MINIMAL SETS FOR RANDOM FLOWS

PEDRO ALADAR TONELLI

1. Introduction

The main purpose of this work is to study some of the classical concepts of topological dynamics in the special context of random dynamical systems, in particular minimal sets. Random dynamical systems is a kind of skew-product flow, where the base flow is a metric dynamical systems and the fibers are topological spaces.
Since it is not pure topological dynamical systems we could ask what objects would replace minimal sets and recurrence in this context. For a recent work on random fixed point one can look the work of Arnold and Schmalfuß [AS96]. Minimal set is a classical concept of topological dynamics, its relation with recurrence is given by the following result that can be found, for instance, in Auslander [Au88]:

Proposition 1.1. *A topological dynamical system on a compact Hausdorff space M is minimal if and only if all its points are recurrent points.*

This is the kind of result we are pursuing here. Next we give the precise definition of random dynamical systems and its invariant sets. Note that the concept of invariant set is not as natural as it seems to be at a first glance. Afterwards we define the minimal sets and relates this concept with a kind of weak recurrence.

2. Random Dynamical Systems

First of all we explain our setup and fix some notations for all text, some assumptions are too strong in some results, I hope that this becomes clear in each proposition, but certainly we will need the assumptions in all strength in the main theorem. The definition of a random dynamical system follows the premises of L. Arnold (see [Ar95]), $\{\Omega, \mathcal{F}, P\}$ is a complete probability space, G is a Lie group and M is a compact homogeneous space of G, that is $M = G/H$, with H closed subgroup of G. We provided M with a metric d compatible with the quotient topology making the action of the group G on the metric space M continuous.

Let $\theta_t : \Omega \to \Omega$ be a metric dynamical system on the probability space $\{\Omega, \mathcal{F}, P\}$, where $t \in T = \mathbb{R}$ or $\mathbb{Z}$ (typically this action is a shift in the space of curves on M).
A cocycle on G for the dynamical system θ_t is a measurable mapping:

$$\sigma : T \times \Omega \to G$$

satisfying:
(i) For all $t_1, t_2 \in T$ we have $\sigma(t_1.t_2, \omega) = \sigma(t_1, \theta_{t_2}(\omega)).\sigma(t_2, \omega)$ almost everywhere.
(ii) $\sigma : T \times \Omega \to G$ is stochastic continuous in the sense of Skorokhod.
For a discussion on the condition (i) one could see for instance Zimmer [Zi84].
Condition (ii) is technical to ensure that there is a full measure set where (i) is accomplished for all points ω. We recall that (ii) means that the application

$$t \in T \mapsto \sigma(t, \cdot) \in \mathcal{L}(\Omega, G)$$

is continuous, taking on the space $\mathcal{L}(\Omega, G)$ the topology of convergence in measure. With this assumption and taking in account that T has a countable dense subgroup we ensure that there is a full set where (i) holds for all elements (cf. Furstenberg [Fu81] or Skorokhod [Sk65]), in other words, this assures that σ is a perfect cocycle (cf. Arnold [Ar95])
As a cocycle we mean one that satisfies conditions (i) and (ii).
Next, we construct with the cocycle a kind of skew-product dynamical system on the product space $(\Omega \times_\sigma M, \mathcal{F} \times \mathcal{B})$ as

$$\begin{aligned} F \quad : \quad & T \times \Omega \times_\sigma M \to \Omega \times_\sigma M \\ & (t, \omega, x) \mapsto (\theta_t(\omega), \sigma(t, \omega).x) \end{aligned}$$

which is an extension of the dynamical system θ_t.
Here $\mathcal{B}$ denotes the Borel σ-algebra of the topological space M. This dynamical system we call a random dynamical system on M. The interesting point here is the mixing of topological structures in one component and probability concepts on the other component. We define in the next sections some dynamical objects related to this dynamical system.
There are already a vast literature considering this sort of dynamical systems [Ar95, Cr85], the main direction seems to be to study this systems from the point of view of ergodic theory, where the topology of M contributes just to gives the Borel algebra on M, we will try to make a detour from this main stream.

