$\odot$ $|a_{ij}| \leq \|A\| \leq C_N \max_{i,j} |a_{ij}|$, $A = ((a_{ij}))_{1 \leq i,j \leq N} \in M_N(\mathbb{R})$ (1)

($\star$) $\|AB\| \leq \|A\| \cdot \|B\|$

$\circledast$ $f_n \in C^1([a,b])$, $f_n' \rightrightarrows g$, $f_n(x_0) \to L \Rightarrow f_n \rightrightarrows f$, $f' = g$

Daí: $f_n \in C^1([a,b])$, $f_n' \rightrightarrows g$, $f_n \rightrightarrows f \Rightarrow f' = g$

— x —

Lema: $A \in C^1(I, M_N(\mathbb{R}))$

$\|(A(t)^n)'\| \leq n \|A(t)\|^{n-1} \|A'(t)\|$

Dem $n = 2$ $(A(t)^2)' = A(t)A'(t) + A'(t)A(t)$

$\|(A(t)^{n+1})'\| = \|[A(t)^n A(t)]'\| = \|A(t)^n A'(t) + (A(t)^n)' A(t)\|$

$\leq \|A(t)^n A'(t)\| + \|(A(t)^n)' . A(t)\|$ $\leq$ hipótese de indução

$\leq \|A(t)^n A'(t)\| + n \|A(t)\|^{n-1} . \|A'(t)\| . \|A(t)\| =$

$= (n+1) \|A(t)\|^n . \|A'(t)\|$

— x —

Vimos: $A \in C([a,b], M_N(\mathbb{R})) \Rightarrow (t \mapsto e^{A(t)}) \in C([a,b], M_N(\mathbb{R}))$

e $\underbrace{\sum_{k=0}^{n} \frac{A(t)^k}{k!}}_{S_n(t) =} \underset{n}{\rightrightarrows} e^{A(t)}$ i.e. $\sup_t \|e^{A(t)} - S_n(t)\| \underset{n}{\to}$

Proposição: $A \in C^1([a,b], M_N(\mathbb{R})) \Rightarrow (t \mapsto e^{A(t)}) \in C^1([a,b], M_N(\mathbb{R}))$

e $\underbrace{\sum_{k=0}^{n} \frac{(A(t)^k)'}{k!}}_{S_n'(t) =} \underset{n}{\rightrightarrows} (e^{A(t)})'$

$\forall t \in [a,b]$ $\forall n$.

Dem: $S_n(t) = ((s_{ij,n}(t)))_{1 \leq i,j \leq N}$.

$$\left|s_{ij,m}'(t) - s_{ij,n}'(t)\right| \underset{\odot}{\leq} \left\| \sum_{k=0}^{m} \frac{(A(t)^k)'}{k!} - \sum_{k=0}^{n} \frac{(A(t)^k)'}{k!} \right\|$$

$m > n$

$$= \left\| \sum_{k=n+1}^{m} \frac{(A(t)^k)'}{k!} \right\| \leq \sum_{k=n+1}^{m} \frac{\|(A(t)^k)'\|}{k!} \underset{\text{Lema}}{\leq}$$

$$\leq \left(\sum_{k=n+1}^{m} \|A(t)\|^{k-1} \cdot \frac{k}{k!} \right) \|A'(t)\| \leq M_1 \cdot \sum_{k=n+1}^{m} \frac{M^{k-1}}{(k-1)!}$$

$M_0 = \sup \{ \|A(t)\| \,;\, t \in [a,b] \}$

$M_1 = \sup \{ \|A'(t)\| \,;\, t \in [a,b] \}$

$$= M_1 \sum_{k=n}^{m-1} \frac{M^k}{k!} = M_1 \left| \sigma_{m-1} - \sigma_{n-1} \right|$$

$$\sigma_n = \sum_{k=0}^{n} \frac{M^k}{k!} \to e^M \therefore (\sigma_m)_n \text{ é de Cauchy}$$

Dado $\varepsilon > 0$, tome $N_0 \in \mathbb{N}$; $n, m \geq N_0 \Rightarrow \left| \sigma_{m-1} - \sigma_{n-1} \right| < \frac{\varepsilon}{M_1}$

Daí, $\forall i,j$, $\forall t \in [a,b]$, $n, m \geq N_0 \Rightarrow$

$$\left| s_{ij,m}'(t) - s_{ij,n}'(t) \right| < \varepsilon$$

$\therefore s_{ij,n}' \rightrightarrows r_{ij}$ em $C([a,b])$ (pois este é completo)

Como $s_{ij,n} \rightrightarrows s_{ij}$, segue de $(\circledast)$ que $s_{ij} \in C^1([a,b])$

e $s_{ij}' = r_{ij}$. Segue de $\odot$ a conclusão CQD.

