuous.

The map p is called *projection*, and each E_x is called a *fiber of* E *at* x or simply a *fiber*.

We often denote such a family by $p\colon E \to X$, or $E \to X$, or simply E.

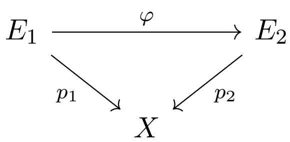
Example 2.2. Let N be a nonnegative integer and consider $X \times \mathbb{C}^N$ with the product topology. The coordinate projection $\pi_1 \colon X \times \mathbb{C}^N \to X$, given by $\pi_1(x, v) = x$, is a surjective continuous map such that, for every $x \in X$, the set $\pi_1^{-1}(x) = \{x\} \times \mathbb{C}^N$ has the structure of a vector space (isomorphic to $\mathbb{C}^N$). Besides, the operation maps

$$(\lambda; x, \xi) \longmapsto (x, \lambda\xi) \qquad \text{and} \qquad \big((x, \xi_1), (x, \xi_2)\big) \longmapsto (x, \xi_1 + \xi_2)$$

are continuous, meaning that the linear structures vary continuously with $x \in X$. This shows that $\pi_1 \colon X \times \mathbb{C}^N \to X$ is a family of vector spaces.

More generally (and for the same reasons), if V is a finite dimensional vector space[1], then $X \times V \to X$ is a family of vector spaces.

Definition 2.3. Let $p_1 \colon E_1 \to X$ and $p_2 \colon E_2 \to X$ be families of vector spaces. A *morphism of families of vector spaces* is a continuous map $\varphi \colon E_1 \to E_2$ which commutes with the projections (i.e., φ satisfies $p_2 \circ \varphi = p_1$, so that the diagram



is commutative) and $\varphi_x \doteq \varphi|_{(E_1)_x} \colon (E_1)_x \to (E_2)_x$ is a linear map for every $x \in X$.

We say that a morphism of families of vector spaces $\varphi \colon E_1 \to E_2$ is an *isomorphism* if it is a homeomorphism. In this case, E_1 is said to be *isomorphic* to E_2, and we write $E_1 \cong E_2$.

Remark 2.4. If $\varphi \colon E_1 \to E_2$ is a morphism of families of vector spaces over X, then $\varphi(E_1)$ is itself a family of vector spaces over X with topology and operations inherited from E_2, and whose projection is precisely the restriction of the projection of E_2 to $\varphi(E_1)$.

We can *"pull back"* families of vector spaces, as in the following

Proposition 2.5. *Let X and Y be topological spaces, and $f \colon Y \to X$ be a continuous map. If $p \colon E \to X$ is a family of vector spaces over X, then*

$$f^*E \doteq \{(y, e) \in Y \times E : f(y) = p(e)\}$$

is a family of vector spaces over Y with projection given by $f^*p \colon f^*E \to Y$, $f^*p(y, e) = y$. Moreover, if $f \colon Y \to X$ and $g \colon Z \to Y$ are continuous maps, then $(f \circ g)^*E \cong g^*f^*E$.

Proof. Note that f^*p is precisely the restriction to f^*E of the projection onto the first factor $Y \times E \to Y$, being therefore continuous. Also, it is surjective since for $y \in Y$ we can choose $e \in E$ with $p(e) = f(y)$ and notice that $f^*p(y, e) = y$.

For $y_0 \in Y$, we have

$$\begin{aligned} (f^*p)^{-1}(y_0) &= \{(y, e) \in Y \times E : f(y) = p(e), y = f^*p(y, e) = y_0\} \\ &= \{y_0\} \times \{e \in E : p(e) = f(y_0)\} \\ &= \{y_0\} \times p^{-1}(f(y_0)) = \{y_0\} \times E_{f(y_0)}\,. \end{aligned}$$

[1]Here V is assumed to be equipped with the unique Hausdorff topological vector space structure (which is linearly homeomorphic to $\mathbb{C}^k$, $k = \dim V$); for references, see [Rud91], Theorem 1.21

Besides, the operation maps

$$\mathbb{C} \times f^*E \to f^*E, \qquad (\lambda; y, e) \longmapsto (y, \lambda e)$$

and

$$\big\{\big((y,e),(y',e')\big) \in f^*E \times f^*E : y = y'\big\} \to f^*E, \qquad \big((y,e),(y,e')\big) \longmapsto (y, e+e')$$

are continuous by the continuity of operations in E. This shows that $f^*p\colon f^*E \to Y$ is a family of vector spaces.

Moreover,

$$(f \circ g)^*E = \{(z,e) \in Z \times E : (f \circ g)(z) = p(e)\}$$

and

$$\begin{aligned} g^*f^*E &= \{(z,e') \in Z \times f^*E : g(z) = f^*p(e')\} \\ &= \{(z,y,e) \in Z \times Y \times E : f(y) = p(e), g(z) = y\} \end{aligned}$$

are isomorphic since $\varphi\colon (f\circ g)^*E \to g^*f^*E$, given by $\varphi(z,e) = (z, g(z), e)$, is a bijective morphism with continuous inverse $\varphi^{-1}(z,y,e) = (z,e)$. □

Corollary 2.6. *Let $p\colon E \to X$ be a family of vector spaces over X and U a subspace of X. The* restriction of E to U, *defined by $E|_U \doteq p^{-1}(U)$, is a family of vector spaces over U.*

Proof. If $i\colon U \to X$ is the inclusion of U into X, then $i^*E = \{(x,e) \in U \times E : p(e) = x\} \cong p^{-1}(U)$, where such isomorphism can be given by $\varphi(x,e) = e$ whose inverse is $\varphi^{-1}(e) = (p(e), e)$. □

Definition 2.7. A family of vector spaces $E \to X$ is *trivial* if there exists a vector space V such that $E \cong X \times V$. A family of vector spaces $p\colon E \to X$ is said to be *locally trivial* if every $x \in X$ admits an open neighborhood U such that $E|_U$ is trivial. An isomorphism $E|_U \cong U \times V$ will be called a *local trivialization* for the family $E \to X$. A *vector bundle over* X is a locally trivial family of vector spaces over X.

If $\varphi\colon E_1 \to E_2$ is a morphism of families of vector spaces and if E_1 and E_2 are vector bundles over X, we shall call φ a *bundle morphism* (or *bundle map*). The definition of *bundle isomorphism* is similar.

It follows directly from the definition of "local triviality" that the dimension of the fibers of a vector bundle is locally constant: if $E \to X$ is vector bundle and if $x \in X$, the existence of the open neighborhood U of x such that $E|_U$ is trivial tells us that $\dim E_y = \dim E_x$ for every $y \in U$ simply because these vector spaces are isomorphic.

The *rank of a vector bundle $E \to X$ at a connected component of X* is the dimension of the fibers of E at any point of the component. If the rank of E at every connected component of X is the same, we call this integer number simply *rank of $E \to X$*.

Remark 2.8. Let $E \to X$ be a vector bundle and let $\varphi_i\colon E|_{U_i} \to U_i \times \mathbb{C}^N$, $i = 1,2$, be local trivializations. The *transition map* $\varphi_{12} \doteq \varphi_2 \circ \varphi_1^{-1}\colon (U_1 \cap U_2) \times \mathbb{C}^N \to (U_1 \cap U_2) \times \mathbb{C}^N$ is of the form $(x,v) \longmapsto (x, g_{12}(x)v)$ for a unique $g_{12}(x) \in GL(\mathbb{C}^N)$. This gives a map $g_{12}\colon U_1 \cap U_2 \to GL(\mathbb{C}^N)$ that must be continuous by the continuity of φ_{12}. We will explore this fact in Section 2.1.3.

Example 2.9. 1. The family of vector spaces $X \times V \to X$ is a vector bundle.

2. The pullback object, described in Proposition 2.5, remains a vector bundle if the *"pulled-back"* object is a vector bundle. Indeed, let $f\colon Y \to X$ be a continuous map and $E \to X$ be a vector bundle. For $y \in Y$, consider an open neighborhood U of $f(y)$ in X such that $E|_U$ is trivial. Let $\varphi\colon E|_U \to U \times V$ be an isomorphism. Then $f^{-1}(U)$ is an open neighborhood of y in Y and $\psi\colon f^*E|_{f^{-1}(U)} \to f^{-1}(U) \times V$, given by $\psi(z, e) = (z, \pi_2 \circ \varphi(e))$, is an isomorphism, whose inverse is precisely $\psi^{-1}(z, v) = (z, \varphi^{-1}(f(z), v))$.

Proposition 2.10. *Let $p\colon E \to X$ be a vector bundle. If X is Hausdorff, so is E.*

Proof. Let $v_1, v_2 \in E$, $v_1 \neq v_2$. If $p(v_1) \neq p(v_2)$, there exist disjoint open subsets $U_1, U_2 \subseteq X$ such that $p(v_i) \in U_i$, and therefore $V_i \doteq p^{-1}(U_i)$, $i = 1, 2$, are disjoint open subsets of E with $v_i \in V_i$. Suppose $p(v_1) = p(v_2) \doteq x$. There exist $N \in \mathbb{N}$, an open neighborhood U of x in X and a bundle isomorphism $\varphi\colon E|_U \to U \times \mathbb{C}^N$. Write $\varphi(v_i) = (x, \xi_i)$ and notice that $\xi_1 \neq \xi_2 \in \mathbb{C}^N$, from where we get disjoint open subsets $\Omega_1, \Omega_2 \subseteq \mathbb{C}^N$ such that $\xi_i \in \Omega_i$. Then $V_i \doteq \varphi^{-1}(U \times \Omega_i)$ are disjoint open subsets of E such that $v_i \in V_i$, $i = 1, 2$. □

Proposition 2.10 does not hold for general families of vector spaces, as one can see in the following

Example 2.11. Let V be the vector space $\mathbb{C}^k$ equipped with the trivial topology $\{\emptyset, \mathbb{C}^k\}$. For a Hausdorff topological space X, let E be the product space $X \times V$, which is not Hausdorff. The projection onto the first coordinate $\pi_1\colon E \to X$, $\pi_1(x, v) = x$, is a continuous surjective map such that $\pi_1^{-1}(x) = \{x\} \times V$ has the structure of a vector space for every $x \in X$. One can easily see that the operation maps

$$\mathbb{C} \times E \longrightarrow E, \qquad (\lambda, (x, v)) \longmapsto (x, \lambda v)$$

and

$$\{((x, v), (x, v')) : x \in X, v, v' \in V\} \longrightarrow E, \qquad ((x, v), (x, v')) \longmapsto (x, v + v')$$

are continuous, so that $\pi_1\colon E \to X$ is a family of vector spaces.

The following result shows that a bijective bundle morphism is a bundle isomorphism, proving that the continuity of the inverse morphism is automatic in the context of vector bundles.

Proposition 2.12. *Let $E_1 \to X$ and $E_2 \to X$ be vector bundles. A bundle morphism $\varphi\colon E_1 \to E_2$ is an isomorphism if and only if $\varphi_x\colon (E_1)_x \to (E_2)_x$ is a vector space isomorphism for each $x \in X$.*

Proof. If φ is an isomorphism, it is clear that φ_x is a linear isomorphism for each given $x \in X$.

Conversely, suppose φ_x is a linear isomorphism for every $x \in X$. Then φ is a bijection and it remains to prove that φ^{-1} is continuous. Let U be an open subset of X such that $E_i|_U$ is trivial, $i = 1, 2$. Let $\psi_i\colon E_i|_U \to U \times \mathbb{C}^N$ be bundle isomorphisms. The map $\psi_2 \circ \varphi \circ \psi_1^{-1}\colon U \times \mathbb{C}^N \to U \times \mathbb{C}^N$ is of the form $(x, \xi) \longmapsto (x, f_x(\xi))$, where $x \longmapsto f_x$ is a continuous map $U \to GL(\mathbb{C}^N)$. Since $g \doteq \psi_1 \circ \varphi^{-1} \circ \psi_2^{-1}$ is given by $(x, \xi) \longmapsto (x, f_x^{-1}(\xi))$ and the inversion map $A \longmapsto A^{-1}$ is continuous in $GL(\mathbb{C}^N)$, it follows that g is continuous. This gives the continuity of φ^{-1} in $E_2|_U$. The arbitrariness of U provides the global continuity of φ^{-1}, as required. □

Definition 2.13. A *continuous section* of a family of vector spaces $p\colon E \to X$ is a continuous map $s\colon X \to E$ that satisfies $p \circ s = id_X$.

Remark 2.14. Observe that a section $s\colon X \to E$ of a family of vector spaces $p\colon E \to X$ is a homeomorphism from X to its image $s(X)$, since $p|_{s(X)} \circ s = id_X$ and $s \circ p|_{s(X)} = id_{s(X)}$. In particular, X is homeomorphic to a subspace of E whenever the family $E \to X$ admits a continuous section.

Unfortunately, it is not always true that a family of vector spaces admits a section, as we can see in the following

Example 2.15. Fix some nonnegative integer k and consider $E \doteq \big([0,1) \times \mathbb{C}^k\big) \cup \big([2,3] \times \mathbb{C}^k\big)$, $X \doteq [0,2]$ and $p\colon E \to X$ be given by

$$p(x,\xi) = \begin{cases} x, & \text{if } x \in [0,1) \\ x-1, & \text{if } x \in [2,3] \end{cases}$$

The operation maps

$$\mathbb{C} \times E \to E, \qquad (\lambda; x, \xi) \longmapsto (x, \lambda\xi)\,,$$

and

$$\big\{\big((x,\xi),(y,\eta)\big) \in E \times E : x = y\big\} \to E, \qquad \big((x,\xi),(x,\eta)\big) \longmapsto (x, \xi+\eta)\,,$$

are continuous. Then $p\colon E \to X$ is a family of vector spaces. If $s\colon X \to E$ is such that $p \circ s = id_X$, then necessarily s is of the form

$$s(x) = \begin{cases} (x, f(x)) & \text{if } x \in [0,1) \\ (x+1, g(x)) & \text{if } x \in [1,2] \end{cases}$$

for some $f\colon [0,1) \to \mathbb{C}^k$ and $g\colon [1,2] \to \mathbb{C}^k$, so that s cannot be a continuous map. Thus, $E \to X$ admits no continuous section.

Proposition 2.16. *If a family of vector spaces $p\colon E \to X$ is trivial, then there exists $N \in \mathbb{N}$ and sections $s_1, \cdots, s_N\colon X \to E$ such that $\{s_i(x)\}_{i=1}^N$ forms a basis for E_x for every $x \in X$.*

Proof. If E is trivial, there exists $N \in \mathbb{N}$ and a bundle isomorphism $\varphi\colon X \times \mathbb{C}^N \to E$. Let $\{\omega_i\}_{i=1}^N$ be a basis for $\mathbb{C}^N$. For $i = 1, \cdots, N$, consider $s_i\colon X \to E$ given by $s_i(x) = \varphi(x, \omega_i)$. The continuity of φ and

$$p \circ s_i(x) = p \circ \varphi(x, \omega_i) = x$$

gives that each s_i is a section of E. Since $\{\omega_i\}$ is a basis for $\mathbb{C}^N$ and since $\varphi_x\colon \mathbb{C}^N \to E_x$ is a linear isomorphism, we have that $\{s_i(x)\}$ is a basis for E_x for every $x \in X$. □

Let us proceed to prove a converse of the previous result.

Lemma 2.17. *Let $p\colon E \to X$ be a family of vector spaces, $Z \doteq \{0_x \in E_x : x \in X\}$ and $U \subseteq E$ be an open set such that $Z \subseteq U$. There exists an open set $V \subseteq E$ such that $Z \subseteq V$ and $tv \in U$ for every $(t,v) \in [0,1] \times V$.*

Proof. By the continuity of the linear operations in E, it follows that

$$W \doteq \{(t,v) \in [0,1] \times E : tv \in U\}$$

is an open subset of $[0,1] \times E$. Besides, $[0,1] \times Z \subseteq W$. By the Tube Lemma (Corollary 1.16), there exists an open set $V \subseteq E$ such that

$$[0,1] \times Z \subseteq [0,1] \times V \subseteq W\,.$$

The first inclusion implies $Z \subseteq V$ and the last one implies $tv \in U$ for every $(t, v) \in [0,1] \times V$, as desired. □

Suppose $p \colon E \to X$ is a family of vector spaces and that there exist $N \in \mathbb{N}$ and sections $s_1, \cdots, s_N \colon X \to E$ such that $\{s_i(x)\}_{i=1}^N$ is a basis for E_x for every $x \in X$. The map $L \colon X \times \mathbb{C}^N \to E$, defined by

$$L(x,\xi) = \sum_{i=1}^N \xi_i s_i(x)$$

is a morphism of families of vector spaces and satisfies $L(x, e_j) = s_j(x)$ for every $x \in X$, where $\{e_i\}$ is the standard basis for $\mathbb{C}^N$, so that L is a continuous bijection. Next, we show that L^{-1} is continuous.

Lemma 2.18. *Let $p \colon E \to X$, $\{s_1, \cdots, s_N\}$ and $L \colon X \times \mathbb{C}^N \to E$ be as above. Let $S \doteq \{\xi \in \mathbb{C}^N : |\xi| = 1\}$ be the unit sphere in $\mathbb{C}^N$. If E is Hausdorff and X is locally compact, then $L(X \times S)$ is closed in E.*

Proof. Let $v \in E \setminus L(X \times S)$, $x = p(v)$. Let V be a compact neighborhood of x in X. We have that $L(V \times S)$ is compact and, therefore, closed in E. There exists an open neighborhood W of v in E such that $W \cap L(V \times S) = \emptyset$. Then $W \cap p^{-1}(V)$ is an open neighborhood of v in E such that

$$W \cap p^{-1}(V) \cap L(X \times S) \subseteq W \cap L(V \times S) = \emptyset\,.$$

This concludes the proof. □

Corollary 2.19. *Let $p \colon E \to X$, $\{s_1, \cdots, s_N\}$ and L be as in Lemma 2.18. Let $S \doteq \{\xi \in \mathbb{C}^N : |\xi| = 1\}$ and $B \doteq \{\xi \in \mathbb{C}^N : |\xi| < 1\}$ be the unit sphere and the open unit ball in $\mathbb{C}^N$, respectively. If E is Hausdorff and X is locally compact, then there exists an open set $V \subseteq E$ such that*

$$Z \doteq \{0_x \in E_x : x \in X\} \subseteq V \subseteq L(X \times B)\,.$$

Proof. By Lemma 2.18, $U \doteq E \setminus L(X \times S)$ is open and $Z \subseteq U$. By Lemma 2.17, there exists an open set $V \subseteq E$ such that $Z \subseteq V$ and $tv \in U$ for every $(t, v) \in [0,1] \times V$. It remains to show that $V \subseteq L(X \times B)$. Let $v \in E \setminus L(X \times B) = L(X \times (\mathbb{C}^N \setminus B))$, say $v = L(x, \xi)$ for some $(x, \xi) \in X \times (\mathbb{C}^N \setminus B)$. If $v \in V$, then $tv = tL(x, \xi) = L(x, t\xi) \in U$ for every $t \in [0,1]$. Letting $t = 1/|\xi| \in [0,1]$, we would have

$$\frac{1}{|\xi|} v = L\left(x, \frac{\xi}{|\xi|}\right) \in U \cap L(X \times S)\,,$$

a contradiction. Thus, we must have $V \subseteq L(X \times B)$. □

Lemma 2.20. *Let $p\colon E \to X$ be a family of vector spaces, $s\colon X \to E$ be a section of E and $\lambda \in \mathbb{C} \setminus \{0\}$. The maps*

$$v \longmapsto \lambda v \qquad \text{and} \qquad v \longmapsto v + s(p(v))$$

are homeomorphisms $E \to E$.

Proof. The continuity of these maps follows from the fact that the linear structures of the fibers E_x vary continuously with x. Notice that their inverses

$$v \longmapsto \frac{1}{\lambda} v \qquad \text{and} \qquad v \longmapsto v - s(p(v))$$

are also continuous. □

We are now ready to prove the following converse of Proposition 2.16, whose proof I have learned from Daniel Tausk.

Proposition 2.21 (Daniel Tausk - Private communication)**.** *Let $p\colon E \to X$ be a family of vector spaces and assume that there exist $N \in \mathbb{N}$ and sections $s_1, \cdots, s_N\colon X \to E$ such that $\{s_i(x)\}_{i=1}^N$ is a basis for E_x for every $x \in X$. If E is Hausdorff and X is locally compact, then the continuous bijection $L\colon X \times \mathbb{C}^N \to E$,*

$$L(x, \xi) = \sum_{i=1}^{N} \xi_i s_i(x)\,,$$

is a homeomorphism. In particular, E is trivial.

Proof. Fix $(x, \xi) \in X \times \mathbb{C}^N$, $r > 0$ and V an open neighborhood of x in X. Due to Lemma 1.17, it suffices to prove that $L(V \times B(\xi, r))$ is a neighborhood of $L(x, \xi)$ in E, where $B(\xi, r) \doteq \{\eta \in \mathbb{C}^N : |\xi - \eta| < r\}$. Since $E \to X$ admits sections, we have that X is homeomorphic to a subspace of E (see Remark 2.14), so that X is Hausdorff. From the fact that every open subset of a locally compact Hausdorff space is itself locally compact, one obtain that V is locally compact. The restriction of p to $p^{-1}(V)$, $p\colon p^{-1}(V) \to V$, is a family of vector spaces over V. By Corollary 2.19, there exists an open set $W \subseteq p^{-1}(V)$ such that

$$L(V \times \{0\}) \subseteq W \subseteq L(V \times B(0, 1))\,.$$

Consider the section $s\colon X \to E$ given by $s(y) = L(y, \xi)$. By Lemma 2.20, $W' \doteq \{rw + s(p(w)) : w \in W\}$ is an open subset of E. For $y \in V$, since $L(y, 0) \in W$, we have

$$\begin{aligned} rL(y, 0) + s(p(L(y, 0))) &= L(y, 0) + s(y) \\ &= L(y, 0) + L(y, \xi) \\ &= L(y, \xi) \in W'\,. \end{aligned}$$

For $w \in W$, since $W \subseteq L(V \times B(0,1))$, there exists $(y, \eta) \in B(0,1)$ such that $L(y, \eta) = w$. Then

$$\begin{aligned} rw + s(p(w)) &= rL(y,\eta) + s(p(L(y,\eta))) \\ &= L(y, r\eta) + s(y) \\ &= L(y, r\eta) + L(y, \xi) \\ &= L(y, \xi + r\eta) \in L(V \times B(\xi, r)) \end{aligned}$$

Therefore

$$L(V \times \{\xi\}) \subseteq W' \subseteq L(V \times B(\xi, r)\,.$$

This proves that $L(X \times B(\xi, r))$ is a neighborhood of $L(x, \xi)$ in E, as desired. □

2.1.2 Subbundles

Propositions 2.16 and 2.21 give a useful tool for proving that a family of vector spaces is a vector bundle. Here is an example.

Proposition 2.22. *Let E and F be vector bundles over a locally compact Hausdorff space X, and let $\varphi\colon E \to F$ be a bundle morphism. If $x \longmapsto \dim \varphi(E_x)$ is locally constant, then $\varphi(E)$ is itself a vector bundle.*

Proof. By Proposition 2.10, we have that E is Hausdorff. In Remark 2.4 we have already seen that $\varphi(E)$ is a family of vector spaces over X. It remains to show it is locally trivial, so let $x_0 \in X$ be fixed. Since E is a vector bundle, there exists an open neighborhood U of x_0 such that $E|_U \cong U \times \mathbb{C}^N$ and Proposition 2.16 gives sections $s_1, \cdots, s_N\colon U \to E$ such that $\{s_i(x)\}_{i=1}^N$ is a basis for E_x for every $x \in U$. Then $\{\varphi \circ s_i(x_0)\}_{i=1}^N$ spans $\varphi(E_{x_0})$, and we can choose $1 \le i_1 < \cdots < i_k \le N$ such that $\{\varphi \circ s_{i_j}(x_0)\}_{j=1}^k$ is a basis for $\varphi(E_{x_0})$. By continuity, there exists an open neighborhood $V \subseteq U$ of x_0 such that $\{\varphi \circ s_{i_j}(x)\}_{j=1}^k$ is linear independent for all $x \in V$. Since $\dim \varphi(E_x)$ is locally constant, it turns out that $\{\varphi \circ s_{i_j}(x)\}_{j=1}^k$ is a basis for $\varphi(E_x)$ for every $x \in V$. By Proposition 2.21, it follows that $\varphi(E)|_V$ is trivial, as desired. □

Definition 2.23. A vector bundle $E \to X$ is said to be a (*vector*) *subbundle* of another vector bundle $E' \to X$ if there exists an injective bundle morphism $\sigma\colon E \to E'$.

Let $E' \to X$ be a vector bundle and $E \subseteq E'$ be such that $p|_E\colon E \to X$ is itself a vector bundle. Then the inclusion $i\colon E \to E'$ is an injective bundle morphism. On the other hand, let $\varphi\colon E \to E'$ be an injective bundle morphism. Since $\dim \varphi(E_x) = \dim E_x$ for every $x \in X$, we have that $\dim \varphi(E_x)$ is locally constant. By Proposition 2.22, $\varphi(E)$ is itself a vector bundle whenever X is locally compact Hausdorff, and it is isomorphic to E due to Proposition 2.12.

Therefore, provided that X is locally compact Hausdorff, the previous paragraph allows us to say that a subbundle of a vector bundle $E' \to X$ is simply a subspace $E \subseteq E'$ that is itself a vector bundle.

