SOME GOOD LEMMAS

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Contents

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1. Some Simple Uniformity Lemmas

1.1. **Notation.** On an arbitrary metric space we denote by $\mathrm{B}[x;r]$ the closed ball of center x and radious $r>0$ and by $\mathrm{B}(x;r)$ the open ball of center x and radious $r>0$.

1.2. **Lemma.** *Let X be a topological space, (M,d) be a metric space, U be an open subset of $X\times M$ and $K\subset U$ be a compact subset. Then there exists $\varepsilon>0$ such that for every $(x,y)\in K$, $\{x\}\times\mathrm{B}(y;\varepsilon)$ is contained in U.*

Proof. For every $(x,y)\in K$, choose an open neighborhood $V_{(x,y)}$ of x in X and $r_{(x,y)}>0$ such that $V_{(x,y)}\times\mathrm{B}(y;r_{(x,y)})\subset U$. We have an open cover:

$$K\subset\bigcup_{(x,y)\in K}V_{(x,y)}\times\mathrm{B}\big(y;\tfrac{1}{2}\,r_{(x,y)}\big),$$

from which we can take a finite subcover:

$$K\subset\bigcup_{i=1}^{n}V_{(x_i,y_i)}\times\mathrm{B}\big(y_i;\tfrac{1}{2}\,r_{(x_i,y_i)}\big).$$

Now take $\varepsilon=\frac{1}{2}\min\{r_{(x_i,y_i)}\}_{i=1}^{n}$. For every $(x,y)\in K$ we can find $i=1,\ldots,n$ with $x\in V_{(x_i,y_i)}$ and $d(y,y_i)<\frac{1}{2}\,r_{(x_i,y_i)}$; then $\mathrm{B}(y;\varepsilon)\subset\mathrm{B}(y_i,r_{(x_i,y_i)})$ and therefore:

$$\{x\}\times\mathrm{B}(y;\varepsilon)\subset V_{(x_i,y_i)}\times\mathrm{B}(y_i,r_{(x_i,y_i)})\subset U.$$ □

1.3. **Lemma.** *Let (M,d), (N,d') be metric spaces, $K\subset M$ be compact subset and $f:M\to N$ a continuous function. Then given $\varepsilon>0$, there exists $\delta>0$ such that for all $x\in K$, $y\in M$, $d(x,y)<\delta$ implies $d'\big(f(x),f(y)\big)<\varepsilon$.*

Proof. Otherwise, we would be able to find $\varepsilon>0$ such that for every integer $n\ge 1$, there would exist $x_n\in K$, $y_n\in M$ with $d(x_n,y_n)<\frac{1}{n}$ but $d'\big(f(x_n),f(y_n)\big)\ge\varepsilon$. Some subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges to $x\in K$ and then also $(y_{n_k})_{k\in\mathbb{N}}$ converges to x. By the continuity of f, we have:

$$\lim_{k\to+\infty}d'\big(f(x_{n_k}),f(y_{n_k})\big)=0,$$

which yields a contradiction. □

2. Sections of Quotient Topological Spaces

2.1. **Lemma.** *Let X, Y be topological spaces and $q : X \to Y$ be a continuous map. Assume that X is Hausdorff and that there exists a continuous right inverse $s : Y \to X$ of q. Then the image of s is closed.*

Proof. Let $u \in X \setminus s(Y)$ be given and set $v = s\big(q(u)\big)$. Then $u \neq v$ and therefore there exist disjoint open sets U, V in X with $u \in U$ and $v \in V$. Define:

$$W = U \cap (s \circ q)^{-1}(V).$$

Obviously, W is open and $u \in W$. We claim that $W \cap s(Y) = \emptyset$. Namely, if we had $w \in W \cap s(Y)$ then $(s \circ q)(w) = w$, so that $w \in V$, contradicting $w \in U$. □

3. Uniform Dominated Convergence Theorem

Let Λ be a topological space, A be a subset of Λ, $\lambda_0 \in \Lambda$ be a limit point of A, (M, d) be a metric space and I be an arbitrary set. Let $(p_i^\lambda)_{i \in I, \lambda \in A}$ and $(p_i)_{i \in I}$ be families of points of M. We say that p_i^λ tends to p_i as $\lambda \to \lambda_0$ *uniformly in I* if for every $\varepsilon > 0$ there exists a neighborhood V of λ_0 in Λ such that $d(p_i^\lambda, p_i) < \varepsilon$, for all $i \in I$ and for all $\lambda \in V \cap A$ with $\lambda \neq \lambda_0$.