3. Invariant sets and Minimal sets

Next we deal with with set valued invariant functions which we call simply random invariant set

The following definition was given by Harry Furstenberg [Fu81]. Let $\mathcal{H}(M)$ denote the set of all closed nonempty sets of M with the Hausdorff topology. We are assuming also that M is a compact set

Definition 3.1. *A* **invariant set** *for random dynamical system F is a measurable mapping*

$$\Phi : \Omega \to \mathcal{H}(M)$$

such that

$$\forall t \in T \quad \sigma(t,\omega)\Phi(\omega) = \Phi(\theta_t(\omega)) \ P - \text{a.s.}$$

We recall that $\Phi : \Omega \to \mathcal{H}(M)$is measurable if it is measurable with respect to the Borel σ-algebra on $\mathcal{H}(M)$.

We will use the following result from the book of Castaing and Valadier [CV77, ch.3], for $B \subset M$ we set

$$\Phi^{-}(B) = \{\omega \in \Omega : \Phi(\omega) \cap B \neq \emptyset\}$$

and

$$\Gamma(\Phi) = \{(\omega, x) \in \Omega \times_\sigma X : x \in \Phi(\omega)\}.$$

A measurable selection for Φ is a measurable mapping $f : \Omega \to M$ such that $f(\omega) \in \Phi(\omega)$ for all $\omega \in \Omega$.

Proposition 3.1 (Castaing, Valadier)**.** *The assertions are equivalents if M is a compact space:*
(i) Φ *is measurable.*
(ii) $B \in \mathcal{B} \implies \Phi^{-}(B) \in \mathcal{F}$.
(iii) F *closed* $\implies \Phi^{-}(F) \in \mathcal{F}$.
(iv) U *open* $\implies \Phi^{-}(U) \in \mathcal{F}$.
(v) *There is a sequence f_n of measurable selections such that* $\Phi(\omega) = \text{cl}\{f_n(\omega)\}$.
(vi) $\Gamma(\Phi) \in \mathcal{F} \times B$.
(vii) *If M is a metric space with metric d then for all $x \in X$ $d(x, \Phi(\cdot))$ is measurable.*

The first consequence of this proposition is :

Proposition 3.2. *If $\Phi : \Omega \to \mathcal{H}(M)$ is an invariant set then $\Gamma(\Phi) \in \mathcal{F} \times B$ is an invariant set for F in the usual sense.*

Proof. It follows immediately from the definition that $\Gamma(\Phi)$ is invariant, and from the proposition above we have that $\Gamma(\Phi) \in \mathcal{F} \times B$. ∎

Another important result is given by the next proposition where $\mathcal{P}(M)$ is the space of probability measures over M. The definition of invariant measures can be found at [Ar95, Cr85].

Proposition 3.3. *Let $\mu_{\cdot} : \Omega \to \mathcal{P}(M)$ be an invariant measure, set $\Phi_\mu(\omega) = \text{supp}\mu_\omega$, then Φ is an invariant set.*

Proof. Let $x \in \text{supp}\mu_\omega$, this means that for all neighborhood V of x one has: $\mu_\omega(V) > 0$. We want to prove that $\sigma(t,\omega)x \in \text{supp}\mu_{\theta_t(\omega)}$.
Let U be a neighborhood of $\sigma(t,\omega)x$. Since $\sigma(t,\omega)$ is a homeomorphism there is a neighborhood V of x,such that $\sigma(t,\omega)V = U$. Then

$$\mu_{\theta_t(\omega)}(U) = \sigma(t,\omega)\mu_\omega(U) = \mu_\omega(V) > 0$$

which implies $\sigma(t,\omega)x \in \text{supp}\mu_{\theta_t(\omega)}$.
On the other hand, suppose that x is in $\text{supp}\mu_{\theta_t(\omega)}$. We want to prove that $x \in \sigma(t,\omega)\text{supp}\mu_\omega$. Take a neighborhood V of x and then we have:

$$\begin{aligned} \mu_{\theta_t(\omega)}(V) = \mu_\omega(\sigma^{-1}(t,\omega)V) > 0 &\implies \sigma^{-1}(t,\omega)x \in \text{supp}\mu_\omega \\ &\implies x \in \sigma(t,\omega)\text{supp}\mu_\omega \end{aligned} \tag{1}$$

and μ_ω is invariant. □

We have in general

Proposition 3.4. *Let $X \subset \Omega \times_\sigma M$ and $X \in \mathcal{F} \times \mathcal{B}$ be an invariant set for F. If $P(p_\Omega(X)) = 1$, then the mapping*

$$\begin{aligned} \Phi \ : \ &\Omega \to \mathcal{H}(M) \\ &\omega \mapsto \text{cl}M_\omega \end{aligned}$$

is an invariant set for the random dynamical system.

