(15) $\mathcal{D} = \{ y \in C^2[0,2\pi] : y(0) = y(2\pi),\ y'(0) = y'(2\pi) \}$, $L : \mathcal{D} \longrightarrow C[0,2\pi]$
$\qquad y \longmapsto y'' + y$

A Equação $y'' + y = 0$ tem sol. geral $y(x) = c_1 \cos x + c_2 \operatorname{sen} x$, $c_i \in \mathbb{R}$

Como $\cos 0 = \cos 2\pi$ e $\operatorname{sen} 0 = \operatorname{sen} 2\pi$, temos $\{\cos, \operatorname{sen}\} \equiv \ker L$ .//

Dada $f \in C[0,2\pi]$, o p.v.i. $\begin{cases} y'' + y = f(x) \\ y(0) = y'(0) = 0 \end{cases}$ tem solução $y_p(x) = \int_0^x \operatorname{sen}(x-t) f(t)\, dt$.

Logo, a equação $y'' + y = f(x)$ tem sol. geral

$$y(x) = c_1 \cos x + c_2 \operatorname{sen} x + \int_0^x \operatorname{sen}(x-t) f(t)\, dt .$$

$$\therefore\ y'(x) = -c_1 \operatorname{sen} x + c_2 \cos x + \int_0^x \cos(x-t) f(t)\, dt$$

$y(0) = c_1$, $y(2\pi) = c_1 + \int_0^{2\pi} \operatorname{sen}(2\pi - t) f(t)\, dt = c_1 - \int_0^{2\pi} \operatorname{sen} t\, f(t)\, dt$

$y'(0) = c_2$, $y'(2\pi) = c_2 + \int_0^{2\pi} \cos(2\pi - t) f(t)\, dt = c_2 + \int_0^{2\pi} \cos t\, f(t)\, dt$.

Conclusão. Toda sol. de $y'' + y = f(x)$ satisfaz

$$y(0) - y(2\pi) = \int_0^{2\pi} \operatorname{sen} t\, f(t)\, dt \quad \text{e} \quad y'(2\pi) - y'(0) = \int_0^{2\pi} \cos t\, f(t)\, dt .$$

Se $f \in L(\mathcal{D})$, então a solução de $y'' + y = f(x)$ pertence a $\mathcal{D}$,

logo $\int_0^{2\pi} \operatorname{sen} t\, f(t)\, dt = \int_0^{2\pi} \cos t\, f(t)\, dt = 0$.

A recíproca é imediata. //

(16) $\mathcal{D} = \{ y \in C^2[0,1] : y(0) = y(1) \, , \, y'(0) = y'(1) \}$, $\quad L : \mathcal{D} \longrightarrow C[0,1]$

$$y \longmapsto y'' - y$$

É suficiente mostrar que, dada $f \in C[0,1]$, o problema de contorno $\begin{cases} y'' - y = f(x) \\ y(0) = y(1) \\ y'(0) = y'(1) \end{cases}$ admite uma única solução.

• Equação homogênea associada: $y'' - y = 0$

sol.'s ~~geral~~ : $y_1(x) = \cosh x$, $y_2(x) = \operatorname{senh} x$

sol. geral : $y_h(x) = c_1 \cosh x + c_2 \operatorname{senh} x$

• Sol. part. da não homogênea: da forma $y_p(x) = c_1(x)\, y_1(x) + c_2(x)\, y_2(x)$

Derivando y_p e substituindo em $y'' - y = f(x)$, vê-se que é suficiente que $c_1'(x)$ e $c_2'(x)$ satisfaçam $\begin{cases} c_1'(x)\, y_1(x) + c_2'(x)\, y_2(x) = 0 \\ c_1'(x)\, y_1'(x) + c_2'(x)\, y_2'(x) = f(x) \end{cases}$

$$W[y_1, y_2](x) = \det \begin{pmatrix} \cosh x & \operatorname{senh} x \\ \operatorname{senh} x & \cosh x \end{pmatrix} = 1 \quad \forall x \in [0,1] .$$

$$c_1'(x) = -f(x)\, y_2(x) \rightsquigarrow c_1(x) = -\int_0^x \operatorname{senh} t \, f(t)\, dt$$

$$c_2'(x) = f(x)\, y_1(x) \longrightarrow c_2(x) = \int_0^x \cosh t \, f(t)\, dt$$