Lemma 2.24. *Let $N \in \mathbb{N}$, $E \to X$ be a rank N vector bundle, $V \subseteq X$ be an open set and $s_1, \cdots, s_k\colon V \to E$ such that $\{s_i(x)\}_{i=1}^k$ is linearly independent for each $x \in V$. There exist an open set $U \subseteq V$ and sections $s_{k+1}, \cdots, s_N\colon U \to E$ such that $\{s_i(x)\}_{i=1}^N$ is a basis for E_x for every $x \in U$.*

Proof. Let $x_0 \in V$, $W \subseteq V$ be an open neighborhood of x_0 and $\varphi\colon E|_W \to W \times \mathbb{C}^N$ a local trivialization. For $i \in \{1, \cdots, k\}$, let $v_i\colon W \to \mathbb{C}^N$ be given by $v_i(x) = \pi_2 \circ \varphi \circ s_i(x)$. Fix $v_{k+1}, \cdots, v_N \in \mathbb{C}^N$ such that $\{v_1(x_0), \cdots, v_k(x_0), v_{k+1}, \cdots, v_N\}$ is a basis for $\mathbb{C}^N$. Since $x \overset{f}{\longmapsto} \det[v_1(x), \cdots, v_k(x), v_{k+1}, \cdots, v_N]$ is continuous and $f(x_0) \neq 0$, there exists an open set $U \subseteq W$ such that $f(x) \neq 0$ whenever $x \in U$. Thus $\mathcal{B}_x \doteq \{v_1(x), \cdots, v_k(x), v_{k+1}, \cdots, v_N\}$ is a basis for $\mathbb{C}^N$ for each $x \in U$. For $j \in \{k+1, \cdots, N\}$, define $s_j\colon U \to E$ by $s_j(x) = \varphi^{-1}(x, v_j)$. Each s_j is a section and, for $x \in U$, we have that $\{s_i(x)\}_{i=1}^N$ is a basis for E_x since φ_x is a linear isomorphism that sends $\{s_i(x)\}$ to $\mathcal{B}_x$. □

Proposition 2.25. *Let $E \to X$ be a vector bundle of rank N. For each $x \in X$, suppose we have a k-dimensional linear subspace $F_x \subseteq E_x$. Then $F \doteq \bigsqcup_{x\in X} F_x$ is a subbundle of E if and only if every $x_0 \in X$ admits a neighborhood U and a local trivialization $\varphi\colon E|_U \to U \times \mathbb{C}^N$ such that $\varphi(F_x) = \{x\} \times (\mathbb{C}^k \oplus \{0\})$ for $x \in U$.*

Proof. Fix $x_0 \in X$. Consider U a neighborhood of x_0 and $\varphi\colon E|_U \to U \times \mathbb{C}^N$ a local trivialization such that $\varphi(F_x) = \{x\} \times (\mathbb{C}^k \oplus \{0\})$ for $x \in U$. Let $\pi\colon \mathbb{C}^N \to \mathbb{C}^k$ be the projection onto the first k coordinates, $\pi(v_1, \cdots, v_N) = (v_1, \cdots, v_k)$. The map $\psi\colon F|_U \to U \times \mathbb{C}^k$, given by $\psi \doteq (id_U \times \pi) \circ \varphi$, is a local trivialization for F. Since $x_0 \in X$ was arbitrary, we have proved that $F \to X$ is a vector bundle, begin therefore a subbundle of $E \to X$.

Conversely, assume $F \to X$ is a subbundle. For $x_0 \in X$, let V be a neighborhood of x_0 such that $F|_V \cong V \times \mathbb{C}^k$. Proposition 2.16 gives sections $s_1, \cdots, s_k\colon V \to F$ such that $\{s_i(x)\}_{i=1}^k$ is a basis for F_x $\forall x \in V$. By Lemma 2.24, there exists $U \subseteq V$ and sections $s_{k+1}, \cdots, s_N\colon U \to E$ such that $\{s_i(x)\}_{i=1}^N$ is a basis for E_x $\forall x \in U$. Let $\{e_i\}_{i=1}^N$ be the standard basis for $\mathbb{C}^N$ and $\{e_i^*\}_{i=1}^N$ its dual basis. Define $\chi\colon U \times \mathbb{C}^N \to E|_U$ by $\chi(x, v) = \sum_{i=1}^N e_i^*(v) s_i(x)$. Then χ is a bijective bundle morphism and Proposition 2.12 allows us to say that χ is an isomorphism. The map $\varphi \doteq \chi^{-1}$ is the desired local trivialization, since $\varphi(s_i(x)) = (x, e_i)$ for $x \in U$, $i \in \{1, \cdots, N\}$. □

Notice that X is an arbitrary topological space in the previous result, no hypothesis on compactness or Hausdorffness was necessary.

2.1.3 Transition Data

Let $p\colon E \to X$ be a vector bundle of rank N. By definition, there exist open subsets of X, $\{U_\alpha\}_{\alpha\in A}$, such that $\bigcup_{\alpha\in A} U_\alpha = X$, together with isomorphisms $\varphi_\alpha\colon E|_{U_\alpha} \to U_\alpha \times \mathbb{C}^N$. For each $\alpha, \beta \in A$, the composition

$$\varphi_{\alpha\beta} \doteq \varphi_\beta \circ \varphi_\alpha^{-1}\colon (U_\alpha \cap U_\beta) \times \mathbb{C}^N \to (U_\alpha \cap U_\beta) \times \mathbb{C}^N$$

is an isomophism, which must be of the form $(x, u) \longmapsto (x, g_{\alpha\beta}(x)u)$, for a unique linear bijection $g_{\alpha\beta}(x) \in GL(\mathbb{C}^N)$. This implies that the map $\varphi_{\alpha\beta}$ is determined by $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(\mathbb{C}^N)$ (and, of course, $g_{\alpha\beta}$ is determined by $\varphi_{\alpha\beta}$). From the continuity of $\varphi_{\alpha\beta}$ we obtain the continuity of $g_{\alpha\beta}$.

The collection

$$\{g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(\mathbb{C}^N)\}_{\alpha,\beta\in A}$$

is called *transition data* of the vector bundle $E \to X$. Note that such collection satisfies the *cocycle condition*

$$g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \qquad \forall x \in U_\alpha \cap U_\beta \cap U_\gamma \quad \forall \alpha, \beta, \gamma \in A. \tag{2.1}$$

It turns out that this condition describes vector bundles completely, in the sense of the following

Proposition 2.26. *Suppose we have an open cover $\{U_\alpha\}_{\alpha\in A}$ of a topological space X together with continuous maps $\{g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(\mathbb{C}^N)\}_{\alpha,\beta\in A}$ satisfying cocycle condition (2.1). Then there exists a vector bundle $E \to X$ of rank N whose transition data is precisely the given collection $\{g_{\alpha\beta}\}$.*

Moreover, if the transition data $\{g_{\alpha\beta}\}$ arises from a vector bundle $F \to X$, then the previous bundle E is isomorphic to F.

Proof. First note that condition (2.1) implies $g_{\alpha\alpha}(x)g_{\alpha\alpha}(x) = g_{\alpha\alpha}(x)$, so we have

$$g_{\alpha\alpha}(x) = id_{\mathbb{C}^N} \qquad \forall x \in U_\alpha \quad \forall \alpha \in A. \tag{2.2}$$

It also follows that $g_{\alpha\beta}(x)g_{\beta\alpha}(x) = g_{\alpha\alpha}(x) = id_{\mathbb{C}^N}$, and therefore

$$g_{\alpha\beta}(x)^{-1} = g_{\beta\alpha}(x) \qquad \forall x \in U_\alpha \cap U_\beta \quad \forall \alpha, \beta \in A\,. \tag{2.3}$$

Let $Z \doteq \bigsqcup_{\alpha\in A} U_\alpha \times \mathbb{C}^N$ equipped with the disjoint union topology. In Z, consider the following relation: for $(x,u) \in U_\alpha \times \mathbb{C}^N$ and $(y,v) \in U_\beta \times \mathbb{C}^N$,

$$(x,u) \sim (y,v) \iff \begin{cases} x = y \\ v = g_{\beta\alpha}(x)u \end{cases}$$

We have, for $(x,u) \in U_\alpha \times \mathbb{C}^N$, $(y,v) \in U_\beta \times \mathbb{C}^N$ and $(z,w) \in U_\gamma \times \mathbb{C}^N$,

- $(x,u) \sim (x,u)$ by (2.2),
- $(x,u) \sim (y,v) \implies (y,v) \sim (x,u)$ by (2.3),
- $\begin{cases} (x,u) \sim (y,v) \\ (y,v) \sim (z,w) \end{cases} \implies (x,u) \sim (z,w)$ by (2.1).

This shows that $\sim$ is an equivalence relation in Z and we are allowed to consider the quotient space $E \doteq Z/{\sim}$. Let $q\colon Z \to E$ be the quotient map, $q(x,u) = [x,u]_\alpha$ for $(x,u) \in U_\alpha \times \mathbb{C}^N$, and define $p\colon E \to X$ by $p[x,u]_\alpha = \pi_1(x,u) = x$ (note that p is well defined since $[x,u]_\alpha = [y,v]_\beta$ implies $x = y$). Next, we prove that $p\colon E \to X$ is the desired vector bundle.

For $\alpha, \beta \in A$, let $\varphi_{\alpha\beta}\colon (U_\alpha \cap U_\beta) \times \mathbb{C}^N \to (U_\alpha \cap U_\beta) \times \mathbb{C}^N$ be given by $\varphi_{\alpha\beta}(x,u) = (x, g_{\alpha\beta}(x)u)$. It is easy to see that $\varphi_{\alpha\beta}$ is a bijective bundle morphism and hence an isomorphism, by Proposition 2.12. If $W \subseteq U_\beta \times \mathbb{C}^N$ for some $\beta \in A$, then $q^{-1}(q(W)) = \bigsqcup_{\alpha\in A} \varphi_{\alpha\beta}(W \cap (U_\alpha \times \mathbb{C}^N))$. This, and the fact that each $\varphi_{\alpha\beta}$ is open, imply that q is an open map (to see that, just let W be an open set). For $\alpha \in A$, let $q_\alpha \doteq q|_{U_\alpha\times\mathbb{C}^N}$. Since q_α is an injective open map, it is a homeomorphism between $U_\alpha \times \mathbb{C}^N$ and $q(U_\alpha \times \mathbb{C}^N) = p^{-1}(U_\alpha) = E|_{U_\alpha}$, which is linear in the fibers and clearly commutes with projections. Thus,

$$\varphi_\alpha \doteq q_\alpha^{-1}\colon E|_{U_\alpha} \to U_\alpha \times \mathbb{C}^N$$

is a bundle isomorphism. This proves that $p\colon E \to X$ is a vector bundle.

Note that, for $(x,u) \in (U_\alpha \cap U_\beta) \times \mathbb{C}^N$,

$$\begin{aligned}\varphi_\beta \circ \varphi_\alpha^{-1}(x,u) &= \varphi_\beta[x,u]_\alpha \\ &= \varphi_\beta[x, g_{\alpha\beta}(x)u]_\beta \\ &= (x, g_{\alpha\beta}(x)u)\,,\end{aligned}$$

proving that the transition data of $p\colon E \to X$ is precisely $\{g_{\alpha\beta}\}$, as desired.

Now, let $p'\colon E' \to X$ be a vector bundle with transition data $\{g_{\alpha\beta}\}$. For $\alpha \in A$, consider $\varphi'_\alpha\colon p'^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^N$ an isomorphism such that $\varphi'_\beta \circ \varphi'^{-1}_\alpha$ are given by $(x,u) \longmapsto (x, g_{\alpha\beta}(x)u)$. Define $\psi\colon E' \to E$ by $\psi(\omega) = [\varphi'_\alpha(\omega)]_\alpha$ whenever $p'(\omega) \in U_\alpha$ $\big(\psi$ depends only on ω and not on α, since $[\varphi'_\alpha(\omega)]_\alpha = [\varphi_{\alpha\beta}(\varphi'_\alpha(\omega))]_\beta = [\varphi'_\beta(\omega)]_\beta$ for $p'(\omega) \in U_\alpha \cap U_\beta\big)$. Observe that, if $(x,u) \in U_\alpha \times \mathbb{C}^N$ for some $\alpha \in A$, then

$$\psi \circ \varphi'^{-1}_\alpha(x,u) = [\varphi'_\alpha \circ \varphi'^{-1}_\alpha(x,u)]_\alpha = [x,u]_\alpha\,,$$

so that the map $\rho_\alpha \doteq q_\alpha^{-1} \circ \psi \circ \varphi'^{-1}_\alpha\colon U_\alpha \times \mathbb{C}^N \to U_\alpha \times \mathbb{C}^N$ coincides with the identity. In particular, $q_\alpha \circ \rho_\alpha \circ \varphi'_\alpha$ is continuous. This proves that ψ is continuous. Also,

$$\begin{aligned}p \circ \psi(\omega) &= p[\varphi'_\alpha(\omega)]_\alpha \\ &= \pi_1 \circ \varphi'_\alpha(\omega) \\ &= p'(\omega)\,,\end{aligned}$$

and we conclude that ψ is a bundle morphism. Since q_α and φ'_α are isomorphisms, we have that ψ is a bijective bundle morphism and therefore an isomorphism, by Proposition 2.12. □

2.1.4 Operations with Vector Bundles

Just like in the case of vector spaces, there is a natural way to construct the direct sum of two vector bundles, called *Whitney sum.*

Proposition 2.27. *Let $p\colon E \to X$ and $q\colon F \to X$ be vector bundles. Consider*

$$E \oplus F \doteq \{(e,f) \in E \times F : p(e) = q(f)\}\,.$$

Then $p \oplus q\colon E \oplus F \to X$, given by $(p \oplus q)(e,f) \doteq p(e) = q(f)$, is a vector bundle with fiber at x isomorphic to $E_x \oplus F_x$.

Proof. Note that $p \oplus q$ is precisely the restriction of the composition $E \times F \xrightarrow{\pi_1} E \xrightarrow{p} X$ to the subspace $E \oplus F$, then it is continuous and surjective. Besides, for $x \in X$,

$$\begin{aligned}(E \oplus F)_x = (p \oplus q)^{-1}(x) &= \{(e,f) \in E \oplus F : p(e) = q(f) = x\} \\ &= p^{-1}(x) \times q^{-1}(x) \\ &\cong E_x \oplus F_x\,.\end{aligned}$$

To see local triviality at a fixed point x, let U be an open neighborhood of x and consider local trivializations

$$\varphi\colon E|_U \to U \times \mathbb{C}^m \qquad \text{and} \qquad \psi\colon F|_U \to U \times \mathbb{C}^n$$

and define $\chi\colon (E\oplus F)|_U \to U\times(\mathbb{C}^m\oplus\mathbb{C}^n)$ by

$$\chi(e,f)=\big(p(e);\varphi_{p(e)}(e),\psi_{q(f)}(f)\big)\,.$$

It is not hard to see that χ is continuous, fiberwise linear and bijective, with inverse $\chi^{-1}(x;u,v)=(\varphi_x^{-1}(u),\psi_x^{-1}(v))$. Thus, χ is an isomorphism and the local triviality of $E\oplus F\to X$ follows. □

Remark 2.28. It can be proved that the Whitney sum satisfies the same properties that the direct sum of vector spaces does. For instance, if E_1, E_2 and E_3 are vector bundles over X, one can easily see that $E_1\oplus E_2\cong E_2\oplus E_1$ and $(E_1\oplus E_2)\oplus E_3\cong E_1\oplus(E_2\oplus E_3)$. Besides, if $E\cong E'$ and $F\cong F'$ as vector bundles over X, it follows that $E\oplus F\cong E'\oplus F'$.

Proposition 2.29. *Let $E_1\to X$ and $E_2\to X$ be vector bundles and let $f\colon Y\to X$ be a continuous map. Then $f^*(E_1\oplus E_2)\cong f^*E_1\oplus f^*E_2$ as vector bundles over Y.*

Proof. It follows directly from the definitions. If p_i denotes the projection of the bundle E_i, we have

$$\begin{aligned}E_1\oplus E_2&=\{(e_1,e_2)\in E_1\times E_2 : p_1(e_1)=p_2(e_2)\},\\ f^*E_i&=\{(y,e_i)\in Y\times E_i : p_i(e_i)=f(y)\},\ (i=1,2)\\ f^*(E_1\oplus E_2)&=\{(y;e_1,e_2)\in Y\times(E_1\oplus E_2) : p_1(e_1)=f(y)=p_2(e_2)\}\end{aligned}$$

and

$$\begin{aligned}f^*E_1\oplus f^*E_2&=\{(u,v)\in f^*E_1\times f^*E_2 : f^*p_1(u)=f^*p_2(v)\}\\ &=\{(y,e_1;z,e_2)\in(Y\times E_1)\times(Y\times E_2) : p_1(e_1)=f(y),p_2(e_2)=f(z),y=z\}\\ &=\{(y,e_1;y,e_2)\in(Y\times E_1)\times(Y\times E_2) : p_1(e_1)=f(y)=p_2(e_2)\}\,.\end{aligned}$$

The map $f^*(E_1\oplus E_2)\ni(y;e_1,e_2)\longmapsto(y,e_1;y,e_2)\in f^*E_1\oplus f^*E_2$ is an isomorphism. □

Proposition 2.30. *Let X be a locally compact topological space. Let E_1, E_2 and E_3 be vector bundles over X. Assume that we have bundle morphisms $f\colon E_1\to E_2$ and $g\colon E_2\to E_3$ such that*

$$0\longrightarrow E_1\xrightarrow{\ f\ }E_2\xrightarrow{\ g\ }E_3\longrightarrow 0 \tag{2.4}$$

is a short exact sequence of vector bundles. If there exists a bundle morphism $s\colon E_2\to E_1$ such that $s\circ f=id_{E_1}$, then

(i) there exists a bundle morphism $s'\colon E_3\to E_2$ such that $g\circ s'=id_{E_3}$, and

(ii) $E_2\cong E_1\oplus E_3$.

Proof. Fix $x\in X$. Given $v\in(E_2)_x$, we have $v=(v-f\circ s(v))+f\circ s(v)$, where $v-f\circ s(v)\in\text{Ker }s_x$ and $f\circ s(v)\in f((E_1)_x)$. Thus $(E_2)_x=\text{Ker }s_x+f((E_1)_x)$. Besides, if $v\in\text{Ker }s_x\cap f((E_1)_x)$, there exists $u\in(E_1)_x$ such that $f(u)=v$, and we have $0=s(v)=s\circ f(u)=u$, so that $v=f(u)=f(0)=0$. This proves that $(E_2)_x=\text{Ker }s_x\oplus f((E_1)_x)$. In particular, since f is injective, we have that $x\longmapsto\dim\text{Ker }s_x$ is locally constant. Let us prove that Ker s is a subbundle of E_2. Consider the bundle morphism $\psi\colon E_2\to E_2$ given by $\psi=id_{E_2}-f\circ s$, $\psi(v)=v-f\circ s(v)$. Observe that, for $x\in X$, $\psi((E_2)_x)=\text{Ker }s_x$, so that $x\longmapsto\dim\psi((E_2)_x)$ is locally constant. By

Proposition 2.22, Ker $s = \psi(E_2)$ is a vector bundle over X, and therefore it is a subbundle of E_2. For every $x \in X$,

$$(E_2)_x = \text{Ker } s_x \oplus f((E_1)_x) = \text{Ker } s_x \oplus \text{Ker } g_x \tag{2.5}$$

by exactness of (2.4). Thus, g sends Ker s_x isomorphically onto $(E_3)_x$. Therefore, by Proposition 2.12, the restriction $g|_{\text{Ker } s} \colon \text{Ker } s \to E_3$ is a bundle isomorphism. The map $s' \doteq (g|_{\text{Ker } s})^{-1}$ satisfies (i). To see (ii), observe that $\varphi \colon E_1 \oplus E_3 \to E_2$, given by $\varphi(v_1, v_3) = f(v_1) + s'(v_3)$, is an isomorphism, again by Proposition 2.12 and (2.5). □

We can also construct quotient vector bundles.

Proposition 2.31. *Let $p \colon E \to X$ be a vector bundle and $E' \to X$ be a subbundle of E. There exist a vector bundle $E/E' \to X$ and a bundle map $\sigma \colon E \to E/E'$ such that*

(i) $(E/E')_x \cong E_x/E'_x$ *for every $x \in X$, and*

(ii) *for every vector bundle $E_1 \to X$ and every bundle map $\varphi \colon E \to E_1$, there exists a bundle map $\hat{\varphi} \colon E/E' \to E_1$ such that $\varphi = \hat{\varphi} \circ \sigma$.*

Proof. In E, consider the equivalence relation

$$u \sim v \iff \begin{cases} p(u) = p(v) \\ u - v \in E'_{p(u)} \end{cases}$$

Let $E/E' \doteq E/\!\sim$ and $q \colon E/E' \to X$ be given by $q([u]) = p(u)$ (it is well defined since $[u] = [v]$ implies $p(u) = p(v)$). Continuity and surjectivity of p imply the same properties for q. Also,

$$\begin{aligned} (E/E')_x = q^{-1}(x) &= \{[u] \in E/E' : q([u]) = x\} \\ &= \{[u] \in E/E' : p(u) = x\} \\ &\cong E_x/E'_x \end{aligned}$$

for every $x \in X$. Next we prove local triviality. Fix $x_0 \in X$. By Proposition 2.25, there exist an open neighborhood U of x and a local trivialization

$$\varphi \colon E|_U \to U \times \mathbb{C}^{N+k} = U \times (\mathbb{C}^N \oplus \mathbb{C}^k)$$

such that $\varphi(E'_x) = \{x\} \times (\mathbb{C}^N \oplus \{0\})$ for every $x \in U$. Let $\pi \colon \mathbb{C}^N \oplus \mathbb{C}^k \to \mathbb{C}^k$ be the standard projection. If $u, v \in E|_U$ satisfy $u \sim v$, then

$$(id_U \times \pi) \circ \varphi(u - v) = (id_U \times \pi)(p(u); z, 0) = (p(u), 0)$$

(for some $z \in \mathbb{C}^N$) implying $(id_U \times \pi) \circ \varphi(u) = (id_U \times \pi) \circ \varphi(v)$. Thus, we have a well defined continuous map $\psi \colon q^{-1}(U) \to U \times (\{0\} \oplus \mathbb{C}^k)$ given by $\psi([u]) = (id_U \times \pi) \circ \varphi(u)$. It is easy to check that ψ is bijective and ψ^{-1} is precisely $(x; 0, \omega) \longmapsto [\varphi^{-1}(x; 0, \omega)]$. It follows that ψ is an isomorphism, proving the local triviality of E/E'.

Let $\sigma \colon E \to E/E'$ be the quotient map $u \longmapsto [u]$, which is a bundle map by the very definition of q. By properties of quotient topology (described in Proposition 1.18), one establishes (ii). □

2.1.5 Vector Bundles over Compact Hausdorff Spaces

To study some nice properties that vector bundles over compact Hausdorff spaces satisfy, we begin with a few technical lemmas. In this section, X will always denote a compact Hausdorff topological space unless otherwise specified.

Lemma 2.32. *Let Y be a closed subset of X and $E, F \to X$ be vector bundles. If $\sigma\colon E|_Y \to F|_Y$ is a bundle morphism, there exists a bundle morphism $\tilde{\sigma}\colon E \to F$ such that $\tilde{\sigma}(e) = \sigma(e) \ \ \forall e \in E|_Y$.*

Proof. Arguing in each connected component, we can assume that X is connected. By compactness, there exists a finite open cover $\{U_1, \cdots, U_n\}$ of X and local trivializations $\varphi_k\colon E|_{\overline{U_k}} \to \overline{U_k} \times \mathbb{C}^N$ and $\psi_k\colon F|_{\overline{U_k}} \to \overline{U_k} \times \mathbb{C}^M$. Defining $\sigma_k\colon E|_{Y\cap\overline{U_k}} \to F|_{Y\cap\overline{U_k}}$ to be the restriction of σ to $E|_{Y\cap\overline{U_k}}$, we see that the map $\psi_k \circ \sigma_k \circ \varphi_k^{-1}\colon (Y \cap \overline{U_k}) \times \mathbb{C}^N \to (Y \cap \overline{U_k}) \times \mathbb{C}^M$ has the form $(x, v) \longmapsto \big(x, f_k(x)(v)\big)$ for some continuous map $f_k\colon Y \cap \overline{U_k} \to \mathcal{B}(\mathbb{C}^N, \mathbb{C}^M) \cong \mathbb{C}^{MN}$. By Tietze extension theorem[2] (since X is compact Hausdorff, so is $\overline{U_k}$), there exists a continuous map $\tilde{f}_k\colon \overline{U_k} \to \mathcal{B}(\mathbb{C}^N, \mathbb{C}^M) \cong \mathbb{C}^{MN}$ such that $\tilde{f}_k(x) = f_k(x) \ \ \forall x \in Y \cap \overline{U_k}$. Let $\sigma'_k\colon E|_{\overline{U_k}} \to F|_{\overline{U_k}}$ be the map induced by $\tilde{f}_k$ (σ'_k is a bundle morphism that extends σ_k) and consider $\{\lambda_k\}_{k=1}^n$ a partition of unity subordinate to $\{U_k\}_{k=1}^n$. If $p\colon E \to X$ is the bundle projection, define $\tilde{\sigma}_k\colon E \to F$ by

$$\tilde{\sigma}_k(e) = \begin{cases} \lambda_k(p(e))\sigma'_k(e) & \text{if } e \in p^{-1}(U_k), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tilde{\sigma} \doteq \sum_{k=1}^n \tilde{\sigma}_k\colon E \to F$. We clearly have that $\tilde{\sigma}$ is a bundle morphism and $\forall e \in E|_Y$

$$\begin{aligned} \tilde{\sigma}(e) &= \sum_{k=1}^n \tilde{\sigma}_k(e) \\ &= \sum_{\substack{j \\ e\in p^{-1}(U_j)}} \lambda_j(p(e))\sigma'_j(e) \\ &= \sum_{\substack{j \\ e\in p^{-1}(U_j)}} \lambda_j(p(e))\sigma_j(e) \\ &= \sum_{\substack{j \\ e\in p^{-1}(U_j)}} \lambda_j(p(e))\sigma(e) \\ &= \sigma(e)\,. \end{aligned}$$

□

Lemma 2.33. *Let $E \to X$ and $F \to X$ be vector bundles over an arbitrary topological space X, and let $\sigma\colon E \to F$ be a bundle morphism. Then $\mathcal{O}_\sigma \doteq \{x \in X \mid \sigma_x\colon E_x \to F_x \text{ is an isomorphism}\}$ is open.*

Proof. Let $x \in \mathcal{O}_\sigma$. There exists an open neighborhood U of x such that $E|_U \cong U \times \mathbb{C}^N \cong F|_U$. The restriction of σ to $E|_U$, $\sigma_U\colon E|_U \to F|_U$ induces a continuous map $f_U\colon U \to \mathcal{B}(\mathbb{C}^N, \mathbb{C}^N)$. Since $f_U(x)$ belongs to the open set $GL(\mathbb{C}^N)$, there exists an open set $V \subseteq U$ such that $x \in V$ and $f_U(V) \subseteq GL(\mathbb{C}^N)$. Then clearly $V \subseteq \mathcal{O}_\sigma$, concluding that $\mathcal{O}_\sigma$ is open. □

Lemma 2.34. *Let $p\colon E \to X \times [0,1]$ be a vector bundle and $i_t\colon X \to X \times [0,1]$ be given by $i_t(x) = (x,t)$. Then $i_0^*E \cong i_1^*E$.*

[2] Theorem 1.14

Proof. For $\tau \in [0,1]$ we have $E|_{X\times\{\tau\}} \cong i_\tau^* E$ ($\varphi(x,e) = e$ is an isomorphism). Define $E^\tau \doteq i_\tau^* E \times [0,1]$ and $p_\tau \colon E^\tau \to X \times [0,1]$, $p_\tau(x,e;t) = (x,t)$. Fix $(x,t) \in X \times [0,1]$ and let U be an open neighborhood of x such that $(i_\tau^* E)|_U \cong U \times \mathbb{C}^N$. Then $U \times [0,1]$ is an open neighborhood of (x,t) in $X \times [0,1]$ satisfying $p_\tau^{-1}(U \times [0,1]) = (i_\tau^* E)|_U \times [0,1] \cong (U \times [0,1]) \times \mathbb{C}^N$. This shows that $p_\tau \colon E^\tau \to X \times [0,1]$ is a vector bundle.