3.1. **Lemma.** *Let Λ be a first countable topological space (i.e., every point of Λ has a countable fundamental system of neighborhoods), A be a subset of Λ, $\lambda_0 \in \Lambda$ be a limit point of A, (M, d) be a metric space*[1] *and I be an arbitrary set. Let $(p_i^\lambda)_{i \in I, \lambda \in A}$ and $(p_i)_{i \in I}$ be families of points of M. If for every countable subset I_0 of I we have that p_i^λ tends to p_i as $\lambda \to \lambda_0$ uniformly in I_0 then p_i^λ tends to p_i as $\lambda \to \lambda_0$ uniformly in I.*

Proof. Assume that it is not the case that p_i^λ tends to p_i as $\lambda \to \lambda_0$ uniformly in I. Then, there exists $\varepsilon > 0$ such that for every neighborhood V of λ_0 in Λ, there exists $i \in I$ and $\lambda \in V \cap A$ with $\lambda \neq \lambda_0$ and $d(p_i^\lambda, p_i) \geq \varepsilon$; let such an $\varepsilon > 0$ be fixed. Let $(V_n)_{n \geq 1}$ be a countable fundamental system of neighborhoods of λ_0 in Λ and for each $n \geq 1$ choose $i_n \in I$ and $\lambda^n \in V_n \cap A$ with $\lambda^n \neq \lambda_0$ and $d(p_{i_n}^{\lambda^n}, p_{i_n}) \geq \varepsilon$. Set $I_0 = \big\{i_n : n \geq 1\big\}$. Clearly, it is not the case that p_i^λ tends to p_i as $\lambda \to \lambda_0$ uniformly in I_0. This contradicts our hypothesis. □

Recall that a *measure space* is a triple $(\Omega, \mathcal{A}, \mu)$, where Ω is a set, $\mathcal{A}$ is a σ-algebra of subsets of Ω and $\mu : \mathcal{A} \to [0, +\infty]$ is a countable additive measure on $\mathcal{A}$. We have the following "uniform version" of the Lebesgue's Dominated Convergence Theorem.

3.2. **Lemma.** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, Λ be a topological space, A be a subset of Λ, $\lambda_0 \in \Lambda$ be a limit point of A, and I be an arbitrary set. Let $(f_i^\lambda)_{i \in I, \lambda \in A}$, $(f_i)_{i \in I}$ be families of maps $f_i^\lambda : \Omega \to \mathbb{R}$, $f_i : \Omega \to \mathbb{R}$. Assume that:*

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[1] In fact, we could consider an arbitrary uniform space.

- *f_i^λ is measurable, for all $i \in I$, $\lambda \in A \setminus \{\lambda_0\}$;*
- *$f_i^\lambda(\omega)$ tends to $f_i(\omega)$ as $\lambda \to \lambda_0$ uniformly in I, for every $\omega \in \Omega$;*
- *there exists an integrable map $\phi : \Omega \to [0, +\infty]$ such that*
$$\left|f_i^\lambda(\omega)\right| \le \phi(\omega),$$
for all $\omega \in \Omega$, $i \in I$ and $\lambda \in A \setminus \{\lambda_0\}$.

Then, for all $i \in I$, $\lambda \in A \setminus \{\lambda_0\}$, the maps f_i^λ and f_i are integrable and $\int_\Omega f_i^\lambda \,\mathrm{d}\mu$ tends to $\int_\Omega f_i \,\mathrm{d}\mu$ as $\lambda \to \lambda_0$ uniformly in I.

Proof. Since λ_0 is a limit point of A and Λ is first countable, there exists a sequence $(\lambda^n)_{n\ge1}$ in A with $\lambda^n \ne \lambda_0$ for all $n \ge 1$ and $\lambda^n \to \lambda_0$. Then, for all $i \in I$, we have that $f_i^{\lambda_n} \to f_i$ pointwise in Ω. It follows that f_i is measurable and that $|f_i(\omega)| \le \phi(\omega)$, for all $i \in I$, $\omega \in \Omega$. Thus, the maps f_i^λ and f_i are integrable, for all $i \in I$, $\lambda \in A \setminus \{\lambda_0\}$. Let us prove that $\int_\Omega f_i^\lambda \,\mathrm{d}\mu$ tends to $\int_\Omega f_i \,\mathrm{d}\mu$ as $\lambda \to \lambda_0$ uniformly in I; by Lemma 3.1, it suffices to show that $\int_\Omega f_i^\lambda \,\mathrm{d}\mu$ tends to $\int_\Omega f_i \,\mathrm{d}\mu$ as $\lambda \to \lambda_0$ uniformly in I_0, for any fixed countable subset I_0 of I. For each $\lambda \in A$, we define a map $g^\lambda : \Omega \to [0, +\infty]$ by setting:
$$g^\lambda(\omega) = \sup_{i \in I_0} \left|f_i^\lambda(\omega) - f_i(\omega)\right|,$$
for all $\omega \in \Omega$. Since I_0 is countable, it follows that g^λ is measurable for all $\lambda \in A \setminus \{\lambda_0\}$. Clearly $|g^\lambda(\omega)| \le 2\phi(\omega)$, for all $\omega \in \Omega$, $\lambda \in A \setminus \{\lambda_0\}$. Moreover, the fact that $f_i^\lambda(\omega)$ tends to $f_i(\omega)$ as $\lambda \to \lambda_0$ uniformly in I_0 for all $\omega \in \Omega$ implies that $\lim_{\lambda\to\lambda_0} g^\lambda(\omega) = 0$, for all $\omega \in \Omega$. If $(\lambda^n)_{n\ge1}$ is an arbitrary sequence in A with $\lambda^n \ne \lambda_0$ for all $n \ge 1$ and with $\lambda^n \to \lambda_0$ then, by the standard version of Lebesgue Dominated Convergence Theorem, we get:
$$\lim_{n\to\infty} \int_\Omega g^{\lambda_n} \,\mathrm{d}\mu = 0.$$
Since the sequence $(\lambda^n)_{n\ge1}$ is arbitrary and Λ is first countable, it follows that:
$$\lim_{\lambda\to\lambda_0} \int_\Omega g^\lambda \,\mathrm{d}\mu = 0.$$
From the inequality:
$$\Big|\int_\Omega f_i^\lambda \,\mathrm{d}\mu - \int_\Omega f_i \,\mathrm{d}\mu\Big| \le \int_\Omega g^\lambda \,\mathrm{d}\mu, \quad i \in I,\ \lambda \in A \setminus \{\lambda_0\},$$
it follows that $\int_\Omega f_i^\lambda \,\mathrm{d}\mu$ tends to $\int_\Omega f_i \,\mathrm{d}\mu$ as $\lambda \to \lambda_0$ uniformly in I_0. This concludes the proof. □