Então $y_p(x) = -\cosh x \int_0^x \operatorname{senh} t \, f(t) + \operatorname{senh} x \int_0^x \cosh t \, f(t)\, dt$

$$= \int_0^x [\cosh x \operatorname{senh}(-t) + \operatorname{senh} x \cosh(-t)]\, f(t)\, dt$$

$$= \int_0^x \operatorname{senh}(x-t)\, f(t)\, dt$$

• Sol. geral da não homogênea: $y(x) = c_1 \cosh x + c_2 \operatorname{senh} x + \int_0^x \operatorname{senh}(x-t)\, f(t)\, dt$

$$\therefore \; y'(x) = c_1 \operatorname{senh} x + c_2 \cosh x + \int_0^x \cosh(x-t)\, f(t)\, dt$$

(obs. Isso resolve o 14.b)

Temos $\quad y(0) = c_1 \,, \quad y(1) = c_1 \cosh(1) + c_2 \operatorname{senh}(1) + \overbrace{\int_0^1 \operatorname{senh}(1-t) f(t)\, dt}^{A}$

$\quad\quad\quad y'(0) = c_2 \,, \quad y'(1) = c_1 \operatorname{senh}(1) + c_2 \cosh(1) + \underbrace{\int_0^1 \cosh(1-t) f(t)\, dt}_{B}$

Então

$$\begin{cases} y(0) = y(1) \\ y'(0) = y'(1) \end{cases} \Longrightarrow \begin{cases} (1-\cosh(1))\, c_1 + \operatorname{senh}(1)\, c_2 = A \\ -\operatorname{senh}(1)\, c_1 + (1-\cosh(1))\, c_2 = B \end{cases} \quad (*)$$

$D \doteq (1-\cosh(1))^2 - \operatorname{senh}^2(1) = 2 - 2\cosh(1) \neq 0$ $\quad$ [Em particular, $(*)$ admite uma única solução; isso prova que L é bijeção]

$D_1 \doteq (1-\cosh(1))\, A + \operatorname{senh}(1)\, B \quad ; \quad D_2 \doteq (1-\cosh(1))\, B + \operatorname{senh}(1)\, A$

$$c_1 = \frac{D_1}{D} = \frac{\overset{\cosh(-1)}{1-\cosh(1)}}{2-2\cosh(1)} \int_0^1 \operatorname{senh}(1-t) f(t)\, dt + \frac{\overset{-\operatorname{senh}(-1)}{\operatorname{senh}(1)}}{2-2\cosh(1)} \int_0^1 \cosh(1-t) f(t)\, dt \qquad \operatorname{senh}(1-t+(-1))$$

$$c_2 = \frac{D_2}{D} = \frac{\overset{\cosh(-1)}{1-\cosh(1)}}{2-2\cosh(1)} \int_0^1 \cosh(1-t) f(t)\, dt + \frac{\overset{-\operatorname{senh}(-1)}{\operatorname{senh}(1)}}{2-2\cosh(1)} \int_0^1 \operatorname{senh}(1-t) f(t)\, dt \qquad \cosh(1-t+(-1))$$

$$c_1 = \frac{1}{D}\int_0^1 \operatorname{senh}(1-t) f(t)\, dt - \frac{1}{D}\int_0^1 \operatorname{senh}(-t) f(t)\, dt = \frac{1}{D}\int_0^1 (\operatorname{senh}(1-t) + \operatorname{senh} t) f(t)\, dt$$

$$c_2 = \frac{1}{D}\int_0^1 \cosh(1-t) f(t)\, dt - \frac{1}{D}\int_0^1 \cosh(-t) f(t)\, dt = \frac{1}{D}\int_0^1 (\cosh(1-t) - \cosh t)\, f(t)\, dt$$

$$y_h(x) = c_1 \cosh x + c_2 \operatorname{senh} x = \frac{1}{D}\int_0^1 \Big[\cosh x\,\big(\operatorname{senh}(1-t) + \overset{-\operatorname{senh}(-t)}{\operatorname{senh} t}\big) + \operatorname{senh} x\,\big(\cosh(1-t) - \overset{\cosh(-t)}{\cosh t}\big)\Big] f(t)\, dt$$