Using Proposition 2.12, we see that $\sigma \colon E|_{X\times\{\tau\}} \to E^\tau|_{X\times\{\tau\}}$, given by $\sigma(e) = (\pi_1 p(e), e, \tau)$, is a bundle isomorphism, where $\pi_1 \colon X \times [0,1] \to X$ is the projection onto the first coordinate. By Lemma 2.32, there exists a bundle morphism $\tilde\sigma \colon E \to E^\tau$ extending σ. Lemma 2.33 gives that the set

$$\mathcal{O}_{\tilde\sigma} = \{(x,t) \in X \times [0,1] \mid \tilde\sigma_{(x,t)} \colon E_{(x,t)} \to E^\tau_{(x,t)} \text{ is an isomorphism } \}$$

is open in $X \times [0,1]$. Besides, $X \times \{\tau\} \subseteq \mathcal{O}_{\tilde\sigma}$, since σ itself is an isomorphism. By the Tube Lemma[3], there exists an interval $I_\tau \ni \tau$ open in $[0,1]$ such that $X \times I_\tau \subseteq \mathcal{O}_{\tilde\sigma}$. It follows from Proposition 2.12 that $E|_{X\times I_\tau} \xrightarrow{\tilde\sigma} E^\tau|_{X\times I_\tau}$ is a bundle isomorphism.

Now, for $t \in I_\tau$, we have $i_t^* E \cong E|_{X\times\{t\}} \cong E^\tau|_{X\times\{t\}} \cong i_\tau^* E$ (here, the map $(x,e;t) \longmapsto (x,e)$ provides the latter isomorphism). Thus, $i_t^* E \cong i_\tau^* E$ for every $t \in I_\tau$.

Since $[0,1]$ is compact, we can write $[0,1] = I_0 \cup I_{\tau_1} \cup \cdots I_{\tau_n} \cup I_1$ and conclude that $i_0^* E \cong i_1^* E$, as desired. □

We are now ready to prove the following

Theorem 2.35. *Let X be a compact Hausdorff space, Y be any topological space and $E \to Y$ be a vector bundle. If $f, g \colon X \to Y$ are homotopic, then $f^*E \cong g^*E$.*

Proof. Let $H \colon X \times [0,1] \to Y$ be a homotopy between f and g. For $t \in [0,1]$, denote $i_t \colon X \to X \times [0,1]$, $i_t(x) = (x,t)$. We have $f = H \circ i_0$ and $g = H \circ i_1$. Then Proposition 2.5 and Lemma 2.34 give

$$f^*E = (H \circ i_0)^* E \cong i_0^*(H^*E) \cong i_1^*(H^*E) \cong (H \circ i_1)^* E = g^*E.$$ □

There is also a way to embed a vector bundle into a trivial one.

Proposition 2.36. *Every vector bundle over a compact Hausdorff space is a subbundle of a trivial vector bundle.*

Proof. Let $p \colon E \to X$ be a vector bundle. Arguing in each connected component, we can assume that X is connected. By compactness, there exists a finite open cover $\{U_k\}_{k=1}^n$ of X and local trivializations $\varphi_k \colon E|_{U_k} \to U_k \times \mathbb{C}^N$. Let $\{\lambda_k\}_{k=1}^n$ be a partition of unity subordinate to $\{U_k\}_{k=1}^n$. If $\pi_2 \colon X \times \mathbb{C}^N \to \mathbb{C}^N$ denotes the standard projection, consider $\Phi_k \colon E \to \mathbb{C}^N$ given by

$$\Phi_k(e) = \begin{cases} \lambda_k(p(e))\pi_2(\varphi_k(e)) & \text{if } e \in p^{-1}(U_k), \\ 0 & \text{otherwise.} \end{cases}$$

and define $\Phi \colon E \to (\mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N)$ by $\Phi \doteq \Phi_1 \oplus \cdots \oplus \Phi_n$, that is,

$$\Phi(e) \doteq (\Phi_1(e), \cdots, \Phi_n(e)), \quad e \in E.$$

[3] Lemma 1.15

Let $\sigma\colon E \to X \times (\mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N)$ be given by $\sigma(e) = (p(e), \Phi(e))$. By construction, σ is a bundle morphism. To see that σ is injective, it suffices to prove that Φ is injective. For every $x \in X$, consider $\Phi_x \doteq \Phi|_{E_x}\colon E_x \to (\mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N)$. It suffices to prove that Φ_x is injective. If $e \in \mathrm{Ker}\ (\Phi_x)$, then $0 = \Phi_k(e) = \lambda_k(p(e))\pi_2(\varphi_k(e))$ for all $k \in \{1, \cdots, n\}$. If k is such that $\lambda_k(p(e)) \neq 0$, and hence $p(e) \in U_k$, then $\pi_2(\varphi_k(e)) = 0$ and we have $e = 0$ since φ_k is an isomorphism. □

The following technical Lemma is useful when dealing with continuity of orthogonal projections. It will be used right below and in Chapter 3.

Lemma 2.37. *Let X be an arbitrary topological space and $E \to X$ be a vector subbundle of $X \times \mathbb{C}^N$. For $x \in X$, let $P_x \in \mathcal{B}(\mathbb{C}^N, \mathbb{C}^N)$ be the orthogonal projection onto E_x. The map $P\colon X \to \mathcal{B}(\mathbb{C}^N, \mathbb{C}^N)$, $x \longmapsto P_x$, is continuous.*

Proof. By Proposition 2.25, for $y \in X$ there exists an open neighborhood $U \subseteq X$ of y and a bundle isomorphism $\varphi\colon U \times \mathbb{C}^N \to U \times C^N$ such that $\varphi(E_x) = \{x\} \times (\mathbb{C}^k \oplus \{0\})$ for every $x \in U$. Let $\psi\colon U \times \mathbb{C}^k \to U \times \mathbb{C}^N$ be given by $\psi(x, \xi) = \varphi^{-1}(x; \xi, 0)$. There exists a continuous map $A\colon X \to \mathcal{B}(\mathbb{C}^k, \mathbb{C}^N)$, $x \longmapsto A_x$, such that $\psi(x, \xi) = (x, A_x\xi)$. For $x \in X$, $A_x\colon \mathbb{C}^k \to E_x$ is a linear isomorphism. For each $x \in X$, $A_x^*A_x \in \mathcal{B}(\mathbb{C}^k, \mathbb{C}^k)$ is a positive self-adjoint operator such that $\mathrm{Ker}\ (A_x^*A_x) \subseteq \mathrm{Ker}\ A_x = \{0\}$ (since $\langle A_x^*A_x\xi, \xi\rangle = \|A\xi\|^2$), so that $A_x^*A_x$ is invertible. Consider the map $R \doteq A(A^*A)^{-1/2}\colon X \to \mathcal{B}(\mathbb{C}^k, \mathbb{C}^N)$, which is continuous because taking the square root of a linear operator is a continuous map[4]. We have $(RR^*)^2 = RR^* = (RR^*)^*$ so that $R_xR_x^*$ is the orthogonal projection of $\mathbb{C}^N$ onto $R_x(\mathbb{C}^k) = A_x(\mathbb{C}^k) = E_x$. Thus $P = RR^*$, which concludes the proof. □

Proposition 2.38. *Let $E \to X$ be a vector bundle. There exists a vector bundle $E^\perp \to X$ such that $E \oplus E^\perp$ is trivial.*

Proof. By Proposition 2.36, $E \to X$ is a subbundle of $X \times \mathbb{C}^N$ for some $N \in \mathbb{N}$. Thus, E can be seen as a topological subspace of $X \times \mathbb{C}^N$. Seeing each fiber E_x as a linear subspace of $\mathbb{C}^N$, we can consider

$$E^\perp \doteq \bigsqcup_{x \in X} (E_x)^\perp \subseteq X \times \mathbb{C}^N .$$

First, let us see that $E^\perp$ is a subbundle of $X \times \mathbb{C}^N$. For every $x \in X$, let $P_x \in \mathcal{B}(\mathbb{C}^N, \mathbb{C}^N)$ be the orthogonal projection onto E_x. By Lemma 2.37, the map $x \longmapsto P_x$ is continuous. We have then a bundle morphism $\varphi\colon X \times \mathbb{C}^N \to X \times \mathbb{C}^N$ given by $\varphi(x, \xi) = (x, \xi - P_x(\xi))$. Observe that $\varphi(\{x\} \times \mathbb{C}^N) = (E_x)^\perp$ for every $x \in X$, so that $\varphi(X \times \mathbb{C}^N) = E^\perp$. Since $x \longmapsto \dim E_x$ is locally constant, it follows that $x \longmapsto \dim \varphi(\{x\} \times \mathbb{C}^N)$ is locally constant, and we can apply Proposition 2.22 to conclude that $E^\perp$ is a vector bundle.

Notice that $\psi\colon X \times \mathbb{C}^N \to E \oplus E^\perp$, given by $\psi(x, \xi) = \big((x, P_x(\xi)); (x, \xi - P_x(\xi))\big)$ is a bundle isomorphism, concluding the proof. □

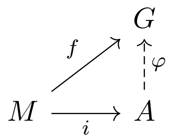
2.2 K-Theory

The construction of topological K-theory relies on the concept of Grothendieck completion of a commutative monoid (see Section 1.1), a notion that generalizes the way we obtain $\mathbb{Z}$ from $\mathbb{N}$.

[4]See Corollary 1.31.

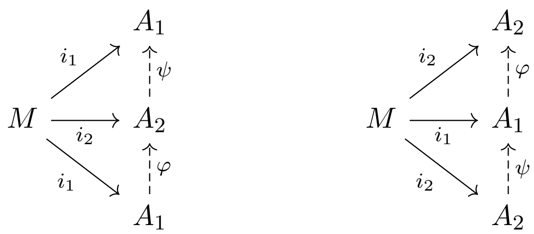
2.2.1 The Functor $\mathcal{G}$

A *Grothendieck completion* of a commutative monoid M is a pair (A, i), where A is an abelian group and $i \colon M \to A$ is a monoid morphism, satisfying the universal property: for every abelian group G and every monoid morphism $f \colon M \to G$ there exists a unique group morphism $\varphi \colon A \to G$ such that $f = \varphi \circ i$, that is, such that the diagram



is commutative.

As usual, this universal property gives uniqueness up to isomorphism of such object. For let M be a commutative monoid and let $(A_1.i_1)$ and (A_2, i_2) be Grothendieck completions of M. We have group morphisms $\varphi \colon A_1 \to A_2$ and $\psi \colon A_2 \to A_1$ that commutes the diagrams



implying $\psi \circ \varphi \circ i_1 = i_1$ and $\varphi \circ \psi \circ i_2 = i_2$. Since $id_{A_j} \colon A_j \to A_j$, $j = 1, 2$, also satisfies $id_{A_j} \circ i_j = i_j$, we conclude that $\psi \circ \varphi = id_{A_1}$ and $\varphi \circ \psi = id_{A_2}$. Thus, φ is an isomorphism between the groups A_1 and A_2 which satisfies $i_2 = \varphi \circ i_1$.

With that said, we can write "the" Grothendieck completion of a commutative monoid M, and we will denote it by $(\mathcal{G}(M), i_M)$.

Theorem 2.39. *Every commutative monoid admits a Grothendieck completion.*

Proof. Let M be a commutative monoid. In $M \times M$, consider the relation

$$(a, b) \sim (a', b') \iff \text{there exists } c \in M \text{ such that } a + b' + c = a' + b + c\,.$$

This is an equivalence relation since

- $(a, b) \sim (a, b)$ as $a + b + 0 = a + b + 0$,
- $(a, b) \sim (a', b') \implies a + b' + c = a' + b + c \implies a' + b + c = a + b' + c \implies (a', b') \sim (a, b)$,
- $\begin{cases} (a, b) \sim (a', b') \\ (a', b') \sim (a'', b'') \end{cases} \implies \begin{cases} a + b' + c = a' + b + c \\ a' + b'' + d = a'' + b' + d \end{cases} \implies a + b'' + z = a'' + b + z$, where

  $z = a' + b' + c + d$, and it follows that $(a, b) \sim (a'', b'')$.

Let $\mathcal{G}(M) \doteq M \times M/\sim$ and denote $[(a, b)]$ the equivalence class of (a, b). By the very definition of the equivalence relation, the operation $+ \colon \mathcal{G}(M) \times \mathcal{G}(M) \to \mathcal{G}(M)$ given by $[(a, b)] + [(a', b')] = [(a + a', b + b')]$ is well defined. Associativity and commutativity of $+$ follow from similar properties of the operation of M. Note that, for every $a, b \in M$,

- $[(a,b)] + [(0,0)] = [(a+0, b+0)] = [(a,b)]$,
- $[(a,b)] + [(b,a)] = [(a+b, a+b)] = [(0,0)]$ since $(a+b)+0+0 = 0+(a+b)+0$.

Thus $\mathcal{G}(M)$ is an abelian group with $[(0,0)]$ as neutral element and $-[(a,b)] = [(b,a)]$ for $a,b \in M$. Let $i_M\colon M \to \mathcal{G}(M)$ be given by $i_M(a) = [(a,0)]$. Since $i_M(0) = [(0,0)]$ and $\forall a,b \in M$

$$i_M(a+b) = [(a+b,0)] = [(a+b,0+0)] = [(a,0)] + [(b,0)] = i_M(a) + i_M(b),$$

we have that i_M is a monoid morphism. Now let G be an abelian group and $f\colon M \to G$ be a monoid morphism. Observe that, if $[(a,b)] = [(a',b')]$, there exists $c \in M$ such that $a+b'+c = a'+b+c$ and therefore $f(a)-f(b) = f(a')-f(b')$ in G. Then the map $\varphi\colon \mathcal{G}(M) \to G$ given by $\varphi([(a,b)]) = f(a)-f(b)$ is well defined. Note that $\varphi([(0,0)]) = f(0)-f(0) = 0$ and for every $a,a',b,b' \in M$

- $\varphi([(a,b)]+[(a',b')]) = \varphi([(a+a',b+b')]) = f(a+a')-f(b+b') = f(a)+f(a')-f(b)-f(b') = f(a)-f(b)+f(a')-f(b') = \varphi([(a,b)]) + \varphi([(a',b')])$
- $\varphi(-[(a,b)]) = \varphi([(b,a)]) = f(b)-f(a) = -(f(a)-f(b)) = -\varphi([(a,b)])$
- $\varphi(i_M(a)) = \varphi([(a,0)]) = f(a)-f(0) = f(a)$

proving that φ is a group morphism satisfying $\varphi \circ i_M = f$. Since $i_M(M)$ generates the group $\mathcal{G}(M)$, we have that φ is the only map with such properties. □

Example 2.40. The Grothendieck completion of $\mathbb{N}$ is isomorphic to $\mathbb{Z}$. An obvious isomorphism is $\varphi\colon \mathcal{G}(\mathbb{N}) \to \mathbb{Z}$, given by $\varphi([(n_+,n_-)]) = n_+ - n_-$.

Example 2.41. The Grothendieck completion of $(\mathbb{Z}^\times,\cdot)$ is $(\mathbb{Q}^\times,\cdot)$. The map $\psi\colon \mathcal{G}(\mathbb{Z}^\times) \to \mathbb{Q}^\times$, given by $\psi([(a,b)]) = a/b$, is easily seen to be a group isomorphism.

Proposition 2.42. *Let M_1 and M_2 be commutative monoids. If $f\colon M_1 \to M_2$ is a monoid morphism, there exists a unique group morphism $\mathcal{G}(f)\colon \mathcal{G}(M_1) \to \mathcal{G}(M_2)$ such that*

$$\begin{array}{ccc} \mathcal{G}(M_1) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(M_2) \\ {\scriptstyle i_{M_1}}\uparrow & & \uparrow{\scriptstyle i_{M_2}} \\ M_1 & \xrightarrow[f]{} & M_2 \end{array}$$

is a commutative diagram. Moreover, if M_3 is a commutative monoid and $g\colon M_2 \to M_3$ is a monoid morphism, we have $\mathcal{G}(g\circ f) = \mathcal{G}(g)\circ\mathcal{G}(f)$. Besides, $\mathcal{G}(id_M) = id_{\mathcal{G}(M)}$ for every commutative monoid M.

Proof. Since $i_{M_2} \circ f\colon M_1 \to \mathcal{G}(M_2)$ is a monoid morphism, there exists a unique group morphism $\varphi\colon \mathcal{G}(M_1) \to \mathcal{G}(M_2)$ such that $i_{M_2} \circ f = \varphi \circ i_{M_1}$. Define $\mathcal{G}(f) \doteq \varphi$.

In case $M_1 = M_2 = M$, we have $i_M \circ id_M = id_{\mathcal{G}(M)} \circ i_M$, and uniqueness of $\mathcal{G}(id_M)$ implies $\mathcal{G}(id_M) = id_{\mathcal{G}(M)}$. Similarly, we have the commutative diagram

$$\begin{array}{ccccc} \mathcal{G}(M_1) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(M_2) & \xrightarrow{\mathcal{G}(g)} & \mathcal{G}(M_3) \\ {\scriptstyle i_{M_1}}\uparrow & & {\scriptstyle i_{M_2}}\uparrow & & \uparrow{\scriptstyle i_{M_3}} \\ M_1 & \xrightarrow[f]{} & M_2 & \xrightarrow[g]{} & M_3 \end{array}$$

implying that $\mathcal{G}(g) \circ \mathcal{G}(f) \circ i_{M_1} = \mathcal{G}(g) \circ i_{M_2} \circ f = i_{M_3} \circ g \circ f$. This, together with uniqueness of $\mathcal{G}(g \circ f)$, gives $\mathcal{G}(g \circ f) = \mathcal{G}(g) \circ \mathcal{G}(f)$. □

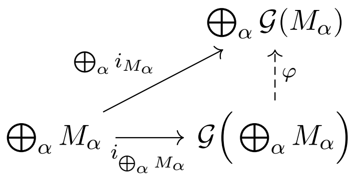
To sum up, Proposition 2.42 shows that $\mathcal{G}$ is a covariant functor from the category of commutative monoids into the category of abelian groups.

The following result is a consequence of the general fact of category theory that left adjoint functors respect limits as well as right adjont functors respect colimits (see [Mac98]). The functor $\mathcal{G}$ is the left adjoint of the *forgetful functor* from the category of abelian groups into the category of commutative monoids, that sends an abelian group to itself seen as a commutative monoid. We will give a more elementary proof though.

Proposition 2.43. *For $\{M_\alpha\}_\alpha$ a family of commutative monoids, it is true that*

$$\mathcal{G}\Big(\bigoplus_\alpha M_\alpha\Big) \cong \bigoplus_\alpha \mathcal{G}(M_\alpha)\,.$$

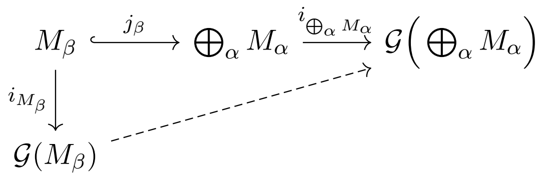
Proof. The universal property of $\mathcal{G}$ gives $\varphi\colon \mathcal{G}\Big(\bigoplus_\alpha M_\alpha\Big) \to \bigoplus_\alpha \mathcal{G}(M_\alpha)$ commuting the diagram



Let us prove that φ is an isomorphism. For each β, consider the natural inclusions

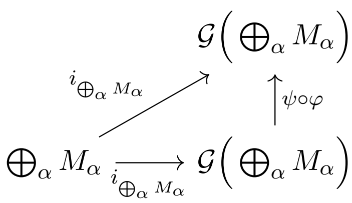
$$j_\beta\colon M_\beta \hookrightarrow \bigoplus_\alpha M_\alpha \qquad \text{and} \qquad k_\beta\colon \mathcal{G}(M_\beta) \hookrightarrow \bigoplus_\alpha \mathcal{G}(M_\alpha)\,.$$

The universal property of direct sum allows us to join the dashed arrows obtained in the commutative diagrams



to get a map $\psi\colon \bigoplus_\alpha \mathcal{G}(M_\alpha) \to \mathcal{G}\Big(\bigoplus_\alpha M_\alpha\Big)$ satisfying $\psi \circ k_\beta \circ i_{M_\beta} = \big(i_{\bigoplus_\alpha M_\alpha}\big) \circ j_\beta$ for all β.

Now, the diagram



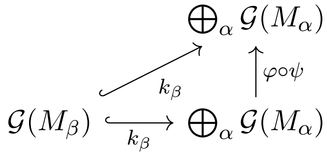
commutes since

$$\psi \circ \varphi \circ \big(i_{\bigoplus_\alpha M_\alpha}\big) \circ j_\beta = \psi \circ \Big(\bigoplus_\alpha i_{M_\alpha}\Big) \circ j_\beta = \psi \circ k_\beta \circ i_{M_\beta} = \big(i_{\bigoplus_\alpha M_\alpha}\big) \circ j_\beta$$

for all β. Notice that this diagram still commutes if we exchange $\psi \circ \varphi$ for $id_{\mathcal{G}\left(\bigoplus_\alpha M_\alpha\right)}$. Thus

$\psi \circ \varphi = id_{\mathcal{G}\left(\bigoplus_\alpha M_\alpha\right)}$.

Furthermore, the diagram



also commutes since

$$\varphi \circ \psi \circ k_\beta \circ i_{M_\beta} = \varphi \circ \left(i_{\bigoplus_\alpha M_\alpha}\right) \circ j_\beta = \left(\bigoplus_\alpha i_{M_\alpha}\right) \circ j_\beta = k_\beta \circ i_{M_\beta}$$

As before, this diagram remains commutative if we exchange $\varphi \circ \psi$ for $id_{\bigoplus_\alpha \mathcal{G}(M_\alpha)}$, and this allows us to conclude that $\varphi \circ \psi = id_{\bigoplus_\alpha \mathcal{G}(M_\alpha)}$.

This proves that φ is an isomorphism between $\mathcal{G}\left(\bigoplus_\alpha M_\alpha\right)$ and $\bigoplus_\alpha \mathcal{G}(M_\alpha)$. □

2.2.2 The Functor Vect

Now that we have constructed the Grothendieck completion of a commutative monoid and established some of its properties, we are ready to introduce the K-theory group of a compact Hausdorff topological space. For such a space X, denote by $\mathsf{Vect}\ X$ the set of all isomorphism classes of vector bundles over X. Formally, an element of $\mathsf{Vect}\ X$ is represented by a vector bundle $E \to X$:

$$[E] = \left\{F \mid F \to X \text{ is a vector bundle over } X \text{ and } F \cong E\right\}.$$

Using the Whitney sum of vector spaces, we can equip $\mathsf{Vect}\ X$ with the operation

$$[E] + [E'] \doteq [E \oplus E']$$

that, due to Remark 2.28, turns out to be well defined, associative and commutative. It obviously has $[X \times \{0\}]$ as neutral element. Thus, $\mathsf{Vect}\ X$ is a commutative monoid.