Proof. Φ is measurable according the proposition of Castaing, Valadier. It is also well defined since the projection of M on Ω has full measure, and the invariance follows immediately from the invariance of X □

Corollary 3.5. *Let Φ_1 and Φ_2 two invariant sets. If*

$$P(\{\omega : \Phi_1(\omega) \cap \Phi_2(\omega) \neq \emptyset\}) = 1,$$

, then

$$\Phi(\omega) = \Phi_1(\omega) \cap \Phi_2(\omega)$$

is an invariant set for the random dynamical system.

Proof. Φ is measurable according the proposition from Castaing, Valadier [CV77] we have that $\Gamma(\Phi) = \Gamma(\Phi_1) \cap \Gamma(\Phi_2)$ is an invariant set of F and $P(p_\Omega(G(\Phi))) = 1$, the result follows from the last proposition. □

We define now an order in the set of invariant sets and introduce the concept of minimal set
If Φ_1 and Φ_2 are invariant sets we say that holds the relation $\Phi_1 \leq \Phi_2$, when
$\Phi_1(\omega) \subset \Phi_2(\omega) \quad P-$ a.e..

Definition 3.2. *An invariant set Φ_m is called* **minimal** *if*

$$\Phi \leq \Phi_m \implies \Phi = \Phi_m$$

for each invariant set Φ.

The following proposition shows the existence of minimal sets in case that M is compact.

Proposition 3.6. *Let M be a compact topological space and Φ an invariant set for the random dynamical system F then there is a minimal invariant set Φ_m such that $\Phi_m \leq \Phi$.*

Proof. Consider the set

$$\mathcal{P} = \{\Psi \in L(\Omega, \mathcal{H}(M)) : \Psi \text{ invariant and } \Psi \leq \Phi\}$$

It is clear that $\mathcal{P}$ is not empty and partially ordered. Let $\mathcal{S}$ be a countable total ordered subset of $\mathcal{P}$. We prove that $\mathcal{S}$ has a minimal element.
We define:

$$\Psi_m(\omega) = \bigcap_{\Psi \in \mathcal{S}} \Psi(\omega)$$

Since M and each $\Psi(\omega)$ are compact, $\Psi_m(\omega)$ is also compact and is not empty.
$\Psi_m : \Omega \to \mathcal{H}(M)$ is also measurable (Castaing, Valadier; [CV77, Prop. III.4]).
It remains to prove that Ψ_m is invariant. Fix $t \in T$:

$$\begin{aligned} x \in \sigma(t,\omega)\Psi_m(\omega) &\implies \exists y \in \Psi_m(\omega) : \sigma(t,\omega)y = x \\ &\implies y \in \Psi(\omega)\ \forall \Psi \in \mathcal{S} \\ &\implies x \in \sigma(t,\omega)\Psi(\omega) \forall \Psi \in \mathcal{S} \\ &\implies x \in \Psi(\theta_t(\omega)) \text{ P- a.e.} \end{aligned}$$

and then

$$x \in \Psi_m(\theta_t(\omega)) \quad P-\text{a.e.}$$

A similar argument proves the inverse relation. Then one has

$$\sigma(t,\omega)\Psi_m(\omega) = \Psi_m(\theta_t(\omega)) \quad P-\text{a.e.}$$

which implies that $\Psi_m \in \mathcal{P}$.