Exercício: $A(t)A'(t) = A'(t)A(t) \ \forall t \quad (1)$

$$\Rightarrow \frac{d}{dt} e^{A(t)} = A'(t) e^{A(t)}$$

Daí, (1) $\Rightarrow \begin{cases} X' = A(t)X \\ X(t_0) = X_0 \end{cases} \Leftrightarrow X(t) = e^{B(t)} X_0$

se $B'(t) = A(t)$ e $B(t_0) = 0$ $\left(e^{B(t_0)} \overset{\text{Daí:}}{=} I \right)$

Caso Particular $A(t) = \varphi(t) A$, $\varphi \in C^1([a,b], \mathbb{R})$
$A \in M_n(\mathbb{R})$ (constante)

$$\Psi(t) = \int_{t_0}^{t} \varphi(s)\, ds, \quad B(t) = \Psi(t) A$$

$$X(t) = e^{B(t)} X_0 \Leftrightarrow \begin{cases} X' = A(t) X_0 \\ X(t_0) = X_0 . \end{cases}$$

Se $AX_0 = \lambda X_0$, $\lambda \in \mathbb{R}$

$$e^{B(t)} X_0 = \sum_{k=0}^{\infty} \frac{(\Phi(t)A)^k}{k!} X_0 =$$

$$= \sum_{k=0}^{\infty} \frac{\Phi(t)^k A^k X_0}{k!} = \left(\sum_{k=0}^{\infty} \frac{\Phi(t)^k \lambda^k}{k} \right) X_0 = e^{\Phi(t)\lambda} X_0$$

Caso particular do caso particular: $A(t) \equiv A$, $\varphi(t) \equiv 1$, $\Psi(t) = t$. Logo $\begin{cases} X' = AX \\ X(t_0) = X_0 \end{cases} \Leftrightarrow X(t) = e^{tA} X_0$

Se $AX_0 = \lambda X_0$, $X(t) = e^{\lambda t} X_0$

<u>Proposição</u> $A, B \in M_n(\mathbb{R})$, $AB = BA \Rightarrow$

$$e^{A+B} = e^A e^B = e^B e^A$$

<u>Demonstração</u>.

- $AB = BA \Rightarrow \underbrace{A^k B = B A^k}_{(2)}$ $\forall k \in \mathbb{N}$ (Prove por indução)

- Daí; $\forall t > 0$

$$e^{tA} B = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \right) B = \lim_{n \to \infty} \left(\sum_{k=0}^{n} \frac{t^k}{k!} A^k \right) B$$

$$= \lim_{n \to \infty} \sum_{k=0}^{n} \frac{t^k A^k B}{k!} \underset{(2)}{=} \lim_{n \to \infty} B \sum_{k=0}^{n} \frac{t^k A^k}{k!}$$

$$= B \lim_{n \to \infty} \sum_{k=0}^{n} \frac{t^k A^k}{k!} = B e^{tA}.$$

- $\frac{d}{dt} e^{At} e^{Bt} = A e^{tA} \cdot B e^{tB} = AB e^{tA} e^{tB}$

- $X_1(t) = e^{t(A+B)} X_0$ e $X_2(t) = e^{tA} e^{tB} X_0$

satisfazem o mesmo pvi, a saber, $\begin{cases} X' = (A+B) X \\ X(t_0) = X_0 \end{cases}$,

$\forall X_0 \in \mathbb{R}^n$. Logo $e^{t(A+B)} X_0 = e^{tA} e^{tB} X_0$ $\forall t \in \mathbb{R}$,

Logo $e^{t(A+B)} = e^{tA} e^{tB}$ $\forall t \in \mathbb{R}$ $\therefore$ $e^{A+B} = e^A \cdot e^B$