If $f\colon Y \to X$ is a continuous map between compact Hausdorff topological spaces, Proposition 2.5 and Example 2.9.2 say that f induces a map $f^*\colon \mathsf{Vect}\ X \to \mathsf{Vect}\ Y$ that sends $[E \to X]$ to $[f^*E \to Y]$. Indeed, if $E_1 \to X$ and $E_2 \to X$ are isomorphic vector bundles and if $\varphi\colon E_1 \to E_2$ is a bundle isomorphism, the map $\psi\colon f^*E_1 \to f^*E_2$, given by $\psi(y, e_1) = (y, \varphi(e_1))$, is an isomorphism. Due to Proposition 2.29, the induced map $f^*\colon \mathsf{Vect}\ X \to \mathsf{Vect}\ Y$ is a monoid morphism. Besides, if $g\colon Y \to X$ is another continuous map homotopic to f, then $f^* = g^*$ by Theorem 2.35. In particular, f^* is a monoid isomorphism between $\mathsf{Vect}\ X$ and $\mathsf{Vect}\ Y$ if f is a homotopy equivalence[5]. All of this discussion can be summarized in the following

Theorem 2.44. Vect *is a contravariant homotopy-invariant functor from the category of compact Hausdorff topological spaces into the category of commutative monoids.*

---

[5] A *homotopy equivalence* is a continuous map $f\colon Y \to X$ such that there exists a continuous map $g\colon X \to Y$ satisfying the property: $f \circ g$ and $g \circ f$ are homotopic to id_X and id_Y, respectively. If there exists a homotopy equivalence $f\colon Y \to X$, the spaces X and Y are said to be *homotopy equivalent* (or are said to have the same homotopy type).

Example 2.45. 1. If X is a point, a vector bundle over X is simply a vector space, so the isomorphism classes of vector bundles are completely characterized by the dimension of such space. Thus $\mathsf{Vect}\ X \cong \mathbb{N}$.

2. If X is contractible[6], the homotopy-invariance of the functor Vect implies immediately that $\mathsf{Vect}\ X \cong \mathsf{Vect}\ \{p\} \cong \mathbb{N}$ for every $p \in X$, so that every vector bundle over X is trivial.

2.2.3 The Functor K

We define the K-theory functor by composing the functor Vect with the functor $\mathcal{G}$.

Definition 2.46. The K-*theory group* (or K-*group*) of a compact Hausdorff topological space X, denoted by $\mathsf{K}(X)$ (or $\mathsf{K}^0(X)$), is defined to be

$$\mathsf{K}(X) \doteq \mathcal{G}(\mathsf{Vect}\ X)\,.$$

For simplicity, we will denote $[E] \in \mathsf{Vect}\ X$ and $i_{\mathsf{Vect}\ X}([E]) \in \mathsf{K}(X)$ by the same symbol $[E]$. So a typical element of $\mathsf{K}(X)$ can be written as $[E] - [F]$, where E and F are vector bundles over X.

If X and Y are compact Hausdorff spaces and $f\colon X \to Y$ is a continuous map, we have a monoid morphism $f^*\colon \mathsf{Vect}\, Y \to \mathsf{Vect}\, X$ and, by Proposition 2.42, a group morphism $\mathsf{K}(f) \doteq \mathcal{G}(f^*)\colon \mathsf{K}(Y) \to \mathsf{K}(X)$.

The main properties about K will be summarized as follows.

Theorem 2.47. *K is a contravariant homotopy-invariant functor from the category of compact Hausdorff topological spaces into the category of abelian groups.*

Proof. Composition of a contravariant functor with a covariant functor gives a contravariant functor. It remains to show the homotopy invariance of K. If X and Y are homotopy equivalent compact Hausdorff topological spaces, there exists a homotopy equivalence $f\colon X \to Y$. In the discussion before Theorem 2.44, $f^*\colon \mathsf{Vect}\ Y \to \mathsf{Vect}\ X$ is a monoid isomorphism. Functoriality of $\mathcal{G}$ implies that $\mathsf{K}(f) = \mathcal{G}(f^*)\colon \mathsf{K}(Y) \to \mathsf{K}(X)$ is a group isomorphism. □

Example 2.48. If X is a contractible space, we have $\mathsf{Vect}\ X \cong \mathbb{N}$ and therefore $\mathsf{K}(X) \cong \mathcal{G}(\mathbb{N}) \cong \mathbb{Z}$. In particular, the K-group of a point space is isomorphic to $\mathbb{Z}$.

Given a family of topological spaces $\{X_{\alpha\in A}\}$, we denote by $\bigsqcup_{\alpha\in A} X_\alpha$ its disjoint union equipped with its canonical topology.

If A is finite and each X_α is compact Hausdorff, the disjoint union $\bigsqcup_{\alpha\in A} X_\alpha$ is compact Hausdorff as well.

Proposition 2.49. *If $\{X_1, \cdots, X_n\}$ is a family of compact Hausdorff spaces, then*

$$\mathsf{K}\Big(\bigsqcup_{k=1}^{n} X_k\Big) \;\cong\; \bigoplus_{k=1}^{n} \mathsf{K}(X_k)\,.$$

[6] A topological space is *contractible* if it has the same homotopy type of $\{p\}$ for every $p \in X$.

Proof. Let $X \doteq \coprod_{k=1}^n X_k$. Observe that a vector bundle over X is just a choice of a vector bundle over each X_k, so that $\mathsf{Vect}\ X \cong \bigoplus_{k=1}^n \mathsf{Vect}\ X_k$. By Proposition 2.43, we have

$$\mathsf{K}(X) = \mathcal{G}(\mathsf{Vect}\ X) \cong \mathcal{G}\Big(\bigoplus_{k=1}^n \mathsf{Vect}\ X_k\Big) \cong \bigoplus_{k=1}^n \mathcal{G}(\mathsf{Vect}\ X_k) = \bigoplus_{k=1}^n \mathsf{K}(X_k)\,,$$

as desired. □

Proposition 2.50. *Let X be a compact Hausdorff space. Every element in $\mathsf{K}(X)$ can be written in the form $[E]-[X\times\mathbb{C}^N]$ for some vector bundle $E\to X$ and some $N\in\mathbb{N}$. Moreover, two elements $[E]-[X\times\mathbb{C}^N]$ and $[F]-[X\times\mathbb{C}^M]$ are equal in $\mathsf{K}(X)$ if and only if there exists $r\geq 0$ such that $E\oplus(X\times\mathbb{C}^{M+r})\cong F\oplus(X\times\mathbb{C}^{N+r})$.*

Proof. Let $[E_1]-[E_2]\in\mathsf{K}(X)$. Proposition 2.38 gives a vector bundle $E_2^\perp\to X$ such that $E_2\oplus E_2^\perp\cong X\times\mathbb{C}^N$ for some $N\in\mathbb{N}$. Then, in $\mathsf{K}(X)$, we have

$$[E_1]-[E_2] = ([E_1]-[E_2]) + ([E_2^\perp]-[E_2^\perp]) = [E_1\oplus E_2^\perp]-[E_2\oplus E_2^\perp] = [E]-[X\times\mathbb{C}^N]$$

where $E\doteq E_1\oplus E_2^\perp$.

Recall that, in $\mathsf{K}(X)$, we have

$$[E]-[X\times\mathbb{C}^N] = [F]-[X\times\mathbb{C}^M] \iff \begin{array}{c}\text{there exists a vector bundle } G\to X \text{ such that}\\ E\oplus(X\times\mathbb{C}^M)\oplus G\cong F\oplus(X\times\mathbb{C}^N)\oplus G\end{array} \tag{2.6}$$

Suppose there exists $r\geq 0$ as in the hypothesis. Letting $G\doteq X\times\mathbb{C}^r$, (2.6) implies that

$$[E]-[X\times\mathbb{C}^N] = [F]-[X\times\mathbb{C}^M] \qquad \text{in } \mathsf{K}(X)\,. \tag{2.7}$$

Conversely, suppose equation (2.7) holds. Then there exists $G\to X$ as in (2.6). Using again Proposition 2.38, there exists $r\geq 0$ such that $G\oplus G^\perp\cong X\times\mathbb{C}^r$. Thus

$$E\oplus(X\times\mathbb{C}^M)\oplus G\oplus G^\perp \cong F\oplus(X\times\mathbb{C}^N)\oplus G\oplus G^\perp\,,$$

which is precisely

$$E\oplus(X\times\mathbb{C}^{M+r})\cong F\oplus(X\times\mathbb{C}^{N+r})\,,$$

concluding the proof. □

Chapter 3

Families-Index and Atiyah-Jänich Theorem

In this chapter we are mainly interested in understand the elements of functional analysis behind the Atiyah-Jänich Theorem. We give details of what was done in [Ati67], [Muk13] and [Bre16].

3.1 Fredholm Operators

Let V_1 and V_2 be arbitrary vector spaces.

Definition 3.1. A linear map $T\colon V_1 \to V_2$ is said to be a *Fredholm operator* if $\operatorname{Ker} T \doteq T^{-1}(0)$ and $\operatorname{Coker} T \doteq V_2/T(V_1)$ are both finite dimensional. In case V_1 and V_2 are normed spaces, a Fredholm operator from V_1 into V_2 is assumed to be continuous unless otherwise specified.

The set of all Fredholm operators from V_1 into V_2 is denoted by $\mathcal{F}(V_1, V_2)$ (so, using the continuity convention, we have $\mathcal{F}(V_1, V_2) \subseteq \mathcal{B}(V_1, V_2)$ whenever V_1 and V_2 are normed spaces). The *Fredholm index* of a $T \in \mathcal{F}(V_1, V_2)$ is the integer

$$\operatorname{ind}(T) \doteq \dim \operatorname{Ker} T - \dim \operatorname{Coker} T.$$

Example 3.2. Let H be an infinite dimensional Hilbert space and let $\mathcal{B} = \{e_i\}_{i\in\mathbb{N}} \sqcup \{e_\alpha\}_{\alpha\in A}$ be a Hilbert basis for H. Let $k \geq 0$.

Consider the linear operator $S_k \doteq S_k^{(\mathcal{B})}\colon H \to H$ (that depends on $\mathcal{B}$) given by

$$S_k^{(\mathcal{B})}(e_j) = \begin{cases} e_{j-k} & \text{if } j \in \mathbb{N} \text{ and } j > k, \\ 0 & \text{if } j \in \mathbb{N} \text{ and } j \leq k, \\ e_j & \text{if } j \in A. \end{cases}$$

We have $\operatorname{Ker} S_k = \operatorname{span}\{e_1, \cdots, e_k\}$ and $S_k(H) = H$, so that $\dim \operatorname{Ker} S_k = k$ and $\dim \operatorname{Coker} S_k = 0$. Thus, S_k is a Fredholm operator and $\operatorname{ind}(S_k) = k$.

Besides, the linear map $S_{-k} \doteq S_{-k}^{(\mathcal{B})}\colon H \to H$, defined by

$$S_{-k}^{(\mathcal{B})}(e_j) = \begin{cases} e_{j+k} & \text{if } j \in \mathbb{N}, \\ e_j & \text{if } j \in A, \end{cases}$$

is also a Fredholm operator and satisfies $\operatorname{ind}(S_{-k}) = -k$ as well.

Lemma 3.3. *Let* $V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3$ *be an exact sequence of vector spaces and linear maps. If* V_1 *and* V_3 *are finite dimensional, so is* V_2.

Proof. By the Rank-Nullity Theorem, it follows that

$$\begin{aligned}\dim V_2 &= \dim \operatorname{Ker}\ f_2 + \dim f_2(V_2)\\ &= \dim f_1(V_1) + \dim f_2(V_2)\\ &= \dim V_1 - \dim \operatorname{Ker}\ f_1 + \dim f_2(V_2)\\ &\leq \dim V_1 + \dim V_3\,.\end{aligned}$$

□

Lemma 3.4. *If*

$$0 \longrightarrow V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \longrightarrow \cdots \longrightarrow V_{n-1} \xrightarrow{f_{n-1}} V_n \longrightarrow 0$$

is an exact sequence of finite dimensional vector spaces and linear maps, then

$$\sum_{k=1}^{n} (-1)^k \dim V_k = 0\,.$$

Proof. The proof follows by induction on the length n of the exact sequence. The cases $n = 1, 2$ are obvious, because in these cases we have respectively $V_1 \cong 0$ and $V_1 \cong V_2$. For the inductive step, fix $n > 2$ and assume the result is true for exact sequences of lengths less than n. We can consider the exact sequences

$$0 \longrightarrow f_2(V_2) \hookrightarrow V_3 \longrightarrow \cdots \longrightarrow V_{n-1} \xrightarrow{f_{n-1}} V_n \longrightarrow 0$$

and

$$0 \longrightarrow V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} f_2(V_2) \longrightarrow 0$$

By the induction hypothesis, we have

$$-\dim f_2(V_2) - \sum_{k=3}^{n} (-1)^k \dim V_k = 0 \tag{3.1}$$

and

$$-\dim V_1 + \dim V_2 - \dim f_2(V_2) = 0. \tag{3.2}$$

The result follows adding equation (3.2) to opposite sign equation (3.1). □

Lemma 3.5. *In a commutative diagram of vector spaces and linear maps with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 & \xrightarrow{f} & V_2 & \xrightarrow{g} & V_3 & \longrightarrow & 0\\ & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle S} & & \\ 0 & \longrightarrow & V_1' & \xrightarrow{f'} & V_2' & \xrightarrow{g'} & V_3' & \longrightarrow & 0\end{array}$$

if the vertical arrows T and S are Fredholm operators, then the middle vertical arrow R is also a Fredholm operator, and $\operatorname{ind}(R) = \operatorname{ind}(T) + \operatorname{ind}(S)$.

Proof. There exists a linear map $\delta\colon \text{Ker } S \to \text{Coker } T$ such that the sequence

$$0 \longrightarrow \text{Ker } T \longrightarrow \text{Ker } R \longrightarrow \text{Ker } S \xrightarrow{\delta} \text{Coker } T \longrightarrow \text{Coker } R \longrightarrow \text{Coker } S \longrightarrow 0$$

is exact, where the other maps are induced by f, g, f' and g' (note that the commutativity of the given diagram implies that $f(\text{Ker } T) \subseteq \text{Ker } R$, $g(\text{Ker } R) \subseteq \text{Ker } S$, $f'(T(V_1)) \subseteq R(V_2)$ and $g'(R(V_2)) \subseteq S(V_3)$). This can be proved by diagram chasing and is known as the Snake Lemma[1]. By Lemma 3.3, Ker R and Coker R are finite dimensional and we can apply Lemma 3.4 to conclude that

$$\begin{aligned} 0 &= -\dim \text{Ker } T + \dim \text{Ker } R - \dim \text{Ker } S \\ &\quad + \dim \text{Coker } T - \dim \text{Coker } R - \dim \text{Coker } S \\ &= -\operatorname{ind}(T) + \operatorname{ind}(R) - \operatorname{ind}(S)\,, \end{aligned}$$

as desired. ☐

Proposition 3.6. *If $T \in \mathcal{F}(V_1, V_2)$ and $T' \in \mathcal{F}(V_1', V_2')$, then $T \oplus T' \in \mathcal{F}(V_1 \oplus V_1', V_2 \oplus V_2')$ and* $\operatorname{ind}(T \oplus T') = \operatorname{ind}(T) + \operatorname{ind}(T')$.

Proof. Observe that we have the commutative diagram of vector spaces and linear maps with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 & \xrightarrow{f} & V_1 \oplus V_2 & \xrightarrow{g} & V_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T \oplus T'} & & \downarrow{\scriptstyle T'} & & \\ 0 & \longrightarrow & V_1' & \xrightarrow{f'} & V_1' \oplus V_2' & \xrightarrow{g'} & V_2' & \longrightarrow & 0 \end{array}$$

where $f(v_1) = (v_1, 0)$, $g(v_1, v_2) = v_2$, and f' and g' are defined similarly. The result follows from Lemma 3.5. ☐

Proposition 3.7. *If $T \in \mathcal{F}(V_1, V_2)$ and $S \in \mathcal{F}(V_2, V_3)$, then $ST \in \mathcal{F}(V_1, V_3)$ and* $\operatorname{ind}(ST) = \operatorname{ind}(T) + \operatorname{ind}(S)$.

Proof. Consider the diagram of vector spaces and linear maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 & \xrightarrow{i} & V_1 \oplus V_2 & \xrightarrow{p} & V_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle ST \oplus id_{V_2}} & & \downarrow{\scriptstyle S} & & \\ 0 & \longrightarrow & V_2 & \xrightarrow{j} & V_3 \oplus V_2 & \xrightarrow{q} & V_3 & \longrightarrow & 0 \end{array}$$

where $i(v_1) = (v_1, T(v_1))$, $p(v_1, v_2) = T(v_1) - v_2$, $j(v_2) = (S(v_2), v_2)$ and $q(v_3, v_2) = v_3 - S(v_2)$. It is straightforward to check that this is a commutative diagram and that it has exact rows. Applying Lemma 3.5 and Proposition 3.6, we have that ST is Fredholm and

$$\operatorname{ind}(ST) = \operatorname{ind}(ST \oplus id_{V_2}) = \operatorname{ind}(T) + \operatorname{ind}(S).$$

☐

[1] For an elementary proof, see Lemma 7.8 of [Alu09]. One can also see [Mac98] for a category theoretical proof.

Lemma 3.8. *Let H_1 and H_2 be Hilbert spaces. If $T \in \mathcal{F}(H_1, H_2)$, then $T(H_1)$ is closed in H_2.*

Proof. Let $V \doteq T(H_1)^\perp$, which is finite dimensional since Coker T is. Hence, V is a Hilbert space. The map $T' \colon (\text{Ker } T)^\perp \oplus V \to H_2$, given by $T'(u, v) \doteq T(u) + v$, is a continuous linear bijection. It follows from the Open Mapping Theorem that T' is a linear homeomorphism. Therefore $T(H_1) = T'((\ker T)^\perp)$ is closed in H_2. □

Proposition 3.9. *A linear map $T \in \mathcal{B}(H_1, H_2)$ is Fredholm if and only if* Ker T *and* Ker T^* *are finite dimensional and $T(H_1)$ is closed in H_2. In this case, we have that* $\text{ind}(T) = \dim \text{Ker } T - \dim \text{Ker } T^*$.

Proof. We have Ker $T^* = T(H_1)^\perp$. Closedness of $T(H_1)$ gives $(\text{Ker } T^*)^\perp = T(H_1)^{\perp\perp} = T(H_1)$. This, together with $\text{Ker } T^* \oplus (\text{Ker } T^*)^\perp = H_2$, gives Coker $T \cong$ Ker T^*.

Take $T \in \mathcal{F}(H_1, H_2)$. By Lemma 3.8, $T(H_1)$ is closed and the above argument shows that Ker T^* is finite dimensional and $\text{ind}(T) = \dim \text{Ker } T - \dim \text{Ker } T^*$.

Conversely, if Ker T and Ker T^* are finite dimensional and if $T(H_1)$ is closed, the above argument shows that Coker $T \cong$ Ker T^* is finite dimensional and therefore T is Fredholm with $\text{ind}(T) = \dim \text{Ker } T - \dim \text{Ker } T^*$. □

Corollary 3.10. *Let H_1 and H_2 be Hilbert spaces. Then $T \in \mathcal{B}(H_1, H_2)$ is Fredholm if and only if $T^* \in \mathcal{F}(H_2, H_1)$. In this case,* $\text{ind}(T) = -\text{ind}(T^*)$.

Proposition 3.11. *Let H_1 and H_2 be infinite dimensional Hilbert spaces. If there exists a Fredholm operator $T \in \mathcal{F}(H_1, H_2)$, then H_1 and H_2 are isomorphic as Banach spaces.*

Proof. By the Open Mapping Theorem, $T|_{(\text{Ker } T)^\perp} \colon (\text{Ker } T)^\perp \to T(H_1)$ is an isomorphism of infinite dimensional Banach spaces (recall that $T(H_1)$ is closed by Lemma 3.8). Since Ker T and $T(H_1)^\perp$ are finite dimensional subspaces, we have that

$$\begin{aligned}
\dim H_1 &= \dim \big(\text{Ker } T \oplus (\text{Ker } T)^\perp\big) \\
&= \dim \text{Ker } T + \dim (\text{Ker } T)^\perp \\
&= \dim (\text{Ker } T)^\perp \\
&= \dim T(H_1) \\
&= \dim T(H_1) + \dim T(H_1)^\perp \\
&= \dim \big(T(H_1) \oplus T(H_1)^\perp\big) \\
&= \dim H_2 \,,
\end{aligned}$$

where $\dim H$ denotes the cardinality of a complete orthonormal system of the Hilbert space H. We have used that $\kappa + n = \kappa$ for every $n \in \mathbb{N}$ and every infinite cardinal κ. □

3.2 Compact Operators

In this section, V_1 and V_2 denote normed vector spaces.

Definition 3.12. A linear map $T \colon V_1 \to V_2$ is called a *compact operator* if it maps the open unit ball $\{v \in V_1 : \|v\| < 1\}$ (and hence any bounded subset of V_1) into a relatively compact set in V_2.

Equivalently, $T\colon V_1 \to V_2$ is compact if every bounded sequence $\{v_n\}$ in V_1 admits a subsequence $\{v_{n_k}\}$ such that $\{T(v_{n_k})\}$ converges to a point in V_2.

The set of all compact operators from V_1 into V_2 is denoted by $\mathcal{K}(V_1, V_2)$.

Example 3.13. If $T\colon V_1 \to V_2$ is a bounded finite rank operator (i.e. if $T(V_1)$ is finite dimensional) and if $\{v_n\}$ is a bounded sequence in V_1, then $\{T(v_n)\}$ is a bounded sequence contained in the finite dimensional subspace $T(V_1)$. It is well known that every finite dimensional linear subspace of a normed space is isomorphic (as normed vector spaces) to an euclidean space $\mathbb{C}^N$. We can then apply Bolzano-Weierstrass Theorem to conclude that $\{T(v_n)\}$ admits a convergent subsequence. Thus, any bounded finite rank operator is compact.

Proposition 3.14. *$\mathcal{K}(V_1, V_2)$ is a closed linear subspace of $\mathcal{B}(V_1, V_2)$.*

Proof. A compact operator is bounded since the image of the unit ball is relatively compact, and hence bounded. Thus $\mathcal{K}(V_1, V_2) \subseteq \mathcal{B}(V_1, V_2)$.

Now, let $T, S \in \mathcal{K}(V_1, V_2)$, $\lambda \in \mathbb{C}$ and let $\{v_n\}$ be a bounded sequence in V_1. Compactness of T gives a subsequence $\{v_{n_k}\}$ such that $\{T(v_{n_k})\}$ is convergent. Compactness of S gives a subsequence $\{v_{n_{k_j}}\}$ of $\{v_{n_k}\}$ such that $\{S(v_{n_{k_j}})\}$ is convergent. Then, since operations in V_2 are continuous, the sequence $\{(T + \lambda S)(v_{n_{k_j}})\}$ is convergent. This proves that $T + \lambda S \in \mathcal{K}(V_1, V_2)$.

Let us prove that $\mathcal{K}(V_1, V_2)$ is closed in $\mathcal{B}(V_1, V_2)$. For let $T \in \mathcal{B}(V_1, V_2)$ and assume that $T \in \overline{\mathcal{K}(V_1, V_2)}$. Given $\varepsilon > 0$, there exists $S \in \mathcal{K}(V_1, V_2)$ such that $\|T - S\| < \varepsilon/3$. This means $\|T(v) - S(v)\| < \varepsilon/3$ for all $v \in V_1$ with $\|v\| \leq 1$. Denote by B_1 the unit ball in V_1 and let $\{v_n\}$ be any sequence in B_1. Compactness of S gives a subsequence $\{v_{n_k}\}$ such that $\{S(v_{n_k})\}$ converges to some $y \in \overline{S(B_1)}$. Consider $x \in B_1$ such that $\|y - S(x)\| < \varepsilon/6$. Since $S(v_{n_k}) \to y$, there exists $k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$

$$\|S(v_{n_k}) - S(x)\| \leq \|S(v_{n_k}) - y\| + \|y - S(x)\| < \varepsilon/6 + \varepsilon/6 = \varepsilon/3$$

Then $\forall k \geq k_0$

$$\|T(v_{n_k}) - T(x)\| \leq \|T(v_{n_k}) - S(v_{n_k})\| + \|S(v_{n_k}) - S(x)\| + \|S(x) - T(x)\| < \varepsilon.$$

This proves that T is compact, as desired. □

Proposition 3.15. *Let V_1, V_2 and V_3 are normed spaces. Consider $T \in \mathcal{K}(V_1, V_2)$, $S \in \mathcal{B}(V_2, V_3)$, $T' \in \mathcal{B}(V_1, V_2)$ and $S' \in \mathcal{K}(V_2, V_3)$. Then $ST, S'T' \in \mathcal{K}(V_1, V_3)$.*

Consequently, $\mathcal{K}(V_1, V_1)$ is a closed two-sided ideal of the normed algebra $\mathcal{B}(V_1, V_1)$.

Proof. Let $\{v_n\}$ be a bounded sequence in V_1.

To see $ST \in \mathcal{K}(V_1, V_3)$, just notice that compactness of T gives a subsequence $\{v_{n_k}\}$ such that $\{T(v_{n_k})\}$ is convergent, and boundedness of S implies convergence of $\{S(T(v_{n_k}))\}$.