$$= \frac{1}{D}\int_0^1 [\operatorname{senh}(1-t+x) - \operatorname{senh}(x-t)]\, f(t)\, dt$$

$$\therefore \quad y(x) = \frac{1}{D}\int_0^1 [\operatorname{senh}(1-t+x) - \operatorname{senh}(x-t)]\, f(t)\, dt + \int_0^x \operatorname{senh}(x-t) f(t)\, dt$$

$$= \int_0^1 \left\{ \frac{\operatorname{senh}(1-t+x) - \operatorname{senh}(x-t)}{2-2\cosh(1)} + \operatorname{senh}(x-t)\, \chi_{[0,x]}(t) \right\} f(t)\, dt$$

$$\text{Então} \quad G(x,t) = \frac{\operatorname{senh}(1-t+x) - \operatorname{senh}(x-t)}{2-2\cosh(1)} + \operatorname{senh}(x-t)\,\chi_{[0,x]}(t) = \begin{cases} \cdots + \operatorname{senh}(x-t) \,, & t < x \\ \cdots \,, & t > x \end{cases}$$

$G : [0,1] \times [0,1] \to \mathbb{R}$

Como as duas expressões coincidem quando $x=t$ $[\text{pois } \operatorname{senh}(x-x)=0]$, temos a continuidade de G. Agora: $\forall s \in [0,1]$

$$G(s,0) = \frac{\operatorname{senh}(1+s) - \operatorname{senh} s}{2-2\cosh(1)} + \operatorname{senh} s = \frac{\operatorname{senh}(1+s) - \operatorname{senh} s + (2-2\cosh(1))\operatorname{senh} s}{\dots}$$

$$= \frac{\operatorname{senh}(1)\cosh s + \operatorname{senh} s \cosh(1) + \operatorname{senh} s - 2\cosh(1)\operatorname{senh} s}{\dots}$$

$$= \frac{\overset{-\operatorname{senh}(-1)}{\operatorname{senh}(1)}\cosh s - \operatorname{senh} s \overset{\cosh(-1)}{\cosh(1)} + \operatorname{senh} s}{\dots} = \frac{\operatorname{senh} s - \operatorname{senh}(s-1)}{\dots} = G(s,1);$$

$$D.\ G(0,s) = \operatorname{senh}(1-s) \overset{+\operatorname{senh}(s)}{- \operatorname{senh}(-s)} = \operatorname{senh}(1)\cosh(-s) + \operatorname{senh}(-s)\cosh(1) + \operatorname{senh} s$$

$$= \operatorname{senh}(1)\cosh s - \operatorname{senh} s \cosh(1) + \operatorname{senh} s$$

$(*)$
$\operatorname{senh}(2) = 2\operatorname{senh}(1)\cosh(1)$
$\cosh(2) = 2\cosh^2(1) - 1$

e

$$D\ G(1,s) = \operatorname{senh}(2-s) - \operatorname{senh}(1-s) + (2-2\cosh(1))\operatorname{senh}(1-s)$$

$$= \operatorname{senh}(2)\cosh s - \operatorname{senh} s \cosh(2) + \operatorname{senh}(1)\cosh s - \operatorname{senh} s \cosh(1) - 2\operatorname{senh}(1-s)\cosh(1)$$

$$= \underline{\qquad\qquad\qquad\qquad} - 2[\operatorname{senh}(1)\cosh s - \operatorname{senh} s \cosh(1)]\cosh(1)$$

$$\overset{(*)}{=} 2\operatorname{senh}(1)\cosh(1)\cosh s - \operatorname{senh} s(2\cosh^2(1) - 1) + \operatorname{senh}(1)\cosh s - \operatorname{senh} s \cosh(1) - 2\operatorname{senh}(1)\cosh(1)\cosh s$$
$$+ 2\operatorname{senh} s \cosh^2(1)$$

$$= \operatorname{senh} s + \operatorname{senh}(1)\cosh s - \operatorname{senh} s \cosh(1)$$

e $\therefore$ $G(0,s) = G(1,s)$. //