Since $L(\Omega, \mathcal{H}(M))$ is a metric second countable space it follows that all totally ordered subset $\mathcal{S}$ has a minimal element, and using Zorn's Lemma follows that $\mathcal{P}$ has a minimal element, which concludes the proposition. $\square$

We would like to give a characterization of a minimal set, that relates this concept with recurrence as in the classical case.
We recall that the definition of a minimal set for a deterministic flow (X, T) could be given in one of the following equivalent ways
(i) M is an invariant closed subset that contains no proper invariant closed subset.
(ii) Each orbit of a point x in M is dense in M. (see Auslander [Au88, p. 7] The analogy between our definition of minimal set for random systems and the first characterization is clear. A characterization of random minimal sets closer to the second definition is the object of the next theorem.
Let

$$\Phi_M = \bigcup_{\omega\in\Omega} \Phi(\omega) \in M$$

Assume the following $\mathrm{int}\Phi_X \neq \emptyset$. Let V an open set in Φ_M and

$$n(\Phi, V) = P\{\omega : \Phi(w) \cap V \neq \emptyset\}.$$

Theorem 3.1. *An invariant set Φ is a minimal set if and only if for all*

- $A \subset \Omega$ *with* $P(A) > 0$
- $u : \Omega \to M$ *measurable selection of* Φ
- V *open set in* Φ_M *such that* $n(\Phi, V) > 0$

then $\exists t \in T, \omega \in A$ *such that* $\sigma(t, \omega)u(\omega) \in V$

Proof. Suppose that Φ is minimal. Take A, u, V as above and assume, by contradiction, the thesis is not true, i.e.:

$$\forall t \in T, \omega \in A \text{ mit } \theta_t(\omega) \in A \implies \sigma(t, \omega)u(\omega) \notin V$$

Denote by $T_\omega = \{t \in T : \theta_t(\omega) \in A\}$.
Then follows

$$u(\omega) \notin \sigma^{-1}(t, \omega)V$$

or

$$u(\omega) \in (\sigma^{-1}(t, \omega)V)^c.$$

for all $\omega \in A$ and $t \in T_\omega$
It is easy to see that $(\sigma^{-1}(t, \omega)V)^c = \sigma^{-1}(t, \omega)V^c$.
Put

$$X(\omega) = \bigcap_{t\in T_\omega} \sigma^{-1}(t, \omega)V^c.$$

We have:
1. $X(\omega)$ is not empty for ω from A, cause: $u(\omega) \in X(\omega)$.
2. $X(\omega)$ is invariant:
Take x in $X(\omega)$, then $x \in \sigma^{-1}(t, \omega)V^c$ for all $t \in T_\omega$. There is $y \in V^c$, such that

$$x = \sigma^{-1}(t, \omega)y = \sigma(-t, \theta_t(\omega))y$$

and

$$\sigma(s, \omega)x = \sigma(s, \omega)\sigma(-t, \theta_t(\omega))y$$

by the properties of a cocycle

$$\sigma(s, \omega)x = \sigma(s - t, \theta_t(\omega)y = \sigma^{-1}(t - s, \theta_s(\omega))y$$

and this $\sigma(s, \omega)x \in X(\theta_s(\omega))$, cause $t \in T_\omega$ means $t - s \in T_{\theta_s(\omega)}$.
3. $X(\omega)$ is compact.
We define now

$$\Phi_m(\omega) = \Phi(\omega) \cap X(\omega)$$

and note that this set is not empty, compact and invariant. Since $n(\Phi, V) > 0$ it follows that $\Phi_m \neq \Phi$ in a set with positive probability contradicting the assumption of minimality of Φ.
For the other implication:
Assume that Φ is not minimal. Let $\Phi_m < \Phi$ minimal and take

$$A = \{\omega : \Phi_m(\omega) \neq \Phi(\omega)\}.$$

Let V with $n(\Phi, V) > 0$ and $\Phi_m(\omega) \cap V = \emptyset \ \forall \omega \in A$.
Let $u(\omega)$ a measurable selection Φ_m, then there is a t and ω, such that $\sigma(t, \omega)u(\omega) \in V$.
but $\sigma(t, \omega)u(\omega) \in \Phi_m(\theta_t(\omega))$, against the choice of V, cause
$\Phi_m(\omega) \cap V = \emptyset \ \forall \omega \in A$. □