To see $S'T' \in \mathcal{K}(V_1, V_3)$, note that boundedness of T' implies boundedness of $\{T'(v_n)\}$, and compactness of S' gives a subsequence $\{v_{n_k}\}$ such that $\{S'(T'(v_{n_k}))\}$ is convergent. □

Proposition 3.16. *Let H_1 and H_2 be Hilbert spaces. $T \in \mathcal{K}(H_1, H_2)$ if and only if $T^* \in \mathcal{K}(H_2, H_1)$.*

Proof. Since $T^{**} = T$, it suffices to show that $T^* \in \mathcal{K}(H_2, H_1)$ whenever $T \in \mathcal{K}(H_1, H_2)$. Let $\{y_n\}$ be a sequence in H_2 such that $\|y_n\| \leq 1$ for every $n \in \mathbb{N}$. Define $x_n \doteq T^*(y_n)$. Since

$$\|x_n\| = \|T^*(y_n)\| \leq \|T^*\|\|y_n\| \leq \|T^*\|,$$

we have that $\{x_n\}$ is a bounded sequence in H_1. By compactness of T, there exists a subsequence $\{x_{n_j}\}$ such that $\{T(x_{n_j})\}$ is convergent (Cauchy). Notice that, for $k, j \in \mathbb{N}$,

$$\begin{aligned}\|T^*(y_{n_j}) - T^*(y_{n_k})\|^2 &= \langle T(x_{n_j}) - T(x_{n_k}), y_{n_j} - y_{n_k}\rangle \\ &\leq \|T(x_{n_j}) - T(x_{n_k})\| \; \|y_{n_j} - y_{n_k}\| \\ &\leq 2 \left\|T(x_{n_j}) - T(x_{n_k})\right\|,\end{aligned}$$

so that $\{T^*(y_{n_j})\}$ is a Cauchy sequence, and therefore it converges. This concludes the proof. □

Corollary 3.17. *Let H be a Hilbert space. Then $\mathcal{K}(H, H)$ is a closed two-sided $*$-ideal of the C^*-algebra $\mathcal{B}(H, H)$.*

In Hilbert spaces, Fredholm operators are precisely those that are invertible modulo the compact operators, as we can see in the following

Theorem 3.18 (Atkinson)**.** *Let H_1 and H_2 be Hilbert spaces. If $T \in \mathcal{F}(H_1, H_2)$, there exists $S \in \mathcal{B}(H_2, H_1)$ such that $ST - id_{H_1} \in \mathcal{K}(H_1, H_1)$ and $TS - id_{H_2} \in \mathcal{K}(H_2, H_2)$.*

Conversely, if $T \in \mathcal{B}(H_1, H_2)$ and there are $S, S' \in \mathcal{B}(H_2, H_1)$ such that $ST - id_{H_1} \in \mathcal{K}(H_1, H_1)$ and $TS' - id_{H_2} \in \mathcal{K}(H_2, H_2)$, then $T \in \mathcal{F}(H_1, H_2)$.

Proof. Assume T is Fredholm. Restricting T to $(\text{Ker } T)^\perp$, one obtains an invertible operator $T_1 \colon (\text{Ker } T)^\perp \to T(H_1)$. Let $S \colon H_2 \to H_1$ be given by

$$S(y) = \begin{cases} T_1^{-1}(y) & \text{if } y \in T(H_1), \\ 0 & \text{if } y \in T(H_1)^\perp. \end{cases}$$

Let $P \in \mathcal{B}(H_1, H_1)$ and $Q \in \mathcal{B}(H_2, H_2)$ be the orthogonal projections to Ker T and $T(H_1)^\perp$, respectively. Then $ST + P = id_{H_1}$ and $TS + Q = id_{H_2}$. Since Ker T and $T(H_1)^\perp \cong \text{Coker } T$ are finite dimensional, it follows that P and Q are finite rank operators and, therefore, compact.

Conversely, assume that there exist such S and S'. Then there exists $K \in \mathcal{K}(H_1, H_1)$ such that $id_{H_1} - ST = K$. For $x \in \text{Ker } T$, we have $K(x) = x - ST(x) = x$. So, if $x \in \text{Ker } T$ and B is a bounded neighborhood of x in Ker T, it is true that $K(B) = B$ is relatively compact. This proves that Ker T is locally compact. Thus, Ker T is finite dimensional. By Proposition 3.16, the operator $(id_{H_2} - TS')^* = id_{H_2} - (S')^*T^*$ is compact and we can repeat the previous argument to show that Ker T^* is finite dimensional.

By Proposition 3.9, it remains to show that $T(H_1)$ is closed. Let $y_n \in T(H_1)$ and suppose $y_n \to y$ for some $y \in H_2$. We may write $y_n = T(x_n)$ with $x_n \in (\text{Ker } T)^\perp$. Note that $\{x_n\}$ admits a bounded subsequence. In fact, if $\|x_n\| \to \infty$, we can consider $x'_n = x_n/\|x_n\| \in (\text{Ker } T)^\perp$ and observe that $T(x'_n) = y_n/\|x_n\| \to 0$. On the other hand, boundedness of $\{x'_n\}$ and compactness of K gives a subsequence $\{x'_{n_k}\}$ such that $K(x'_{n_k})$ converges to some $x' \in H_1$. Since $K = id_{H_1} - ST$ and $T(x'_{n_k}) \to 0$, we have that $\lim x'_{n_k} = \lim K(x'_{n_k}) = x'$. But then $0 = \lim T(x'_{n_k}) = T(x')$ and this is impossible since $\|x'\| = 1$ and $x' \in (\text{Ker } T)^\perp$.

Let $\{x_{n_k}\}$ be a bounded subsequence of $\{x_n\}$. Again, compactness of K gives a subsequence $\{x_{n_{k_j}}\}$ such that $K(x_{n_{k_j}}) = x_{n_{k_j}} - S(y_{n_{k_j}})$ is convergent. Since $S(y_n) \to S(y)$, the subsequence $\{x_{n_{k_j}}\}$ converges to some $x \in H_1$, and we conclude that $y = \lim y_{n_{k_j}} = \lim T(x_{n_{k_j}}) = T(x)$, as desired. $\square$

Corollary 3.19. *Let H_1 and H_2 be Hilbert spaces. If $T \in \mathcal{F}(H_1, H_2)$ and $K \in \mathcal{K}(H_1, H_2)$, then $T + K \in \mathcal{F}(H_1, H_2)$.*

Proof. Choose $S \in \mathcal{B}(H_2, H_1)$ such that $ST - id_{H_1}$ and $TS - id_{H_2}$ are compact operators. Then $S(T + K) - id_{H_1} = (ST - id_{H_1}) + SK$ and $(T + K)S - id_{H_2} = (TS - id_{H_2}) + KS$ are compact operators by Corollary 3.17. The result follows from Atkinson's Theorem. $\square$

Lemma 3.20. *The Fredholm index* $\mathrm{ind}\colon \mathcal{F}(H_1, H_2) \to \mathbb{Z}$ *is locally constant. In particular, it is continuous and homotopy invariant.*

Proof. Fix $T \in \mathcal{F}(H_1.H_2)$ and consider $V \doteq (\mathrm{Ker}\ T)^\perp$ and $W \doteq T(H_1)^\perp$. Let $\alpha\colon V \to H_1$ and $\beta\colon H_2 \to T(H_1)$ be the inclusion be the orthogonal projection onto $T(H_1)$, respectively. Then $\mathrm{Ker}\ \alpha = \{0\}$, $\mathrm{Coker}\ \alpha = H_1/V \cong \mathrm{Ker}\ T$, $\mathrm{Ker}\ \beta = T(H_1)^\perp \cong \mathrm{Coker}\ T$ and $\mathrm{Coker}\ \beta = \{0\}$, so that $\mathrm{ind}(\alpha) = -\dim \mathrm{Ker}\ T$ and $\mathrm{ind}(\beta) = \dim \mathrm{Coker}\ T$. Then $\mathrm{ind}(\beta T \alpha) = \mathrm{ind}(\beta) + \mathrm{ind}(T) + \mathrm{ind}(\alpha) = 0$. Also, the map $\beta T \alpha\colon V \to T(H_1)$ is a continuous bijection and, by the Open Mapping Theorem, it is an isomorphism in $\mathcal{B}(V, T(H_1))$. Continuity of $\mathcal{F}(H_1, H_2) \ni S \longmapsto \beta S \alpha \in \mathcal{B}(V, T(H_1))$ gives that $\beta T' \alpha$ is an isomorphism for T' sufficiently close to T, so that $0 = \mathrm{ind}(\beta T' \alpha) = \mathrm{ind}(\beta) + \mathrm{ind}(T') + \mathrm{ind}(\alpha)$, implying $\mathrm{ind}(T') = \mathrm{ind}(T)$. Then ind is locally constant, as desired. $\square$

Corollary 3.21. *If $T \in \mathcal{F}(H_1, H_2)$ and $K \in \mathcal{K}(H_1, H_2)$, then* $\mathrm{ind}(T + K) = \mathrm{ind}(T)$.

Proof. Let $\gamma\colon [0, 1] \to \mathcal{B}(H_1, H_2)$ be given by $\gamma(t) = T + tK$. By Corollary 3.19, we have $\gamma(t) \in \mathcal{F}(H_1, H_2)$. Lemma 3.20 gives that $\mathrm{ind}(\gamma(t))$ does not depend on $t \in [0, 1]$, so that $\mathrm{ind}(T) = \mathrm{ind}(T + K)$. $\square$

3.3 The Families-Index

In this section, H_1 and H_2 denote Hilbert spaces.

Let X be a topological space. Equip $\mathcal{F}(H_1, H_2)$ with the norm topology inherited from $\mathcal{B}(H_1, H_2)$, and consider $T\colon X \to \mathcal{F}(H_1, H_2)$ to be a continuous map (we say that T is a *continuous family of bounded Fredholm operators on* X). For $x \in X$, the vector spaces $\mathrm{Ker}\ T_x$ and $\mathrm{Coker}\ T_x$ are finite dimensional. If the dimension of such spaces were locally constant, one could possibly ask if

$$\mathrm{Ker}\ T \doteq \bigsqcup_{x \in X} \mathrm{Ker}\ T_x \quad \text{and} \quad \mathrm{Coker}\ T \doteq \bigsqcup_{x \in X} \mathrm{Coker}\ T_x$$

are vector bundles over X. One might then consider $[\mathrm{Ker}\ T] - [\mathrm{Coker}\ T] \in \mathsf{K}(X)$ as the index of T. Unfortunately, it does not always work this way.

Example 3.22. The continuous map $T\colon \mathbb{S}^1 \to \mathcal{F}(\mathbb{C}, \mathbb{C})$, given by $T_z(\omega) = (z + 1)\omega$, satisfies

$$\dim \mathrm{Ker}\ T_z = \dim \mathrm{Coker}\ T_z = \begin{cases} 0 & \text{if } z \in \mathbb{S}^1 \setminus \{-1\}, \\ 1 & \text{if } z = -1. \end{cases}$$

In order for the idea of considering [Ker T] − [Coker T] $\in \mathsf{K}(X)$ to work, one needs to make some adjustments. There are at least three approaches for doing this: one given in the Appendix of [Ati67] and in [Muk13], another one discussed in [Bre16] and a third one given in [BB13]. We are concerned with the first two approaches and our main interest in this section is to discuss them and prove their equivalence.

We can operate with family of operators in the same way we operate with single operators. Let $T\colon X \to \mathcal{B}(H_1, H_2)$, $T'\colon X \to \mathcal{B}(H_1', H_2')$ and $S\colon X \to \mathcal{B}(H_2, H_3)$. The direct sum $T \oplus T'\colon X \to \mathcal{B}(H_1 \oplus H_1', H_2 \oplus H_2')$ is defined to be $(T \oplus T)_x \doteq T_x \oplus T_x'$ for $x \in X$. If T and T' are families of Fredholm operators, then $T \oplus T'$ is also a family of Fredholm operators by Proposition 3.6. If T and T' are continuous, then $T \oplus T'$ is also continuous since we have continuity of the direct sum map $\mathcal{B}(H_1, H_2) \times \mathcal{B}(H_1', H_2') \to \mathcal{B}(H_1 \oplus H_1', H_2 \oplus H_2')$, $(A, B) \longmapsto A \oplus B$. We define the product $ST\colon X \to \mathcal{B}(H_1, H_3)$ by $(ST)_x \doteq S_x T_x$ for $x \in X$. If T and S are families of Fredholm operators, Proposition 3.7 gives that ST is also a family of Fredholm operators. Since the composition map $\mathcal{B}(H_1, H_2) \times \mathcal{B}(H_2, H_3) \to \mathcal{B}(H_1, H_3)$, $(A, B) \longmapsto BA$ is continuous, we have that ST is a continuous map if S and T are continuous. The adjoint of T is the map $T^*\colon X \to \mathcal{B}(H_2, H_1)$ given by $(T^*)_x \doteq (T_x)^* = T_x^*$, which is continuous if T is continuous since the adjoint $\mathcal{B}(H_1, H_2) \to \mathcal{B}(H_2, H_1)$, $A \longmapsto A^*$, is continuous. Moreover, since $A \in \mathcal{F}(H_1, H_2)$ if and only if $A^* \in \mathcal{F}(H_2, H_1)$ (see Corollary 3.10), we have that T^* is a family of Fredholm operators if and only if so is T. If T is a family of invertible operators, we can define $T^{-1}\colon X \to \mathcal{B}(H_2, H_1)$ by $(T^{-1})_x \doteq (T_x)^{-1} = T_x^{-1}$. By the continuity of the inversion $A \longmapsto A^{-1}$, we have that T^{-1} is continuous if T is continuous. If $H_1 = H_2$ and if T is a family of nonnegative selfadjoint invertible operators, the square root of T is the map $T^{1/2}\colon X \to \mathcal{B}(H_1, H_1)$ given by $(T^{1/2})_x \doteq (T_x)^{1/2} = T_x^{1/2}$. Since the square root $A \longmapsto A^{1/2}$ is continuous (see Corollary 1.31), it follows that $T^{1/2}$ is continuous if T is continuous. The list of operations could continue.

Let us begin with the constructions of the families-index.

3.3.1 First Approach

In this section, we follow closely what was done in [Bre16].

Let X be a topological space and $T\colon X \to \mathcal{F}(H_1, H_2)$ be a continuous map. In the beginning of this chapter it was wondered if

$$\text{Ker } T \doteq \bigsqcup_{x \in X} \text{Ker } T_x \quad \text{and} \quad \text{Coker } T \doteq \bigsqcup_{x \in X} \text{Coker } T_x$$

would be vector bundles if $\dim \text{Ker } T_x$ and $\dim \text{Coker } T_x$ were locally constant on $x \in X$. Let us answer this question.

Proposition 3.23. *Let $T\colon X \to \mathcal{F}(H_1, H_2)$ be as above and suppose that the dimension of* Ker T_x *is locally constant on $x \in X$. Then*

$$\text{Ker } T = \bigsqcup_{x \in X} \text{Ker } T_x\,,$$

seen as a topological subspace of $X \times H_1$, is a vector bundle over X.

Proof. Denoting by $p\colon \text{Ker } T \to X$ the restriction of the projection onto the first coordinate $X \times$

$H_1 \to X$, we can easily see that Ker T is a family of vector spaces over X. It remains to prove that it is locally trivial.

Fix $x \in X$. Let $P_x\colon H_1 \to \text{Ker } T_x$ and $Q_x\colon H_2 \to T_x(H_1)$ be the orthogonal projections. For $y \in X$, consider the map $\hat{T}^x_y\colon H_1 \to \text{Ker } T_x \oplus T_x(H_1)$ given by $\hat{T}^x_y(u) = (P_x u, Q_x T_y u)$. The map $\hat{T}^x\colon X \to \mathcal{B}(H_1, \text{Ker } T_x \oplus T_x(H_1))$, $y \longmapsto \hat{T}^x_y$, is continuous since for $u \in H_1$

$$\|(\hat{T}^x_y - \hat{T}^x_{y'})u\| = \|(0, Q_x(T_y - T_{y'})u)\| = \|Q_x(T_y - T_{y'})u\| \le \|T_y - T_{y'}\|\|u\|$$

(recall $\|Q_x\| \le 1$), which implies $\|\hat{T}^x_y - \hat{T}^x_{y'}\| \le \|T_y - T_{y'}\|$.

Notice that $\hat{T}^x_x$ is surjective: for $(v, w) \in \text{Ker } T_x \oplus T_x(H_1)$ we can let $u \in H_1$ be such that $T_x u = w$ and see that $P_x(v + u - P_x u) = v$ and $Q_x T_x(v + u - P_x u) = w$, so that $\hat{T}^x_x(v + u - P_x u) = (v, w)$. Besides, $\hat{T}^x_x$ is injective: if $u \in \text{Ker } \hat{T}^x_x$ we have $P_x u = 0$, implying $u \in (\text{Ker } T_x)^\perp$, and $Q_x T_x u = 0$, implying $T_x u \in (T_x(H_1))^\perp$ which is only possible if $u \in \text{Ker } T_x$, hence $u = 0$.

By the Open Mapping Theorem, $\hat{T}^x_x$ is an isomorphism. Since the set of isomorphisms in $\mathcal{B}(H_1, \text{Ker } T_x \oplus T_x(H_1))$ is open (see Proposition 1.25), there exists an open neighborhood U_x of x such that $\hat{T}^x_y$ is an isomorphism for every $y \in U_x$. Replacing U_x by the connected component of U_x that contains x if necessary, we can assume that U_x is connected.

Observe that $\hat{T}^x_y(\text{Ker } T_y) \subseteq \text{Ker } T_x \oplus \{0\}$ since $\hat{T}^x_y(u) = (Pu, 0)$ whenever $u \in \text{Ker } T_y$. By hypothesis and connectedness of U_x, $\dim \text{Ker } T_y = \dim \text{Ker } T_x$ for $y \in U_x$. Thus $\hat{T}^x_y$ induces an isomorphism from Ker T_y onto Ker T_x. Then the continuous map $p^{-1}(U_x) \to U_x \times (\text{Ker } T_x \oplus \{0\})$, given by $(y, u) \longmapsto \big(y, \hat{T}^x_y(u)\big)$, is bijective and its inverse is given by $(y; v, 0) \longmapsto (y, (\hat{T}^x_y)^{-1}(v, 0))$. This proves the local triviality of $p\colon \text{Ker } T \to X$. □

Corollary 3.24. *Let $T\colon X \to \mathcal{F}(H_1, H_2)$ be continuous. If T_x is surjective for every $x \in X$, then* Ker $T \to X$ *is a vector bundle.*

Proof. By Lemma 3.20, $\text{ind}(T_x) = \dim \text{Ker } T_x - \dim \text{Coker } T_x = \dim \text{Ker } T_x$ is locally constant. The result follows from Proposition 3.23. □

We need some technical results before defining a families-index.

Lemma 3.25. *Let V_1 and V_2 be arbitrary vector spaces, $L\colon V_1 \to V_2$ be a linear map and $W \subseteq V_2$ be a linear subspace. Suppose* Ker L *is finite dimensional. If W is finite dimensional, then $L^{-1}(W)$ is finite dimensional.*

Proof. By contradiction, assume that $\{v_n : n \ge 1\}$ is an infinite linearly independent subset of $L^{-1}(W)$. Let $V \doteq \text{span}\{v_n : n \ge 1\}$. We clearly have that $\text{Ker } L|_V \subseteq \text{Ker } L$ and $L(V) \subseteq W$. Applying the Rank-Nullity Theorem for $L|_V\colon V \to V_2$, we have that

$$\dim V = \dim \text{Ker } L|_V + \dim L(V) \le \dim \text{Ker } L + \dim W\,,$$

which is a contradiction since V has infinite dimension. □

Lemma 3.26. *The set $\{T \in \mathcal{B}(H_1, H_2) : T(H_1) = H_2\}$ is open in $\mathcal{B}(H_1, H_2)$.*

Proof. Let $T_0 \in \mathcal{B}(H_1, H_2)$ be such that $T_0(H_1) = H_2$ and denote the inclusion $(\text{Ker } T_0)^\perp \hookrightarrow H_1$ by α. We have that the map $\mathcal{B}(H_1, H_2) \to \mathcal{B}((\text{Ker } T_0)^\perp, H_2)$, $S \longmapsto S\alpha$, is continuous and $T_0\alpha$ is invertible. Since the set of invertible elements of $\mathcal{B}((\text{Ker } T_0)^\perp, H_2)$ is open (see Proposition 1.25), there exists an open set $U \subseteq \mathcal{B}(H_1, H_2)$ such that $T_0 \in U$ and $S\alpha$ is invertible for every $S \in U$. Notice that $H_2 = S((\text{Ker } T_0)^\perp) \subseteq S(H_1)$ for every $S \in U$, so that U is an open neighborhood of T_0 entirely contained in $\{T \in \mathcal{B}(H_1, H_2) : T(H_1) = H_2\}$. □

Proposition 3.27. *Let X be compact and $T\colon X \to \mathcal{F}(H_1, H_2)$ be continuous. There exists a finite dimensional linear subspace $W \subseteq H_2$ such that $T_x(H_1) + W = H_2$ for every $x \in X$.*

Proof. Fix $x \in X$. Let $W_x \doteq T_x(H_1)^\perp$, which is finite dimensional since T_x is Fredholm. Consider $\tilde{T}^x\colon X \to \mathcal{B}(H_1 \oplus W_x, H_2)$, given by $\tilde{T}^x_y(v, w) = T_y(v) + w$. Since $(\tilde{T}^x_y - \tilde{T}^x_{y'})(v, w) = (T_y - T_{y'})(v)$, we have

$$\|(\tilde{T}^x_y - \tilde{T}^x_{y'})(v, w)\| \leq \|T_y - T_{y'}\|\|v\| \leq \|T_y - T_{y'}\|\|(v, w)\|,$$

so that $\|\tilde{T}^x_y - \tilde{T}^x_{y'}\| \leq \|T_y - T_{y'}\|$. This proves that $\tilde{T}^x$ is continuous. Observe that

$$\tilde{T}^x_x(H_1 \oplus W_x) = T_x(H_1) + W_x = H_2$$

because $T_x(H_1)$ is closed. The previous Lemma and the continuity of $\tilde{T}^x$ give an open neighborhood U_x of x in X such that $\tilde{T}^x_y(H_1 \oplus W_x) = H_2$ for every $y \in U_x$. Thus $T_y(H_1) + W_x = H_2$ for every $y \in U_x$.

The collection $\{U_x : x \in X\}$ is an open cover of X. By compactness, there are $x_1, \cdots, x_n \in X$ such that $X = \bigcup_{i=1}^n U_{x_i}$. Consider the finite dimensional subspace $W \doteq W_{x_1} + \cdots + W_{x_n} \subseteq H_2$. If $x \in X$, we have $x \in U_{x_j}$ for some j, so that $T_x(H_1) + W \supseteq T_x(H_1) + W_{x_j} = H_2$. This concludes the proof. □

Let X be compact and $T\colon X \to \mathcal{F}(H_1, H_2)$ be continuous. Consider $W \subseteq H_2$ given by the previous Proposition, and define $T^W\colon X \to \mathcal{B}(H_1 \oplus W, H_2)$ by $T^W_x(v, w) = T_x(v) + w$. Since

$$\|(T^W_x - T^W_y)(v, w)\| = \|(T_x - T_y)(v)\| \leq \|T_x - T_y\|\|v\| \leq \|T_x - T_y\|\|(v, w)\|$$

for every $x, y \in X$, we have that $\|T^W_x - T^W_y\| \leq \|T_x - T_y\|$, from where it follows that T^W is continuous. Let $x \in X$. If $T^W_x(v, w) = 0$, then $T_x(v) = -w$, so that $w \in T_x(H_1) \cap W$ and $v \in T_x^{-1}(W)$. Thus, Ker $T^W_x \subseteq T_x^{-1}(W) \oplus (T_x(H_1) \cap W)$, from where it follows that Ker T^W_x is finite dimensional (the space $T_x^{-1}(W)$ is finite dimensional by Lemma 3.25). Besides, the equality $T_x(H_1) + W = H_2$ gives the surjectivity of T^W_x. In particular, we have that T^W_x is Fredholm for every $x \in X$, and we can write $T^W\colon X \to \mathcal{F}(H_1 \oplus W, H_2)$. Applying Corollary 3.24, we obtain

Theorem 3.28. *Let X be a compact space, $T\colon X \to \mathcal{F}(H_1, H_2)$ be continuous, W be as in Proposition 3.27 and T^W be as above. Then* Ker T^W*, seen as a topological subspace of $X \times (H_1 \oplus W)$, is a vector bundle over X.*

Proposition 3.29. *Let X be a compact Hausdorff space and $T\colon X \to \mathcal{F}(H_1, H_2)$ be continuous. Let $W, W' \subseteq H_2$ be finite dimensional subspaces such that $T_x(H_1) + W = H_2 = T_x(H_1) + W'$ for every $x \in X$, and consider the respective associated continuous maps $T^W\colon X \to \mathcal{F}(H_1 \oplus W, H_2)$ and $T^{W'}\colon X \to \mathcal{F}(H_1 \oplus W', H_2)$. Then the following equality holds in $\mathsf{K}(X)$:*

$$[\text{Ker } T^W] - [X \times W] = [\text{Ker } T^{W'}] - [X \times W'] . \tag{3.3}$$

Proof. First assume we have $W \subseteq W'$ and $\dim W + 1 = \dim W'$. Write $W' = W \oplus \operatorname{span}\{\omega_0\}$, for some $\omega_0 \in W' \setminus W$. We have that $T^W\colon X \to \mathcal{F}(H_1 \oplus W, H_2)$ and $T^{W'}\colon X \to \mathcal{F}(H_1 \oplus W', H_2)$ are defined by

$$T_x^W(v,w) = T_x(v) + w \quad \text{and} \quad T_x^{W'}(v, w + \lambda\omega_0) = T_x(v) + w + \lambda\omega_0 \,.$$

Define $F\colon X \times [0,1] \to \mathcal{F}(H_1 \oplus W', H_2)$ by $F_{(x,t)}(v, w + \lambda\omega_0) \doteq T_x(v) + w + t\lambda\omega_0$. Since $T_x(H_1) + W = H_2$, we have that $F_{(x,t)}$ is surjective for every $(x,t) \in X \times [0,1]$. By Corollary 3.24 we have a vector bundle $p\colon \text{Ker } F \to X \times [0,1]$. For $t \in [0,1]$, consider $i_t\colon X \to X \times [0,1]$ given by $i_t(x) \doteq (x,t)$. Lemma 2.34 gives

$$i_0^*(\text{Ker } F) \;\cong\; i_1^*(\text{Ker } F).$$

If we let $S\colon X \to \mathcal{F}(H_1 \oplus W \oplus \mathbb{C}, H_2)$ be given by $S_x(v,w,\lambda) \doteq T_x(v) + w = F_{(x,0)}(v, w+\lambda\omega_0)$, then Corollary 3.24 implies that Ker S, seen as a topological subspace of $X \times (H_1 \oplus W \oplus \mathbb{C})$, is a vector bundle, because S_x is surjective $\forall x \in X$. Observe that Ker S is isomorphic to

$$\begin{aligned}
i_0^*(\text{Ker } F) &= \{(x,e) \in X \times \text{Ker } F : p(e) = (x,0)\} \\
&= \{(x,(y,t),(v,w')) \in X \times (X \times [0,1]) \times (H_1 \oplus W') : (y,t) = (x,0), (v,w') \in \text{Ker } F_{(y,t)}\} \\
&= \{(x,(x,0),(v,w+\lambda\omega_0)) : x \in X, v \in H_1, w \in W, \lambda \in \mathbb{C}, (v, w+\lambda\omega_0) \in \text{Ker } F_{(x,0)}\}
\end{aligned}$$

via

$$\text{Ker } S \ni (x,(v,w,\lambda)) \longmapsto (x,(x,0),(v,w+\lambda\omega_0)) \in i_0^*(\text{Ker } F).$$

Besides, notice that $T_x^{W'} = F_{(x,1)}$ for every $x \in X$, so that Ker $T^{W'}$ is isomorphic to

$$\begin{aligned}
i_1^*(\text{Ker } F) &= \{(x,e) \in X \times \text{Ker } F : p(e) = (x,1)\} \\
&= \{(x,(y,t),(v,w')) \in X \times (X \times [0,1]) \times (H_1 \oplus W') : (y,t) = (x,1), (v,w') \in \text{Ker } F_{(y,t)}\} \\
&= \{(x,(x,1),(v,w+\lambda\omega_0)) : x \in X, v \in H_1, w \in W, \lambda \in \mathbb{C}, (v, w+\lambda\omega_0) \in \text{Ker } F_{(x,1)}\}
\end{aligned}$$

via

$$\text{Ker } T^{W'} \ni (x,(v,w+\lambda\omega_0)) \longmapsto (x,(x,1),(v,w+\lambda\omega_0)) \in i_1^*(\text{Ker } F).$$

Moreover, we have that Ker S is isomorphic to

$$\begin{aligned}
\text{Ker } T^W \oplus (X \times \mathbb{C}) &= \{((x,(v,w)),(y,\lambda)) \in \text{Ker } T^W \times (X \times \mathbb{C}) : x = y\} \\
&= \{((x,(v,w)),(x,\lambda)) : x \in X, \lambda \in \mathbb{C}, (v,w) \in \text{Ker } T_x^W\}
\end{aligned}$$

via

$$\text{Ker } S \ni (x,(v,w,\lambda)) \longmapsto ((x,(v,w)),(x,\lambda)) \in \text{Ker } T^W \oplus (X \times \mathbb{C}).$$

Considering these isomorphisms together, we have

$$\begin{aligned}\text{Ker } T^W \oplus (X \times W') &\cong \text{Ker } T^W \oplus (X \times (\mathbb{C} \oplus W))\\ &\cong \text{Ker } T^W \oplus (X \times \mathbb{C}) \oplus (X \times W)\\ &\cong \text{Ker } S \oplus (X \times W)\\ &\cong i_0^*(\text{Ker } F) \oplus (X \times W)\\ &\cong i_1^*(\text{Ker } F) \oplus (X \times W)\\ &\cong \text{Ker } T^{W'} \oplus (X \times W)\,,\end{aligned}$$

from where it follows that $[\text{Ker } T^W] + [X \times W'] = [\text{Ker } T^{W'}] + [X \times W]$ in $\mathsf{K}(X)$. This proves (3.3) for this case.

The second step is to drop the assumption $\dim W + 1 = \dim W'$, so that we assume only that $W \subseteq W'$. Write $W' = W \oplus \text{span}\{\omega_1, \cdots, \omega_n\}$ for a linearly independent set $\{\omega_1, \cdots, \omega_n\} \subseteq W' \backslash W$. Using induction on n we can apply the first step to prove that (3.3) also holds for this case.

Finally, dropping all extra assumptions about W and W' that we have made in steps one and two, we observe that both W and W' are contained in the finite dimensional subspace $W + W' \subseteq H_2$. Applying the second step, we have

$$\begin{aligned}[\text{Ker } T^W] - [X \times W] &= [\text{Ker } T^{W+W'}] - [X \times (W + W')]\\ &= [\text{Ker } T^{W'}] - [X \times W']\,,\end{aligned}$$

concluding the proof. □

We are ready to define the notion of index of a family of Fredholm operators given in [Bre16].

Definition 3.30. Let X be a compact Hausdorff space and $T\colon X \to \mathcal{F}(H_1, H_2)$ be continuous. The *B-index bundle of T* is

$$\text{ind}_B(T) \doteq [\text{Ker } T^W] - [X \times W] \in \mathsf{K}(X)\,,$$

where $W \subseteq H_2$ is a finite dimensional subspace such that $T_x(H_1) + W = H_2$ for every $x \in X$ and $T^W\colon X \to \mathcal{F}(H_1 \oplus W, H_2)$ is defined by $T_x^W(v, w) = T_x(v) + w$.

Proposition 3.29 shows that the above definition of index depends only on T and not on the choice of W.

Remark 3.31. In the case one has a single point space $X = \{x_0\}$, the map $T\colon \{x_0\} \to \mathcal{F}(H_1, H_2)$ can be seen as a single Fredholm operator $T \doteq T_{x_0} \in \mathcal{F}(H_1, H_2)$. Choosing $W \doteq T(H_1)^\perp$, we have $\text{Ker } T \cong \text{Ker } T^W$ and $\text{Coker } T \cong W$ as vector spaces. Since $\mathsf{K}(\{x_0\}) = \mathbb{Z}$, it follows that $\text{ind}_B(T)$ equals $\dim \text{Ker } T - \dim \text{Coker } T$. This proves that the B-index bundle coincides with the classical Fredholm index under these assumptions.

We finish this section proving some properties about the B-index bundle.

Lemma 3.32. *Let X and Y be compact Hausdorff spaces, and let $f\colon Y \to X$ and $T\colon X \to \mathcal{F}(H_1, H_2)$ be continuous maps. Then we have the equality $\text{ind}_B(T \circ f) = f^*(\text{ind}_B(T))$ in $\mathsf{K}(Y)$.*

Proof. Let $W \subseteq H_2$ be a finite dimensional linear subspace such that $T_x(H_1) + W = H_2$ for every $x \in X$. Then $T_{f(y)}(H_1) + W = H_2$ for every $y \in Y$, so that

$$\mathrm{ind}_B(T \circ f) = [\mathrm{Ker}\ (T \circ f)^W] - [Y \times W].$$

The associated map $(T \circ f)^W \colon Y \to \mathcal{F}(H_1 \oplus W, H_2)$ is given by

$$\begin{aligned}(T \circ f)^W_y(v, w) &= (T \circ f)_y(v) + w \\ &= T_{f(y)}(v) + w \\ &= (T^W \circ f)_y(v, w)\,,\end{aligned}$$

so that $(T \circ f)^W = T^W \circ f$. Denoting by $p \colon \mathrm{Ker}\ T^W \to X$ the bundle projection, we have that

$$\begin{aligned}f^*(\mathrm{Ker}\ T^W) &= \{(y, e) \in Y \times \mathrm{Ker}\ T^W : p(e) = f(y)\} \\ &= \{(y, x, (v, w)) \in Y \times X \times (H_1 \oplus W) : f(y) = x, (v, w) \in \mathrm{Ker}\ T^W_x\} \\ &= \{(y, f(y), (v, w)) : y \in Y, (v, w) \in \mathrm{Ker}\ T^W_{f(y)}\}\end{aligned}$$

is isomorphic to

$$\begin{aligned}\mathrm{Ker}\ (T^W \circ f) &= \{(y, (v, w)) \in Y \times (H_1 \oplus W) : (v, w) \in \mathrm{Ker}\ (T^W \circ f)_x\} \\ &= \{(y, (v, w)) : y \in Y, (v, w) \in \mathrm{Ker}\ T^W_{f(y)}\}\end{aligned}$$

via

$$f^*(\mathrm{Ker}\ T^W) \ni (y, f(y), (v, w)) \longmapsto (y, (v, w)) \in \mathrm{Ker}\ (T^W \circ f)\,.$$

Similarly, $Y \times W$ is isomorphic to

$$\begin{aligned}f^*(X \times W) &= \{(y, x, w) \in Y \times X \times W : x = f(y)\} \\ &= \{(y, f(y), w) : y \in Y, w \in W\}\end{aligned}$$

via

$$Y \times W \ni (y, w) \longmapsto (y, f(y), w) \in f^*(X \times W)\,.$$

In conclusion, we have

$$\begin{aligned}\mathrm{ind}_B(T \circ f) &= [\mathrm{Ker}\ (T \circ f)^W] - [Y \times W] \\ &= [\mathrm{Ker}\ (T^W \circ f)] - [Y \times W] \\ &= [f^*(\mathrm{Ker}\ T^W)] - [f^*(X \times W)] \\ &= f^*([\mathrm{Ker}\ T^W] - [X \times W]) \\ &= f^*(\mathrm{ind}_B(T))\,,\end{aligned}$$

as desired. □

The following result concerns the invariance of $\mathrm{ind}_B(T)$ under homotopies, in analogy to Lemma 3.20.

Proposition 3.33. *Let X be a compact Hausdorff space and $S, T\colon X \to \mathcal{F}(H_1, H_2)$ be continuous maps. If S and T are homotopic, then $\mathrm{ind}_B(S) = \mathrm{ind}_B(T)$.*

Proof. As usual, for $t \in [0,1]$ let $i_t\colon X \to X \times [0,1]$ be given by $i_t(x) \doteq (x,t)$. Let $F\colon X \times [0,1] \to \mathcal{F}(H_1, H_2)$ be a homotopy between S and T, that is, $F \circ i_0 = S$ and $F \circ i_1 = T$. We have then

$$\begin{aligned} \mathrm{ind}_B(S) &= \mathrm{ind}_B(F \circ i_0) \\ &\overset{(1)}{=} i_0^*(\mathrm{ind}_B(F)) \\ &\overset{(2)}{=} i_1^*(\mathrm{ind}_B(F)) \\ &\overset{(1)}{=} \mathrm{ind}_B(F \circ i_1) \\ &= \mathrm{ind}_B(T)\,, \end{aligned}$$

where (1) follows from Lemma 3.32 and (2) follows from Lemma 2.34. □

Proposition 3.34. *Let X be a compact Hausdorff space, H_1, H_2, H_1' and H_2' be Hilbert spaces. For continuous maps $S\colon X \to \mathcal{F}(H_1, H_2)$ and $T\colon X \to \mathcal{F}(H_1', H_2')$, we have*

$$\mathrm{ind}_B(S \oplus T) = \mathrm{ind}_B(S) + \mathrm{ind}_B(T)\,.$$

Proof. Let $W \subseteq H_2$ and $W' \subseteq H_2'$ be finite dimensional subspaces such that $S_x(H_1) + W = H_2$ and $T_x(H_1') + W' = H_2'$ for every $x \in X$. We have, for all $x \in X$,

$$(S \oplus T)_x(H_1 \oplus H_1') + W \oplus W' = (S_x(H_1) + W) \oplus (T_x(H_1') + W') = H_2 \oplus H_2'\,.$$

Also, $(S \oplus T)^{W \oplus W'}\colon X \to \mathcal{F}((H_1 \oplus H_1') \oplus (W \oplus W'), H_2 \oplus H_2')$ satisfies

$$\begin{aligned} (S \oplus T)_x^{W \oplus W'}((v,v'),(w,w')) &= (S \oplus T)_x(v,v') + (w,w') \\ &= (S_x(v) + w, T_x(v') + w') \\ &= (S^W \oplus T^{W'})_x((v,w),(v',w')) \end{aligned}$$

Observe that

$$\begin{aligned} \mathrm{Ker}\ (S \oplus T)^{W \oplus W'} = \{(x,((v,v'),(w,w'))) \in X \times ((H_1 \oplus H_1') \oplus (W \oplus W')) : \\ (v,w) \in \mathrm{Ker}\ S_x^W, (v',w') \in \mathrm{Ker}\ T_x^{W'}\} \end{aligned}$$

is isomorphic to

$$\begin{aligned} \mathrm{Ker}\ (S^W \oplus T^{W'}) = \{(x,((v,w),(v',w'))) \in X \times ((H_1 \oplus W) \oplus (H_1' \oplus W')) : \\ (v,w) \in \mathrm{Ker}\ S_x^W, (v',w') \in \mathrm{Ker}\ T_x^{W'}\} \end{aligned}$$

via

$$\mathrm{Ker}\ (S \oplus T)^{W \oplus W'} \ni (x,((v,v'),(w,w'))) \longmapsto (x,((v,w),(v',w'))) \in \mathrm{Ker}\ (S^W \oplus T^{W'})\,,$$

which in turn is isomorphic to

$$\begin{aligned}\text{Ker } S^W \oplus \text{Ker } T^{W'} &= \{((x,(v,w)),(y,(v',w'))) \in \text{Ker } S^W \times \text{Ker } T^{W'} : x = y\} \\ &= \{((x,(v,w)),(x,(v',w'))) : x \in X, (v,w) \in \text{Ker } S^W, (v',w') \in \text{Ker } T^{W'}\}\end{aligned}$$

via

$$\text{Ker } (S^W \oplus T^{W'}) \ni (x,((v,w),(v',w'))) \longmapsto (x,((v,w),(v',w'))) \in \text{Ker } S^W \oplus \text{Ker } T^{W'} .$$

Therefore

$$\begin{aligned}\text{ind}_B(S \oplus T) &= [\text{Ker } (S \oplus T)^{W \oplus W'}] - [X \times (W \oplus W')] \\ &= [\text{Ker } (S^W \oplus T^{W'})] - [X \times (W \oplus W')] \\ &= [\text{Ker } S^W \oplus \text{Ker } T^{W'}] - [(X \times W) \oplus (X \times W')] \\ &= [\text{Ker } S^W] + [\text{Ker } T^W] - ([X \times W] + [X \times W']) \\ &= ([\text{Ker } S^W] - [X \times W]) + ([\text{Ker } T^{W'}] - [X \times W']) \\ &= \text{ind}_B(S) + \text{ind}_B(T) .\end{aligned}$$

□

Corollary 3.35. *Let X, S and T be as in Proposition 3.34. Then* $\text{ind}_B(S \oplus T) = \text{ind}_B(T \oplus S)$.

Proof. $\mathsf{K}(X)$ is an abelian group. □

Proposition 3.7 can be generalized by the as follows.

Proposition 3.36. *Let X be a compact Hausdorff space and H be a Hilbert space. If $S, T \colon X \to \mathcal{F}(H,H)$ are continuous maps, then*

$$\text{ind}_B(ST) = \text{ind}_B(S) + \text{ind}_B(T).$$

Proof. Let $I \colon X \to \mathcal{F}(H,H)$, $x \longmapsto id_H$. We obviously have $\text{ind}_B(I) = 0 \in \mathsf{K}(X)$ and Proposition 3.34 gives

$$\text{ind}_B(ST) = \text{ind}_B(ST) + \text{ind}_B(I) = \text{ind}_B(ST \oplus I) .$$

For $t \in \mathbb{R}$, consider

$$R_t = \begin{pmatrix} \cos(t) id_H & \sin(t) id_H \\ -\sin(t) id_H & \cos(t) id_H \end{pmatrix} \in \mathcal{B}(H \oplus H, H \oplus H).$$

It is clear that R_t is an invertible operator, which implies $R_t \in \mathcal{F}(H \oplus H, H \oplus H)$ for $t \in \mathbb{R}$. Proposition 3.7 allows us to define the continuous map $F \colon X \times [0, \pi/2] \to \mathcal{F}(H \oplus H, H \oplus H)$ by $F(x,t) = (S \oplus I)_x \circ R_t \circ (T \oplus I)_x \circ R_{-t}$. Straightforward calculations give $F(x,0) = (ST \oplus I)_x$ and $F(x,\pi/2) = (S \oplus T)_x$ for every $x \in X$, so that F is a homotopy between $ST \oplus I$ and $S \oplus T$. Thus, by Propositions 3.33 and 3.34,

$$\text{ind}_B(ST) = \text{ind}_B(ST \oplus I) = \text{ind}_B(S \oplus T) = \text{ind}_B(S) + \text{ind}_B(T),$$

as desired. □

Corollary 3.37. *Let X, S and T be as in Proposition 3.36. Then* $\mathrm{ind}_B(ST) = \mathrm{ind}_B(TS)$.

Proof. $\mathsf{K}(X)$ is an abelian group. □

Lemma 3.38. *Let X be a compact Hausdorff space, H_1, H_2 and H_3 be Hilbert spaces.*

(*a*) *If $T\colon X \to \mathcal{F}(H_1, H_2)$ is continuous and $I \in \mathcal{B}(H_2, H_3)$ is invertible, then*

$$\mathrm{ind}_B(IT) = \mathrm{ind}_B(T).$$

(*b*) *If $I \in \mathcal{B}(H_1, H_2)$ is invertible and $T\colon X \to \mathcal{F}(H_2, H_3)$ is continuous, then*

$$\mathrm{ind}_B(TI) = \mathrm{ind}_B(T).$$

Proof. (*a*) Let $W \subseteq H_2$ be a finite dimensional subspace such that $W + T_x(H_1) = H_2$ for every $x \in X$. Consider $W' \doteq I(W) \subseteq H_3$, which has the same dimension of W because I is invertible. We have, for every $x \in X$,

$$W' + IT_x(H_1) = I(W) + IT_x(H_1) = I(W + T_x(H_1)) = I(H_2) = H_3\,.$$

Then

$$\mathrm{ind}_B(T) = [\mathrm{Ker}\ T^W] - [X \times W] \qquad \text{and} \qquad \mathrm{ind}_B(IT) = [\mathrm{Ker}\ (IT)^{W'}] - [X \times W']\,,$$

where $T^W : X \to \mathcal{F}(H_1 \oplus W, H_2)$ is given by $T^W_x(v, w) = T_x(v) + w$, and $(IT)^{W'} : X \to \mathcal{F}(H_1, H_3)$ is given by $(IT)^{W'}_x(v, w') = IT_x(v) + w'$.

Since $\dim W' = \dim W$, we have $[X \times W] = [X \times W']$. Besides, the vector bundle

$$\begin{aligned}\mathrm{Ker}\ T^W &= \{(x,(v,w)) \in X \times (H_1 \oplus W) : (v,w) \in \mathrm{Ker}\ T^W_x\}\\ &= \{(x,(v,w) \in X \times (H_1 \oplus W) : T_x(v) = -w\}\end{aligned}$$

is isomorphic to

$$\begin{aligned}\mathrm{Ker}\ (IT)^{W'} &= \{(x,(v,w')) \in X \times (H_1 \oplus W') : (v,w') \in \mathrm{Ker}\ (IT)^{W'}_x\}\\ &= \{(x,(v,w') \in X \times (H_1 \oplus W') : IT_x(v) = -w'\}\end{aligned}$$

via

$$\mathrm{Ker}\ T^W \ni (x,(v,w)) \longmapsto (x,(v,I(w))) \in \mathrm{Ker}\ (IT)^{W'}\,.$$

This shows that $[\mathrm{Ker}\ T^W] = [\mathrm{Ker}\ (IT)^{W'}]$, from where the result follows.

(*b*) Let $W \subseteq H_3$ be a finite dimensional subspace such that $W + T_x(H_2) = H_3$ for every $x \in X$. We have, for every $x \in X$, $W + T_x I(H_1) = W + T_x(H_2) = H_3$. Then

$$\mathrm{ind}_B(T) = [\mathrm{Ker}\ T^W] - [X \times W] \qquad \text{and} \qquad \mathrm{ind}_B(TI) = [\mathrm{Ker}\ (TI)^W] - [X \times W]\,,$$

where $T^W\colon X \to \mathcal{F}(H_2 \oplus W, H_3)$ is given by $T^W_x(v_2, w) = T_x(v_2) + w$, and $(TI)^W\colon X \to \mathcal{F}(H_1, H_3)$ is given by $(TI)^W_x(v_1, w) = T_x I(v_1) + w$.

The vector bundle

$$\text{Ker } (TI)^W = \{(x, (v_1, w)) \in X \times (H_1 \oplus W) : T_x I(v_1) = -w\}$$

is isomorphic to

$$\text{Ker } T^W = \{(x, (v_2, w)) \in X \times (H_2 \oplus W) : T_x(v_2) = -w\}$$

via

$$\text{Ker } (TI)^W \ni (x, (v_1, w)) \longmapsto (x, (I(v_1), w)) \in \text{Ker } T^W .$$

Therefore, we have $[\text{Ker } T^W] - [X \times W] = [\text{Ker } (TI)^W] - [X \times W]$, concluding the proof. □

Proposition 3.39. *Let X be a compact Hausdorff space, and H_1, H_2 and H_3 be infinite dimensional Hilbert spaces. If $T\colon X \to \mathcal{F}(H_1, H_2)$ and $S\colon X \to \mathcal{F}(H_2, H_3)$ are continuous maps, then*

$$\text{ind}_B(ST) = \text{ind}_B(S) + \text{ind}_B(T) .$$

Proof. The existence of Fredholm operators between H_1, H_2 and H_3 gives Banach space isomorphisms $I_{12}\colon H_2 \to H_1$ and $I_{13}\colon H_3 \to H_1$ (see Proposition 3.11). Notice that $I_{13}ST$ is a family of Fredholm operators from H_1 to itself, as well as $I_{13}SI_{12}^{-1}$ and $I_{12}T$. This allows us to apply Proposition 3.36 to obtain the equality

$$\text{ind}_B(I_{13}ST) = \text{ind}_B((I_{13}SI_{12}^{-1})(I_{12}T)) = \text{ind}_B(I_{13}SI_{12}^{-1}) + \text{ind}_B(I_{12}T) . \tag{3.4}$$

By Lemma 3.38, we have that $\text{ind}_B(I_{13}ST) = \text{ind}_B(ST)$, $\text{ind}_B(I_{13}SI_{12}^{-1}) = \text{ind}_B(S)$ and $\text{ind}_B(I_{12}T) = \text{ind}_B(T)$, so that equation (3.4) becomes

$$\text{ind}_B(ST) = \text{ind}_B(S) + \text{ind}_B(T) .$$ □

Proposition 3.40. *Let X be a compact Hausdorff space and $T\colon X \to \mathcal{F}(H_1, H_2)$ be continuous. If T_x is surjective for every $x \in X$, then $\text{ind}_B(T) = [\text{Ker } T]$.*

Proof. By Corollary 3.24 we have that Ker T, seen as a topological subspace of $X \times H_1$, is a vector bundle over X. Since T_x is surjective for every $x \in X$, we can choose $W \doteq \{0\} \subseteq H_2$ to see that $T_x(H_1) + W = H_2 + \{0\} = H_2$ for every $x \in X$. Therefore,

$$\text{ind}_B(T) = [\text{Ker } T^W] - [X \times W] = [\text{Ker } T^{\{0\}}] - [X \times \{0\}] = [\text{Ker } T^{\{0\}}] ,$$

where $T^{\{0\}}\colon X \to \mathcal{F}(H_1 \oplus \{0\}, H_2)$ is given by $T^{\{0\}}_x(v, 0) = T_x(v) + 0 = T_x(v)$. Observe that

$$\text{Ker } T = \{(x, v) \in X \times H_1 : v \in \text{Ker } T_x\}$$

is isomorphic to

$$\text{Ker } T^{\{0\}} = \{(x, (v, 0)) \in X \times (H_1 \oplus \{0\}) : (v, 0) \in \text{Ker } T^{\{0\}}_x\}$$

via

$$\text{Ker } T \ni (x,v) \longmapsto (x,(v,0)) \in \text{Ker } T^{\{0\}} .$$

Therefore $\text{ind}_B(T) = [\text{Ker } T^{\{0\}}] = [\text{Ker } T]$. □

Corollary 3.41. *Let X be a compact Hausdorff space and $T: X \to \mathcal{F}(H_1, H_2)$ be a continuous family of invertible operators. Then* $\text{ind}_B(T) = 0$.

Proof. By Proposition 3.40, we have $\text{ind}_B(T) = [\text{Ker } T]$. Notice that

$$\text{Ker } T = \bigsqcup_{x \in X} \text{Ker } T_x = \bigsqcup_{x\in X} \{0\} = X \times \{0\} ,$$

from where it follows that $\text{ind}_B(T) = [X \times \{0\}] = 0 \in \mathsf{K}(X)$. □

Proposition 3.42. *Let X be a compact Hausdorff space, H be a Hilbert space and $T: X \to \mathcal{F}(H, H)$ be a continuous family of self-adjoint operators. Then* $\text{ind}_B(T) = 0$.

Proof. Preliminarily, note that a bounded operator $L \in \mathcal{B}(H, H)$ satisfies $\lambda - (z + L) = (\lambda - z) - L$ for every $\lambda, z \in \mathbb{C}$, so that

$$\lambda \in \sigma(z + L) \iff \lambda - z \in \sigma(L)$$

and therefore $\sigma(z + L) = z + \sigma(L)$ for every $z \in \mathbb{C}$ (as subsets of $\mathbb{C}$).