Finally we present some observations in case the basic dynamical system (Ω, T) has some special features, and where the concept of random minimal set coincide with the deterministic one in the sense that will be clear after the following propositions

Definition 3.3. *A point $\omega_0 \in \Omega$ is called essentially transitive, if $P(\mathrm{Orb}(\omega_0)) = 1$.*

Where $\mathrm{Orb}(\omega) = \{\theta_t(\omega) : \forall t \in T\}$

Definition 3.4. *The dynamical system Ω is called strictly ergodic, if there is no essentially transitive .*

One can find examples and more information on this matter in Robert J. Zimmer [Zi84]. We just want to connect this concepts with our definition.

Proposition 3.7. *Let Φ be a minimal set and $x \in \Phi(\omega_0)$, where ω_0 is an essentially transitive point. Set*

$$\mathrm{Orb}(\omega_0, x) = \{F(t, \omega_0, x) : \forall t \in T\}$$

and if A is a subset of $\Omega \times X$ we denote

$$A_\omega = \{y \in X : (\omega, y) \in A\}$$

then

$$\Phi(\omega) = \mathrm{cl}(\mathrm{Orb}(\omega_0, x)_\omega) \;\; P - \text{a.s.}$$

Proof. $\mathrm{Orb}(\omega_0, x)$ is an invariant set and belongs to $\mathcal{F} \times B$.
Since ω_0 is essentially transitive it follows

$$P(p_\Omega(\mathrm{Orb}(\omega_0, x))) = P(\mathrm{Orb}(\omega_0)) = 1$$

Then the mapping $\omega \in \Omega \mapsto \mathrm{cl}(\mathrm{Orb}(\omega_0, x)_\omega)$ is an invariant set.
It remains to prove that

$$\mathrm{cl}(\mathrm{Orb}(\omega_0, x)_\omega) \subset \Phi(\omega) \quad P - \text{almostalways}$$

But this is clear cause at least for the points ω in $\mathrm{Orb}(\omega_0)$ holds

$$\mathrm{Orb}(\omega_0, x)_\omega \subset \Phi(\omega)$$

and since $\Phi(\omega)$ is closed :

$$\mathrm{cl}(\mathrm{Orb}(\omega_0, x)_\omega) \subset \Phi(\omega)$$

Recalling that $\mathrm{Orb}(\omega_0)$ has full measure we have proved the proposition. □

This proposition shows that in case we can find an essentially transitive point we have a similar characterization of minimal sets as in the deterministic case.

References

[Ar95] L. Arnold. "Random Dynamical Systems"- in Lect. Notes in Math. 1609, Springer Verlag, Berlin, 1995. 1, 2, 2, 4

[AS96] L. Arnold and B. Schmalfuß. "Fixed Points and Attractors for Random Dynamical Systems"- Solid Mech. Appl., 47, Kluwer Acad. Publ., Dordrecht, 1996. 1

[Au88] J. Auslander. "Minimal Flows and their Extensions"- North Holland Mathematics Studies 153, North-Holland, Amsterdam, 1988. 1, 6

[CV77] C. Castaing and M. Valadier. " Convex Analysis and Measurable Functions" . Lect. Notes in Math. 580, Springer Verlag, Berlin, 1977. 3, 4, 5

[Cr85] H. Crauel. “Lyapunov Exponents and Invariant Measures of Stochastic Systems on Manifolds” in Lect. Notes in Math. 1186, Springer Verlag, Berlin, 1985. 2, 4

[Fu81] H. Fustenberg. “Rigidity and Cocycles for Ergodic Actions of Semi-simple Lie Groups”. In *Seminaire Bourbaki* 559, Lect. Notes in Math. 842, Springer Verlag, Berlin, 1981. 2, 3

[Sk65] A. V. Skorokhod. “ Studies in the Theory of Random Processes”. Addison Wesley, Reading, 1965. 2

[Zi84] R. J. Zimmer. “Ergodic Theory and Semi-simple Groups”. Monographs in Mathematics, Birkhäuser, Basel, 1984. 2, 7

Instituto de Matemática e Estatística, Caixa postal 66281, 05315-970, São Paulo, SP, Brazil.