For $x \in X$, we have a self-adjoint operator $T_x \in \mathcal{F}(H, H)$. It is a well known fact that the spectrum of a selfadjoint operator is contained in $\mathbb{R}$ (see [Sch12], for example), so that $\sigma(T_x) \subseteq \mathbb{R}$. Then, for $t \in [0, 1]$ it is true that $\sigma(it+T_x) = it+\sigma(T_x) \subseteq it+\mathbb{R}$. This gives $0 \notin \sigma(it+T_x)$ whenever $t \in (0, 1]$, which means that $it + T_x$ is invertible for every $t \in (0, 1]$. The map $F: X \times [0, 1] \to \mathcal{F}(H, H)$, $F(x, t) = it + T_x$, is a homotopy between T and the family of invertible operators $i + T$. Applying Propositions 3.33 and 3.41, we obtain $\text{ind}_B(T) = \text{ind}_B(i + T) = 0$. □

Corollary 3.43. *Let X be a compact Hausdorff space and $T: X \to \mathcal{F}(H_1, H_2)$ be continuous. Suppose that either $H_1 = H_2$ or that both H_1 and H_2 are infinite dimensional. The following equality holds in* $\mathsf{K}(X)$*:*

$$\text{ind}_B(T) = -\text{ind}_B(T^*) .$$

Proof. It follows from the fact that the product $T^*T: X \to \mathcal{F}(H_1, H_1)$ is a family of selfadjoint Fredholm operators and from either Proposition 3.36 or Proposition 3.39 that

$$0 = \text{ind}_B(T^*T) = \text{ind}_B(T^*) + \text{ind}_B(T) ,$$

as desired. □

3.3.2 Second Approach

In this section, we follow closely what was done in the Appendix of [Ati67] and in [Muk13].

We begin with some preliminary results. I learned the following proof from Ruy Exel in [1].

Lemma 3.44 (Ruy Exel)**.** *Let X be a topological space and $T: X \to \mathcal{F}(H_1, H_2)$ be continuous. Let $V \subseteq H_1$ be a closed linear subspace of finite codimension such that $V \cap \text{Ker } T_x = \{0\}$ for every $x \in X$. For $x \in X$, let $P_x \in \mathcal{B}(H_2, H_2)$ be the orthogonal projection onto $T_x(V)$. Then the map $P: X \to \mathcal{B}(H_2, H_2)$, $x \longmapsto P_x$, is continuous.*

Proof. For $x \in X$, let $S_x \doteq T_x|_V \colon V \to H_2$. Since V has finite codimension and T_x induces a surjective linear map $H_1/V \to T_x(H_1)/T_x(V)$, we have that $T_x(H_1)/T_x(V)$ is finite dimensional. We have Ker $S_x = \{0\}$ because $V \cap \text{Ker } T_x = \{0\}$, and Coker $S_x = H_2/S_x(V) = H_2/T_x(V)$ is finite dimensional because $\dim H_2/T_x(V) = \dim H_2/T_x(H_1) + \dim T_x(H_1)/T_x(V)$, so that S_x is Fredholm. Then $S_x^* \colon H_2 \to V$ is also Fredholm, and $S_x^*(H_2) = \overline{S_x^*(H_2)} = (\text{Ker } S_x)^\perp = V$. The map $S \colon X \to \mathcal{F}(V, H_2)$, $x \longmapsto S_x$, is continuous since

$$\|S_x - S_y\| = \|T_x|_V - T_y|_V\| \leq \|T_x - T_y\| \,.$$

Consider the product $S^*S \colon X \to \mathcal{F}(V, V)$. For $x \in X$, we have that $S_x^* S_x$ is a nonnegative selfadjoint Fredholm operator. Also, $\langle S_x^* S_x(v), v\rangle = \|S_x(v)\|^2$ gives Ker $S_x^* S_x \subseteq \text{Ker } S_x = \{0\}$. So $S_x^* S_x(V) = (\text{Ker } S_x^* S_x)^\perp = V$. By the Open Mapping Theorem, $S_x^* S_x$ is an isomorphism. This allows us to define the continuous map $R \doteq S(S^*S)^{-1/2} \colon X \to \mathcal{B}(V, H_2)$. Notice that, for $x \in X$, $R_x^* = (S_x^* S_x)^{-1/2} S_x^*$ and

$$R_x^* R_x = (S_x^* S_x)^{-1/2} S_x^* S_x (S_x^* S_x)^{-1/2} = id_V$$

and this implies

$$(R_x R_x^*)^2 = R_x R_x^* R_x R_x^* = R_x R_x^* \,.$$

Since $(R_x R_x^*)^* = R_x R_x^*$, we have that $R_x R_x^*$ is the orthogonal projection onto

$$\begin{aligned} R_x R_x^*(H_2) &= S_x (S_x^* S_x)^{-1/2} (S_x^* S_x)^{-1/2} S_x^*(H_2) \\ &= S_x (S_x^* S_x)^{-1} S_x^*(H_2) \\ &= S_x (S_x^* S_x)^{-1}(V) \\ &= S_x(V) \\ &= T_x(V) \,. \end{aligned}$$

Thus, $RR^* = P$ and we conclude that P is continuous. □

Proposition 3.45. *Let $V \subseteq H_1$ and $W \subseteq H_2$ be closed linear subspaces. For $S \in \mathcal{B}(H_1, H_2)$, define $\phi_S \colon V \oplus W \to H_2$ by $\phi_S(v, w) = S(v) + w$. The map $\phi \colon \mathcal{B}(H_1, H_2) \to \mathcal{B}(V \oplus W, H_2)$, $S \longmapsto \phi_S$, is continuous. Moreover, if $S_0 \in \mathcal{B}(H_1, H_2)$ is such that ϕ_{S_0} is a Banach space isomorphism, then there exists an open set $U \subseteq \mathcal{B}(H_1, H_2)$, with $S_0 \in U$, such that for all $S \in U$,*

(*i*) $V \cap \text{Ker } S = \{0\}$

(*ii*) *$S(V)$ is closed in H_2*

(*iii*) *the map $W \to H_2/S(V)$, given by $w \longmapsto w + S(V)$, is a Banach space isomorphism.*

Proof. The map ϕ is continuous because from

$$\|(\phi_S - \phi_{S'})(v, w)\| = \|(S - S')(v)\| \leq \|S - S'\|\|v\| \leq \|S - S'\|\|(v, w)\|$$

one obtains $\|\phi_S - \phi_{S'}\| \leq \|S - S'\|$.

Now, since the set of invertible elements of $\mathcal{B}(V \oplus W, H_2)$ is open (see Proposition 1.25) and ϕ_{S_0} is invertible, there exists U open in $\mathcal{B}(H_1, H_2)$ with $S_0 \in U$ and such that ϕ_S is invertible for every $S \in U$. Let $S \in U$. One can easily see that $(V \cap \operatorname{Ker} S) \oplus \{0\} \subseteq \operatorname{Ker} \phi_S$, showing that $V \cap \operatorname{Ker} S = \{0\}$ since ϕ_S is injective. This proves (i). (ii) follows from $S(V) = \phi_S(V \oplus \{0\})$. Note that

$$W \to \frac{V \oplus W}{V \oplus \{0\}}, \qquad w \longmapsto (0, w) + (V \oplus \{0\}),$$

is a linear isomorphism, as well as the map $\dfrac{V \oplus W}{V \oplus \{0\}} \longrightarrow \dfrac{\phi_S(V \oplus W)}{\phi_S(V \oplus \{0\})} = \dfrac{H_2}{S(V)}$, given by

$$(v, w) + (V \oplus \{0\}) \longmapsto \phi_S(v, w) + \phi_S(V \oplus \{0\}) = w + S(V)\,.$$

The composition of these maps is precisely the map described in (iii), concluding the proof. □

Proposition 3.46. *Let $T \in \mathcal{F}(H_1, H_2)$ and let $V \subseteq H_1$ be a closed linear subspace of finite codimension such that $V \cap \operatorname{Ker} T = \{0\}$. Then $H_2/T(V)$ is finite dimensional and $T(V)$ is closed in H_2.*

Moreover, there is an open set $U \subseteq \mathcal{B}(H_1, H_2)$, with $T \in U$, such that for all $S \in U$,

(i) *$V \cap \operatorname{Ker} S = \{0\}$*

(ii) *$S(V)$ is closed in H_2*

(iii) *the map $T(V)^{\perp} \to H_2/S(V)$, given by $w \longmapsto w + S(V)$, is a Banach space isomorphism.*

Proof. Since V has finite codimension and T induces a surjective linear map $H_1/V \to T(H_1)/T(V)$, we have that $T(H_1)/T(V)$ is finite dimensional. The sequence

$$0 \longrightarrow T(H_1)/T(V) \longrightarrow H_2/T(V) \longrightarrow H_2/T(H_1) \longrightarrow 0$$

is a short exact sequence of vector spaces, and therefore it splits. Then $H_2/T(V)$ is isomorphic to $T(H_1)/T(V) \oplus H_2/T(H_1)$ and one has $\dim H_2/T(V) < \infty$. The operator $T_1 \doteq T|_V \colon V \to H_2$ is Fredholm, so that $T(V) = T_1(V)$ is closed in H_2.

Now, let $W \doteq T(V)^{\perp}$. The linear map $T' \colon V \oplus W \to H_2$, $T'(v, w) = T(v) + w$, is a continuous bijection since $V \cap \operatorname{Ker} T = \{0\}$ and $H_2 = T(V) \oplus W$. By the Open Mapping Theorem, T' is an isomorphism. We can then apply Proposition 3.45 noting that $T' = \phi_T$ to obtain the neighborhood U of T in $\mathcal{B}(H_1, H_2)$ satisfying the desired properties. □

Let $T \in \mathcal{F}(H_1, H_2)$. Take $V \doteq (\operatorname{Ker} T)^{\perp}$ and let U be given by Proposition 3.46. Fix $S \in U$. We have a surjection $H_2/S(V) \longrightarrow H_2/S(H_1)$, $y + S(V) \longmapsto y + S(H_1)$, so that Coker S has finite dimension (recall that $H_2/S(V) \cong T(V)^{\perp}$ is finite dimensional). Similarly, taking $\tilde{V} \doteq (\operatorname{Ker} T^*)^{\perp}$, we obtain an open set $\tilde{U} \subseteq \mathcal{B}(H_2, H_1)$ satisfying properties corresponding to (i), (ii) and (iii) of Proposition 3.46. Denote by U^* the set $\{S^* : S \in U\}$, which is an open set because the adjoint map is a homeomorphism. Without loss of generality, eventually replacing $\tilde{U}$ by $\tilde{U} \cap U^*$ and U by $(\tilde{U})^* \cap U = (\tilde{U} \cap U^*)^*$, we can assume that $\tilde{U} = U^*$. By the above argument, Coker S^* is finite dimensional, so that

$$\operatorname{Ker} S = S^*(H_2)^{\perp} \cong H_1/S^*(H_2) = \operatorname{Coker} S^*$$

is also finite dimensional. Therefore, every $S \in U$ is Fredholm.

Remark 3.47. The above discussion shows that we can eventually shrink the open subset U of $\mathcal{B}(H_1, H_2)$ given by Proposition 3.46 to obtain a neighborhood of T that is entirely contained in $\mathcal{F}(H_1, H_2)$.

In particular, we obtain the following

Corollary 3.48. *$\mathcal{F}(H_1, H_2)$ is open in $\mathcal{B}(H_1, H_2)$.*

Theorem 3.49. *Let $T \in \mathcal{F}(H_1, H_2)$ and let $V \subseteq H_1$ be a closed linear subspace of finite codimension such that $V \cap \operatorname{Ker} T = \{0\}$. Then there exists an open set $U \subseteq \mathcal{B}(H_1, H_2)$, with $T \in U$, such that the disjoint union*

$$\bigsqcup_{S \in U} H_2/S(V)\,,$$

topologized as a quotient space of $U \times H_2$, has the structure of a vector bundle over U.

Proof. Consider U as described in Proposition 3.46. By Remark 3.47, we can assume that U is entirely contained in $\mathcal{F}(H_1, H_2)$.

Let us be more explicit about the topology of $\bigsqcup_{S \in U} H_2/S(V)$. In $U \times H_2$, consider the equivalence relation

$$(S, x) \sim (T, y) \iff \big(S = T \text{ and } y - x \in S(V)\big).$$

The quotient space $(U \times H_2)/{\sim}$ is precisely $\bigsqcup_{S \in U} H_2/S(V)$. It is clear that the projection onto the first coordinate $U \times H_2 \to U$, $(S, x) \longmapsto S$, induces a continuous map $p\colon \bigsqcup_{S \in U} H_2/S(V) \longrightarrow U$, acting as $p(x + S(V)) = S$. For every $S \in U$, we have that $p^{-1}(S) = H_2/S(V)$.

Given $S \in U$, let $S|_V \in \mathcal{B}(V, H_2)$ be the restriction of S to V. Since $V \cap \operatorname{Ker} S = \{0\}$ and $S(V)$ has finite codimension, we have that $S|_V$ is Fredholm. The dimension of $\operatorname{Ker}\,(S|_V)^* = S(V)^\perp$ is constant on $S \in U$ because $S(V)^\perp \cong H_2/S(V) \cong T(V)^\perp$ (due to Proposition 3.46), so that we can apply Proposition 3.23 to conclude that the disjoint union

$$\bigsqcup_{S \in U} S(V)^\perp = \bigsqcup_{S \in U} \operatorname{Ker}\,(S|_V)^*,$$

seen as a topological subspace of $U \times H_2$, is a vector bundle over U (its projection arises from restricting the projection onto the first coordinate $U \times H_2 \to U$).

To conclude that $\bigsqcup_{S \in U} H_2/S(V)$ is a vector bundle over U, we are going to prove that there exists a homeomorphism between

$$E \doteq \bigsqcup_{S \in U} S(V)^\perp \subseteq U \times H_2 \quad \text{and} \quad F \doteq \bigsqcup_{S \in U} H_2/S(V) = (U \times H_2)/{\sim}$$

that commutes with projections and is linear on fibers. The restriction of the quotient projection $U \times H_2 \to F$ to E is a continuous map $\varphi\colon E \to F$, given by $\varphi(S, x) = x + S(V)$, that clearly commutes with projections and is linear on fibers. Let us construct a continuous inverse for φ. For $S \in U$, let $P_S\colon H_2 \to H_2$ be the orthogonal projection onto $S(V)$. By Lemma 3.44, the map $U \ni S \longmapsto P_S \in \mathcal{B}(H_2, H_2)$ is continuous, from where we obtain the continuity of the map $U \times H_2 \to E$, $(S, x) \longmapsto (S, x - P_S(x))$. Notice that the former map is compatible with the equivalence relation $\sim$, so that it induces a continuous map $\psi\colon F \to E$ satisfying $\psi(x + S(V)) = (S, x - P_S(x))$. It is straightforward to see that ψ is the desired inverse for φ. $\square$

Proposition 3.50. *Let X be a compact topological space and $T\colon X \to \mathcal{F}(H_1, H_2)$, $x \longmapsto T_x$, be a continuous map. Then*

- (i) *there exists a closed subspace $V \subseteq H_1$ of finite codimension such that $V \cap \operatorname{Ker} T_x = \{0\}$ for every $x \in X$.*
- (ii) *the set*

$$\bigsqcup_{x\in X} H_2/T_x(V)\,,$$

topologized as a quotient space of $X \times H_2$, is a vector bundle over X. This vector bundle will be denoted by $H_2/T(V)$.

Proof. Fix $x \in X$ and let $V_x \doteq (\operatorname{Ker} T_x)^{\perp}$. From Proposition 3.46 and Remark 3.47, there exists an open neighborhood $\mathcal{U}_x$ of T_x in $\mathcal{F}(H_1, H_2)$ such that each $S \in \mathcal{U}_x$ satisfies $V_x \cap \operatorname{Ker} S = \{0\}$. Consider $U_x \doteq T^{-1}(\mathcal{U}_x) \subseteq X$. Notice that $V_x \cap \operatorname{Ker} T_y = \{0\}$ for every $y \in U_x$. Compactness of X provides a finite cover $U_{x_1}, \cdots, U_{x_n}$ of X. We can then define $V \doteq \bigcap_{k=1}^{n} V_{x_k}$, which satisfies (i).

Now, for each $x \in X$ one can apply Theorem 3.49 to T_x to obtain an open neighborhood U of x such that $\bigsqcup_{y\in X} H_2/T_y(V)\big|_U = \bigsqcup_{y\in U} H_2/T_y(V)$ is trivial, proving (ii). □

We need one more technical result before we are able to give Atiyah's definition of the index of a family of Fredholm operators.

Proposition 3.51. *Let X be a compact space and $T\colon X \to \mathcal{F}(H_1, H_2)$ be continuous. Let $V, V' \subseteq H_1$ be closed linear subspaces with finite codimension such that $V \cap \operatorname{Ker} T_x = \{0\} = V' \cap \operatorname{Ker} T_x$ for every $x \in X$. Then we have the following equality in $\mathsf{K}(X)$:*

$$[X \times (H_1/V)] - [H_2/T(V)] = [X \times (H_1/V')] - [H_2/T(V')]\,.$$

Proof. First assume $V' \subseteq V$. We have the short exact sequence of trivial vector bundles

$$0 \longrightarrow X \times (V/V') \xrightarrow{\;f\;} X \times (H_1/V') \xrightarrow{\;g\;} X \times (H_1/V) \longrightarrow 0$$

where f and g are bundle morphisms induced by the inclusion $V \hookrightarrow H_1$ and the surjective map $H_1/V' \twoheadrightarrow H_1/V$, respectively. This sequence splits since these vector bundles are trivial and every short exact sequence of vector spaces splits, so that

$$X \times (H_1/V') \cong (X \times (H_1/V)) \oplus (X \times (V/V'))\,. \tag{3.5}$$

Notice that T induces a bundle isomorphism between $X \times (V/V')$ and

$$T(V)/T(V') = \bigsqcup_{x\in X} T_x(V)/T_x(V')$$

as follows: we compose the continuous map $T'\colon X \times V \to X \times H_2$ given by $T'(x, v) = (x, T_x(v))$ with the quotient projection $X \times H_2 \longrightarrow H_2/T(V')$, obtaining a continuous map that sends $(x, v) \in X \times V$ to $T_x(v) + T_x(V') \in H_2/T(V')$. Notice that $(x, v_1 - v_2) \in X \times V'$ is sent to $T_x(v_1 - v_2) + T_x(V') = 0$ via this composition. This allows us to define $\varphi\colon X \times (V/V') \to H_2/T(V')$ by $\varphi(x, v + V') =$

$T_x(v) + T_x(V')$, making the diagram

$$\begin{array}{ccc} X \times V & \xrightarrow{T'} & X \times H_2 \\ \downarrow & \searrow & \downarrow \\ X \times (V/V') & \overset{\varphi}{\dashrightarrow} & H_2/T(V') \end{array}$$

commutative. Since $V \cap \operatorname{Ker} T = \{0\}$, we have that $T_x(v) \in T_x(V') \implies v \in V'$, implying that φ is injective. Thus φ is a bundle isomorphism between $X \times V/V'$ and $\varphi(X \times V/V') = T(V)/T(V')$ due to Propositions 2.22 and 2.12. Besides, we have the short exact sequence

$$0 \longrightarrow T(V)/T(V') \xrightarrow{\ i\ } H_2/T(V') \xrightarrow{\ j\ } H_2/T(V) \longrightarrow 0 \tag{3.6}$$

where i is an inclusion and j is defined via the diagram (just like the map φ above)

$$\begin{array}{ccc} X \times H_2 & \xrightarrow{id_{X\times H_2}} & X \times H_2 \\ \downarrow & \searrow & \downarrow \\ H_2/T(V') & \overset{j}{\dashrightarrow} & H_2/T(V) \end{array}$$

(explicitly, we have $j(v+T_x(V')) = v+T_x(V)$). Let us prove that sequence (3.6) splits. For $x \in X$, we consider $P_x \in \mathcal{B}(H_2, H_2)$ to be the orthogonal projection onto $T_x(V)$. Lemma 3.44 shows that $P\colon X \to \mathcal{B}(H_2, H_2)$, $x \longmapsto P_x$, is continuous. The continuous map $p\colon H_2/T(V') \to H_2/T(V')$ defined by the commutative diagram

$$\begin{array}{ccc} X \times H_2 & \xrightarrow{P} & X \times H_2 \\ \downarrow & \searrow & \downarrow \\ H_2/T(V') & \overset{p}{\dashrightarrow} & H_2/T(V') \end{array}$$

satisfies $p(v + T_x(V')) = P_x(v) + T_x(V')$. Thus

$$p \circ i(T_x(v) + T_x(V')) = p(T_x(v) + T_x(V')) = P_x(T_x(v)) + T_x(V') = T_x(v) + T_x(V')\,.$$

This shows that p defines a splitting at the left of the sequence (3.6). Therefore, by Proposition 2.30,

$$H_2/T(V') \cong H_2/T(V) \oplus T(V)/T(V') \cong H_2/T(V) \oplus (X \times (V/V'))\,. \tag{3.7}$$

By (3.5) and (3.7), we have

$$[X \times (H_1/V')] = [X \times H_1/V] + [X \times (V/V')]$$

and

$$[H_2/T(V')] = [H_2/T(V)] + [X \times (V/V')],$$

from where it follows that

$$[X \times (H_1/V')] - [X \times (H_1/V)] = [X \times (V/V')] = [H_2/T(V')] - [H_2/T(V)]\,,$$

which proves the desired result.

Now we can drop the assumption $V' \subseteq V$ simply noticing that $V \cap V'$ is a closed subspace of H_1 and

$$\frac{H_1}{V \cap V'} \cong \frac{H_1}{V} \oplus \frac{V}{V \cap V'} \cong \frac{H_1}{V} \oplus \frac{V + V'}{V'}$$

so that $V \cap V'$ has finite codimension. Besides, $(V \cap V') \cap \text{Ker } T_x \subseteq V \cap \text{Ker } T_x = \{0\}$ for every $x \in X$. From what we have proved above, it follows that

$$\begin{aligned}[][X \times (H_1/V)] - [H_2/T(V)] &= [X \times (H_1/(V \cap V'))] - [H_2/T(V \cap V')] \\ &= [X \times (H_1/V')] - [H_2/T(V')]\,.\end{aligned}$$

This concludes the proof. □

Definition 3.52. Let X be a compact space and $T\colon X \to \mathcal{F}(H_1, H_2)$ be continuous.The *A-index bundle of T* is

$$\text{ind}_A(T) = [X \times (H_1/V)] - [H_2/T(V)] \in \mathsf{K}(X)\,,$$

where V is a closed linear subspace of H_1 with finite codimension such that $V \cap \text{Ker } T_x = \{0\}$ for every $x \in X$.

Proposition 3.51 shows that the above definition of index depends only on T and not on the choice of V.

Remark 3.53. As in Remark 3.31, let $X = \{x_0\}$ and see the map $T\colon \{x_0\} \to \mathcal{F}(H_1, H_2)$ as a single Fredholm operator $T \doteq T_{x_0} \in \mathcal{F}(H_1, H_2)$. Choosing $V \doteq (\text{Ker } T)^\perp$, it follows that $H_1/V \cong \text{Ker } T$ and $H_2/T(V) \cong \text{Coker } T$. Recall that $\mathsf{Vect}\,\{x_0\} = \mathbb{N}$ (see Examples 2.45 and 2.48), so $\mathsf{K}(\{x_0\}) = \mathbb{Z}$ and this allows us to say that $\text{ind}_A(T)$ is the difference of integers $\dim \text{Ker } T - \dim \text{Coker } T$. This proves that the A-index bundle coincides with the classical Fredholm index in these conditions.

3.3.3 Equivalence of the two Indices

Our goal in this section is to prove that our two approaches to define the index bundle give the same object.

Let X be a compact Hausdorff space, H_1 and H_2 be Hilbert spaces, and $T\colon X \to \mathcal{F}(H_1, H_2)$ be a continuous map. Suppose that either $H_1 = H_2$ or that both H_1 and H_2 are infinite dimensional. We shall prove that $\text{ind}_A(T) = \text{ind}_B(T)$ in $\mathsf{K}(X)$.

Let $W \subseteq H_2$ be a finite dimensional linear subspace such that $T_x(H_1) + W = H_2$ for every $x \in X$. Recall that the B-index bundle of T is

$$\text{ind}_B(T) = [\text{Ker } T^W] - [X \times W]\,, \tag{3.8}$$

where $T^W\colon X \to \mathcal{F}(H_1 \oplus W, H_2)$ is defined by $T^W_x(v, w) = T_x(v) + w$.

Observe that $W^\perp$ is a closed linear subspace of finite codimension and, for $x \in X$,

$$W^\perp \cap \text{Ker } T^*_x = W^\perp \cap T_x(H_1)^\perp \subseteq (W + T_x(H_1))^\perp = H_2^\perp = \{0\}\,.$$

Therefore, we can take $V = W^\perp$ in Definition 3.52 and write

$$\mathrm{ind}_A(T^*) = [X \times H_2/W^\perp] - [H_1/T^*(W^\perp)] = [X \times W] - [H_1/T^*(W^\perp)]\,, \tag{3.9}$$

where $H_1/T^*(W^\perp) = \bigsqcup_{x\in X} H_1/T^*_x(W^\perp)$ is topologized as the quotient space of $X \times H_1$ obtained via the equivalence relation $(x,v) \sim (x',v') \iff \big(x = x' \text{ and } v - v' \in T^*_x(W^\perp)\big)$.

Consider the continuous map $\psi\colon \mathrm{Ker}\, T^W \to X \times H_1$ given by $\psi(x,(v,w)) = (x,v)$. Composing ψ with the quotient map $X \times H_1 \longrightarrow H_1/T^*(W^\perp)$, we obtain the continuous map $\varphi\colon \mathrm{Ker}\, T^W \to H_1/T^*(W^\perp)$ given by $\varphi(x,(v,w)) = v + T^*_x(W^\perp)$. Clearly, φ is fiberwise linear, being therefore a bundle morphism.

We are going to prove that φ is a bundle isomorphism. By Proposition 2.12, it suffices to prove that φ is fiberwise bijective. Fix $x \in X$. Since W is closed, we have that $T_x^{-1}(W) = T^*(W^\perp)^\perp$. Besides, every $(v,w) \in \mathrm{Ker}\, T^W_x \subseteq H_1 \oplus W$ satisfies $T_x(v) = -w$, so that $v \in T_x^{-1}(W) = T^*_x(W^\perp)^\perp$. Therefore, an element of $\mathrm{Ker}\, T^W_x$ is of the form $(v, T_x(-v))$ for a suitable $v \in T^*_x(W^\perp)^\perp$.

Since W has finite dimension, $T^*|_{W^\perp}\colon W^\perp \to H_1$ is also Fredholm, and therefore $T^*(W^\perp)$ is closed in H_1. We can then write

$$H_1 = T^*_x(W^\perp) \oplus T^*_x(W^\perp)^\perp = T^*_x(W^\perp) \oplus T_x^{-1}(W).$$

For every $v = v_1 + v_2 \in T^*_x(W^\perp) \oplus T_x^{-1}(W)$, with $v_1 \in T^*_x(W^\perp)$ and $v_2 \in T_x^{-1}(W)$, we have

$$v + T^*_x(W^\perp) = v_2 + T^*_x(W^\perp) = \varphi(x,(v_2, T_x(-v_2)))$$

which proves that $\varphi_x\colon \mathrm{Ker}\, T^W_x \to H_1/T^*_x(W^\perp)$ is surjective.

On the other hand, if we let $(v, T_x(-v)), (v', T_x(-v')) \in \mathrm{Ker}\, T^W_x$ be such that $v, v' \in T^*_x(W^\perp)^\perp$, it follows that

$$\begin{aligned} \varphi(x,(v,T_x(-v))) = \varphi(x,(v',T_x(-v'))) &\implies v + T^*_x(W^\perp) = v' + T^*_x(W^\perp) \\ &\implies v - v' \in T^*_x(W^\perp) \\ &\implies v = v'\,, \end{aligned}$$

proving that φ_x is injective.

In conclusion, $\varphi\colon \mathrm{Ker}\, T^W \to H_1/T^*(W^\perp)$ is a bundle isomorphism.

Comparing equations (3.8) and (3.9) and using Corollary 3.43, it follows that

$$\mathrm{ind}_B(T^*) = -\mathrm{ind}_B(T) = \mathrm{ind}_A(T^*)\,.$$

Applying the above argument to T^*, we obtain

Theorem 3.54. *Let X be a compact Hausdorff space. Assume that either $H_1 = H_2$ or that both H_1 and H_2 have infinite dimension. Every continuous map $T\colon X \to \mathcal{F}(H_1,H_2)$ satisfies* $\mathrm{ind}_A(T) = \mathrm{ind}_B(T)$.

Definition 3.55. Let X be a compact Hausdorff topological space. Assume that either $H_1 = H_2$ or that both H_1 and H_2 have infinite dimension. The *index bundle* of a continuous map $T\colon X \to$

$\mathcal{F}(H_1, H_2)$ is

$$\operatorname{ind}(T) \doteq \operatorname{ind}_A(T) = \operatorname{ind}_B(T)\,.$$

To sum up, given a compact Hausdorff topological space X, to each pair of Hilbert spaces (H_1, H_2) such that either $H_1 = H_2$ or both H_1 and H_2 are infinite dimensional, we have a map $\operatorname{ind}\colon C(X, \mathcal{F}(H_1, H_2)) \to \mathsf{K}(X)$, from the set of continuous maps $X \to \mathcal{F}(H_1, H_2)$ into the K-group of X, satisfying the following equations in $\mathsf{K}(X)$:

(a) $\operatorname{ind}(S) = \operatorname{ind}(T)$ if S is homotopic to T;

(b) $\operatorname{ind}(S \oplus T) = \operatorname{ind}(S) + \operatorname{ind}(T)$;

(c) $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$;

(d) $\operatorname{ind}(T) = 0$ if T_x is invertible for every $x \in X$;

(e) $\operatorname{ind}(T) = 0$ if T_x is self-adjoint for every $x \in X$;

(f) $\operatorname{ind}(T) = -\operatorname{ind}(T^*)$.

From (a), it follows that $\operatorname{ind}\colon C(X, \mathcal{F}(H_1, H_2)) \to \mathsf{K}(X)$ induces a map $[X, \mathcal{F}(H_1, H_2)] \to \mathsf{K}(X)$, defined on the set of homotopy classes of continuous maps $X \to \mathcal{F}(H_1, H_2)$, which we will also denote by ind. This latter map is called *families-index.*

3.4 The Atiyah-Jänich Theorem

In this section, X will denote a compact Hausdorff topological space as well as H will denote an infinite dimensional Hilbert space. For simplicity, we will denote $\mathcal{B}(H, H)$ by $\mathcal{B}(H)$, and $\mathcal{F}(H, H)$ by $\mathcal{F}(H)$.

The monoid structure on $\mathcal{F}(H)$ induces a monoid structure in $C(X, \mathcal{F}(H))$ under the pointwise product of families of operators. If $F\colon X \times [0,1] \to \mathcal{F}(H)$ is a homotopy between $S, S' \in C(X, \mathcal{F}(H))$ and $G\colon X \times [0,1] \to \mathcal{F}(H)$ is a homotopy between $T, T' \in C(X, \mathcal{F}(H))$, then FG is a homotopy between ST and $S'T'$. Therefore, the pointwise product on $C(X, \mathcal{F}(H))$ induces a monoid structure in $[X, \mathcal{F}(H)]$, the set of homotopy classes of continuous maps $X \to \mathcal{F}(H)$.

From (a) and (c) above, we have

Theorem 3.56. *The families-index*

$$\operatorname{ind}\colon [X, \mathcal{F}(H)] \longrightarrow \mathsf{K}(X)$$

is a monoid morphism.

Lemma 3.57. *Let $\mathcal{B} = \{e_i\}_{i\in\mathbb{N}} \sqcup \{e_\alpha\}_{\alpha\in A}$ be a Hilbert basis for H (we could take A to be the empty set in case H is separable). For $k \in \mathbb{Z}$, let $S_k = S_k^{(\mathcal{B})} \in \mathcal{F}(H)$ be the operator defined in Example 3.2 and denote by $S_k\colon X \to \mathcal{F}(H)$ the constant map $x \longmapsto S_k$. If $k \geq 0$, we have*

$$\operatorname{ind}(S_k) = [X \times \mathbb{C}^k] \qquad \textit{and} \qquad \operatorname{ind}(S_{-k}) = -[X \times \mathbb{C}^k]\,.$$

Proof. Let $k \geq 0$. Recall that the linear maps $S_k = S_k^{(\mathcal{B})} \in \mathcal{F}(H)$ and $S_{-k} = S_{-k}^{(\mathcal{B})} \in \mathcal{F}(H)$ are defined by

$$S_k^{(\mathcal{B})}(e_j) = \begin{cases} e_{j-k} & \text{if } j \in \mathbb{N} \text{ and } j > k, \\ 0 & \text{if } j \in \mathbb{N} \text{ and } j \leq k, \\ e_j & \text{if } j \in A. \end{cases} \qquad \text{and} \qquad S_{-k}^{(\mathcal{B})}(e_j) = \begin{cases} e_{j+k} & \text{if } j \in \mathbb{N}, \\ e_j & \text{if } j \in A, \end{cases}$$

Since the operator $S_k \in \mathcal{F}(H)$ is surjective, we have that the index of the family S_k is given by $\operatorname{ind}(S_k) = [\operatorname{Ker} S_k]$. Since $\operatorname{Ker} S_k = \operatorname{span}\{e_1, \cdots, e_k\} \cong \mathbb{C}^k$, it follows that

$$\operatorname{Ker} S_k = \bigsqcup_{x \in X} \operatorname{Ker} S_k = \bigsqcup_{x \in X} \operatorname{span}\{e_1, \cdots, e_k\} = X \times \operatorname{span}\{e_1, \cdots, e_k\} \cong X \times \mathbb{C}^k,$$

proving that $\operatorname{ind}(S_k) = [X \times \mathbb{C}^k]$.

Let $I \colon X \to \mathcal{F}(H)$ denote the constant map $x \longmapsto id_H$. Since $S_k S_{-k} = I$, it follows that

$$0 = \operatorname{ind}(I) = \operatorname{ind}(S_k S_{-k}) = \operatorname{ind}(S_k) + \operatorname{ind}(S_{-k}),$$

from where we obtain $\operatorname{ind}(S_{-k}) = -[X \times \mathbb{C}^k]$. □

Lemma 3.58. *Let V and W be vector spaces, $\{v_1, \cdots, v_N\}$ be a linearly independent subset of V and $\{w_1, \cdots, w_N\}$ be an arbitrary subset of W. If $\sum_{i=1}^N v_i \otimes w_i = 0$, then $w_j = 0$ for every $j = 1, \cdots, N$.*

Proof. Let $\{u_1, \cdots, u_n\}$ be a basis for $\operatorname{span}\{w_1, \cdots, w_N\}$. Complete $\{v_1, \cdots, v_N\}$ and $\{u_1, \cdots, u_n\}$ and get basis for V and W, respectively, say $\mathcal{B}_V$ and $\mathcal{B}_W$. Recall that $\{v \otimes w : v \in \mathcal{B}_V, w \in \mathcal{B}_W\}$ forms a basis for $V \otimes W$. Writing $w_i = \sum_{j=1}^n \lambda_{ij} u_j$, we have

$$0 = \sum_{i=1}^N v_i \otimes w_i = \sum_{i=1}^N v_i \otimes \Big(\sum_{j=1}^n \lambda_{ij} u_j\Big) = \sum_{i=1}^N \sum_{j=1}^n \lambda_{ij} (v_i \otimes u_j)$$

so that $\lambda_{ij} = 0$ for every i, j, finishing the proof. □

Proposition 3.59. *The bundle index* $\operatorname{ind} \colon [X, \mathcal{F}(H)] \to \mathsf{K}(X)$ *is surjective.*

Proof. Let $\mathcal{B} = \{e_i\}_{i \in \mathbb{N}} \sqcup \{e_\alpha\}_{\alpha \in A}$ be a Hilbert basis for H. Let

$$H_1 \doteq \overline{\operatorname{span}}\{e_i\}_{i \in \mathbb{N}} \qquad \text{and} \qquad H_2 \doteq \overline{\operatorname{span}}\{e_\alpha\}_{\alpha \in A},$$

so that $H = H_1 \oplus H_2$.[2]

By Proposition 2.50, every element in $\mathsf{K}(X)$ is of the form $[E] - [X \times \mathbb{C}^k]$ for some vector bundle $E \to X$ and some $k \geq 0$. Let $E \to X$ be a vector bundle. It suffices to construct a family T such that $\operatorname{ind}(T) = [E]$ since then it will follow that

$$\operatorname{ind}(T S_{-k}) = \operatorname{ind}(T) + \operatorname{ind}(S_{-k}) = [E] - [X \times \mathbb{C}^k],$$

where $S_{-k} = S_{-k}^{(\mathcal{B})}$ is the operator associated to $\mathcal{B}$ as in Lemma 3.57. By Proposition 2.36 there exists $N > 0$ such that E is a subbundle of $X \times \mathbb{C}^N$. For $x \in X$, let $P_x \colon \mathbb{C}^N \to \mathbb{C}^N$ be the orthogonal

[2]Here, $\overline{\operatorname{span}}\, W$ stands for the closure of the linear space generated by a subset $W \subseteq H$.

projection onto E_x and let $Q_x \doteq id_{\mathbb{C}^N} - P_x$ (Q_x is the orthogonal projection onto $E_x^\perp$). Let $S_1 = S_1^{(\mathcal{B})}$ be as in Lemma 3.57:

$$S_1(e_j) = \begin{cases} e_{j-1} & \text{if } j \in \mathbb{N} \text{ and } j > 1, \\ 0 & \text{if } j = 1, \\ e_j & \text{if } j \in A. \end{cases}$$

Define $T\colon X \to \mathcal{B}(\mathbb{C}^N \otimes H)$ by

$$T_x \doteq P_x \otimes S_1 + Q_x \otimes id_H$$

By Lemma 2.37, $x \longmapsto P_x$ is continuous, so that T is a continuous map. Fix $x \in X$. First note that T_x restricts to the identity map over $\mathbb{C}^N \otimes H_2$: if $\xi \in \mathbb{C}^N$ and $v \in H_2$, then

$$T_x(\xi \otimes v) = P_x(\xi) \otimes S_1(v) + Q_x(\xi) \otimes id_H(v) = (P_x(\xi) + Q_x(\xi)) \otimes v = \xi \otimes v$$

Moreover, T_x restricts to a surjective map over $\mathbb{C}^N \otimes H_1$: given $\xi \in \mathbb{C}^N$ and $i \in \mathbb{N}$, we have

$$\begin{aligned} T_x(Q_x(\xi) \otimes e_i + P_x(\xi) \otimes e_{i+1}) &= (P_x \otimes S_1 + Q_x \otimes id_H)(Q_x(\xi) \otimes e_i + P_x(\xi) \otimes e_{i+1}) \\ &= P_x Q_x(\xi) \otimes S_1(e_i) + P_x^2(\xi) \otimes S_1(e_{i+1}) + Q_x^2(\xi) \otimes e_i + Q_x P_x(\xi) \otimes e_{i+1} \\ &= P_x(\xi) \otimes e_i + Q_x(\xi) \otimes e_i \\ &= (P_x(\xi) + Q_x(\xi)) \otimes e_i \\ &= \xi \otimes e_i \end{aligned}$$

(boundedness of each T_x gives surjectivity because $\overline{\operatorname{span}}\{\xi \otimes e_i : \xi \in \mathbb{C}^N, i \in \mathbb{N}\} = \mathbb{C}^N \otimes H_1$). Therefore, T_x is a surjective map.

Now, let $\sum_{i=1}^n \xi_i \otimes v_i \in \operatorname{Ker} T_x$. We can assume $\{v_i\}_{i=1}^n$ is linearly independent. Using the decomposition $\mathbb{C}^N = E_x \oplus E_x^\perp$, write $\xi_i = \omega_i + \eta_i$ with $\omega_i \in E_x$ and $\eta_i \in E_x^\perp$. Let $\{a_j\}_{j=1}^m$ and $\{b_j\}_{j=1}^{N-m}$ be bases for E_x and $E_x^\perp$, respectively. Write $\omega_i = \sum_{j=1}^m \alpha_{ij} a_j$ and $\eta_i = \sum_{j=1}^{N-m} \beta_{ij} b_j$. Applying T_x to

$$\begin{aligned} \sum_{i=1}^n \xi_i \otimes v_i &= \sum_{i=1}^n \omega_i \otimes v_i + \sum_{i=1}^n \eta_i \otimes v_i \\ &= \sum_{i=1}^n \sum_{j=1}^m \alpha_{ij} a_j \otimes v_i + \sum_{i=1}^n \sum_{j=1}^{N-m} \beta_{ij} b_j \otimes v_i \end{aligned}$$

leads us to (observe that $P_x(\omega_i) = \omega_i$, $P_x(\eta_i) = 0 = Q_x(\omega_i)$ and $Q_x(\eta_i) = \eta_i$)

$$\begin{aligned} 0 &= \sum_{i=1}^n \omega_i \otimes S_1(v_i) + \sum_{i=1}^n \eta_i \otimes v_i \\ &= \sum_{i=1}^n \sum_{j=1}^m \alpha_{ij} a_j \otimes S_1(v_i) + \sum_{i=1}^n \sum_{j=1}^{N-m} \beta_{ij} b_j \otimes v_i \\ &= \sum_{j=1}^m a_j \otimes S_1\left(\sum_{i=1}^n \alpha_{ij} v_i\right) + \sum_{j=1}^{N-m} b_j \otimes \left(\sum_{i=1}^n \beta_{ij} v_i\right). \end{aligned}$$

By Lemma 3.58, it follows that

$$\sum_{i=1}^{n} \alpha_{ij} v_i \in \text{Ker } S_1 = \text{span}\{e_1\} \quad \forall j = 1, \cdots, m$$

and

$$\sum_{i=1}^{n} \beta_{ij} v_i = 0 \quad \forall j = 1, \cdots, N-m\,.$$

Since $\{v_i\}_{i=1}^n$ is linearly independent, one has $\beta_{ij} = 0$ for every $j = 1, \cdots, N$ and every $i = 1, \cdots, n$, which imply $\eta_i = 0$ for all $i = 1, \cdots, n$. Putting all this together, we have

$$\begin{aligned}\sum_{i=1}^{n} \xi_i \otimes v_i &= \sum_{i=1}^{n} \omega_i \otimes v_i \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} a_j \otimes v_i \\ &= \sum_{j=1}^{m} a_j \otimes \left(\sum_{i=1}^{n} \alpha_{ij} v_i \right) \; \in \; E_x \otimes \text{span}\{e_1\}\,.\end{aligned}$$

This shows that Ker $T_x \subseteq E_x \otimes \text{span}\{e_1\}$. On the other hand, for $\xi \in E_x$,

$$T_x(\xi \otimes e_1) = P_x(\xi) \otimes S_1(e_1) + Q_x(\xi) \otimes e_1 = 0\,.$$

Thus Ker $T_x = E_x \otimes \text{span}\{e_1\}$. This proves in particular that T_x is a Fredholm operator for each $x \in X$. We can then regard T as a continuous map $X \to \mathcal{F}(\mathbb{C}^N \otimes H)$. Since each T_x is surjective, Proposition 3.40 gives that

$$\text{ind}(T) = [\text{Ker } T]\,.$$

Notice that $E \subseteq X \times \mathbb{C}^N$ is isomorphic to

$$\begin{aligned}\text{Ker } T &= \{(x,u) \in X \times (\mathbb{C}^N \otimes H) : u \in \text{Ker } T_x\} \\ &= \{(x,u) \in X \times (\mathbb{C}^N \otimes H) : u \in E_x \otimes \text{span}\{e_1\}\}\end{aligned}$$

via

$$E \ni (x,\xi) \longmapsto (x, \xi \otimes e_1) \in \text{Ker } T\,.$$

Therefore, we obtain $\text{ind}(T) = [\text{Ker } T] = [E]$.

We have obtained a continuous map $T\colon X \to \mathcal{F}(\mathbb{C}^N \otimes H)$ with $\text{ind}(T) = [E]$. Let $I\colon \mathbb{C}^N \otimes H \to H$ be an isomorphism of Banach spaces. The product $ITI^{-1}\colon X \to \mathcal{F}(H)$, given by $x \longmapsto IT_xI^{-1}$, is a continuous family of Fredholm operators in H that, by Lemma 3.38, satisfies

$$\text{ind}(ITI^{-1}) = \text{ind}(T) = [E]\,,$$

as desired. □

Denote by $GL(H) \subseteq \mathcal{B}(H)$ the set of invertible bounded linear operators in H.

Lemma 3.60. *Let $T\colon X \to \mathcal{F}(H)$ be continuous. If $\text{ind}(T) = 0$, then T is homotopic to a continuous* *map* $X \to GL(H)$.

Proof. Let $V \subseteq H$ be a closed linear subspace of finite codimension such that $V \cap \operatorname{Ker} T_x = \{0\}$ for every $x \in X$. We have

$$0 = \operatorname{ind}(T) = \operatorname{ind}_A(T) = [X \times (H/V)] - [H/T(V)] .$$

Now, since $[X \times (H/V)] = [H/T(V)]$, by Proposition 2.50 there exists $k \geq 0$ such that

$$X \times \big((H/V) \oplus \mathbb{C}^k\big) \cong (X \times \mathbb{C}^k) \oplus H/T(V) .$$

Let W be a closed linear subspace of V with $\dim V/W = k$. We clearly have

$$X \times (H/W) \cong X \times \big((H/V) \oplus (V/W)\big) \cong X \times \big((H/V) \oplus \mathbb{C}^k\big) .$$

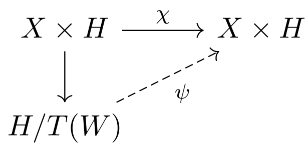
Now we need to read again the proof of Proposition 3.51 and invoke Equation (3.7) with W instead of V', obtaining

$$H/T(W) \cong H/T(V) \oplus (X \times (V/W)) \cong H/T(V) \oplus (X \times \mathbb{C}^k) .$$

Putting together the three previous isomorphisms, we get a bundle isomorphism

$$\alpha \colon X \times (H/W) \longrightarrow H/T(W) . \tag{3.10}$$

For $x \in X$, let $P_x \in \mathcal{B}(H)$ be the orthogonal projection onto $T_x(W)$. Lemma 3.44 gives the continuity of $P \colon X \to \mathcal{B}(H)$, $x \longmapsto P_x$. Consider the continuous map $\chi \colon X \times H \to X \times H$ given by $\chi(x, v) = (x, v - P_x(v))$. Note that for $(x, v), (x, v') \in X \times H$ with $v - v' \in T_x(W)$ we have that $P_x(v - v') = v - v'$, so that $\chi(x, v) = \chi(x, v')$. Thus, χ induces a continuous map $\psi \colon H/T(W) \to X \times H$ making the diagram



commutative, that is, for $(x, v) \in X \times H$ the map ψ sends $v + T_x(W)$ to $(x, v - P_x(v))$ continuously. Note that, for every $x \in X$, ψ maps $H/T_x(W)$ isomorphically onto $T_x(W)^\perp$. The composition $\psi \circ \alpha \colon X \times (H/W) \to X \times H$ induces a continuous map $S \colon X \to \mathcal{B}(H/W, H)$ such that S_x is a linear isomorphism from H/W onto $T_x(W)^\perp$ for every $x \in X$. Since $W \cap \operatorname{Ker} T_x = \{0\}$ for every $x \in X$, we have that T_x is a linear isomorphism from W onto $T_x(W)$.

Let $Q \in \mathcal{B}(H)$ be the orthogonal projection onto $W^\perp$, and define $F \colon X \times [0, 1] \to \mathcal{F}(H)$ by

$$F_{(x,t)}(v) = T_x(v - tQ(v)) + S_x(tQ(v) + W) .$$

We have that F is a homotopy between T and $T' \colon X \to \mathcal{F}(H)$, $T'_x(v) = T_x(v - Q(v)) + S_x(Q(v) + W)$. Fix $x \in X$ and let us prove that T'_x is an isomorphism from H onto itself. Since

$$H = T_x(W) \oplus T_x(W)^\perp = T_x(W) \oplus S_x(H/W) ,$$

we have that T'_x is surjective. Besides, if $v \in H$ is such that $T'_x(v) = 0$, then $T_x(v - Q(v)) = -S_x(Q(v) + W) \in T_x(W) \cap T_x(W)^\perp$, so that $T_x(v - Q(v)) = 0 = S_x(Q(v) + W)$. Since $T|_W$ is injective, we obtain $v = Q(v)$. On the other hand, injectivity of S_x gives $Q(v) \in W$, which is only possible if $Q(v) = 0$. Thus $v = 0$. In conclusion, T is homotopic to the continuous family of invertible operators T'. □

Let $i\colon [X, GL(H)] \to [X, \mathcal{F}(H)]$ be the map induced by the inclusion $GL(H) \hookrightarrow \mathcal{F}(H)$. Corollary 3.41 gives that the index bundle of a family of invertible operators is trivial, so that $i([X, GL(H)]) \subseteq \operatorname{Ker} \operatorname{ind}$. On the other hand, Lemma 3.60 shows that $\operatorname{Ker} \operatorname{ind} \subseteq i([X, GL(H)])$. Therefore

$$\operatorname{Ker} \operatorname{ind} = i([X, GL(H)])\,. \tag{3.11}$$

A theorem by Kuiper [Kui65] states that $[X, GL(H)] = 0$ whenever H is a separable Hilbert space. This result remains true for arbitrary infinite dimensional Hilbert spaces (see [Ill65]).

Applying this to equation (3.11), one obtain that the index bundle ind has trivial kernel. Since ind was already shown to be surjective (Proposition 3.59), Proposition 1.6 allows us to say that ind is also injective. We have proved the main result of this text.

Theorem 3.61 (Atiyah-Jänich)**.** *Let X be a compact Hausdorff topological space and H be an infinite dimensional Hilbert space. The families-index*

$$\operatorname{ind}\colon [X, \mathcal{F}(H)] \longrightarrow \mathsf{K}(X)$$

is an isomorphism.

Corollary 3.62. *If X is a compact Hausdorff space and H is an infinite dimensional Hilbert space, then the monoid $[X, \mathcal{F}(H)]$ is actually an abelian group.*

Let us interpret the Atiyah-Jänich Theorem from the viewpoint of Category Theory.

Fix an infinite dimensional Hilbert space H. For every compact Hausdorff topological space X, we can associate the monoid $[X, \mathcal{F}(H)]$. Let X and Y be compact Hausdorff spaces and $f\colon X \to Y$ be continuous. We have the associated map

$$C(Y, \mathcal{F}(H)) \longrightarrow C(X, \mathcal{F}(H)), \qquad T \longmapsto T \circ f\,. \tag{3.12}$$

Notice that this map is compactible with the pointwise product, that is,

$$(ST) \circ f = (S \circ f)(T \circ f), \qquad \text{for every } S, T \in C(Y, \mathcal{F}(H))\,,$$

so that it is a monoid morphism. Now, suppose that $S, T \in C(Y, \mathcal{F}(H))$ are homotopic, and let $G\colon Y \times [0,1] \to \mathcal{F}(H)$ be a homotopy between them. The continuous map $H\colon X \times [0,1] \to \mathcal{F}(H)$, given by $H(x,t) = G(f(x), t)$, is then a homotopy between $S \circ f$ and $T \circ f$. Thus, (3.12) induces a monoid morphism

$$f^*\colon [Y, \mathcal{F}(H)] \longrightarrow [X, \mathcal{F}(H)]\,.$$

Therefore, this association provides a contravariant functor from the category of compact Hausdorff topological spaces into the category of monoids, which we shall call F.

Recalling that every abelian group can be seen as a monoid, the K-theory functor can be seen as a contravariant functor from the category of compact Hausdorff topological spaces into the category of monoids. It is a consequence of Lemma 3.32 that the families-index

$$\mathrm{ind}\colon [X, \mathcal{F}(H)] \longrightarrow \mathsf{K}(X)$$

is a natural transformation between the functors F and K. In this context, what the Atiyah-Jänich Theorem actually states is that ind is a natural isomorphism.

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