4. A Nice Lemma that Implies Tychonoff's Theorem

4.1. **Lemma.** *Let (X, τ) be a topological space and let $\mathcal{S} \subset \tau$ be a* subbasis *for τ, i.e., every $U \in \tau$ is a union of finite intersections of elements of $\mathcal{S}$. If every open cover of X by elements of $\mathcal{S}$ has a finite subcover then X is compact.*

Proof. Let σ denote the set of all open covers $\mathcal{C} \subset \tau$ which does not have a finite subcover, i.e.:

$$\sigma = \Big\{ \mathcal{C} \subset \tau : \bigcup \mathcal{C} = X \text{ and } \bigcup \mathcal{C}' \neq X \text{ if } \mathcal{C}' \subset \mathcal{C} \text{ is finite} \Big\}.$$

If one considers σ to be ordered by inclusion then it is easy to see that every non empty chain in σ has an upper bound; if X were not compact, then σ would be non empty and by Zorn's Lemma we would be able to find a maximal element $\mathcal{C} \in \sigma$. We will show that $\mathcal{C} \cap \mathcal{S}$ is a cover of X, which will yield a contradiction, since $\mathcal{C} \cap \mathcal{S}$ does not have a finite subcover. Let $x \in X$ be fixed and choose $U \in \mathcal{C}$ with $x \in U$. Since $\mathcal{S}$ is a subbasis, we can find $S_1, \ldots, S_n \in \mathcal{S}$ with $x \in \bigcap_{i=1}^n S_i \subset U$. We will show that some S_i is in $\mathcal{C}$; this will yield $x \in S_i \in \mathcal{C} \cap \mathcal{S}$ and will complete the proof. Assume that $S_i \notin \mathcal{C}$ for every $i = 1, \ldots, n$; then, for each i, $\mathcal{C} \cup \{S_i\}$ has a finite subcover and therefore we can find $V_{ij} \in \mathcal{C}$, $j = 1, \ldots, n_i$, with $S_i \cup \bigcup_{j=1}^{n_i} V_{ij} = X$. It is easy to see that the latter implies:

$$X = \Big(\bigcap_{i=1}^n S_i \Big) \cup \bigcup_{i=1}^n \bigcup_{j=1}^{n_i} V_{ij} \subset U \cup \bigcup_{i=1}^n \bigcup_{j=1}^{n_i} V_{ij};$$

thus $\mathcal{C}$ has a finite subcover, which is a contradiction. □

5. Topology for Sets of Maps

5.1. **Notation.** We denote by $\wp(X)$ the power set of X, i.e., the set of all subsets of X.

If X is a topological space then the set:

$$\big\{ \wp(U) : U \subset X \text{ open} \big\}$$

is a covering of $\wp(X)$ which is closed under finite intersections; therefore, such set is the basis of a topology for $\wp(X)$ which we call the *power set topology* on $\wp(X)$ induced from the topology of X.

If X and Y are topological spaces and $\mathfrak{F}(X, Y) = Y^X$ denotes the set of all maps $f : X \to Y$ then we consider the *graphing map*:

$$\mathfrak{F}(X, Y) \ni f \longmapsto \mathrm{gr}(f) \in \wp(X \times Y)$$

which assigns to every map $f : X \to Y$ its graph. The *strong map topology* on $\mathfrak{F}(X, Y)$ is the topology induced by the graph map, where $\wp(X \times Y)$ has the power set topology induced from the product topology on $X \times Y$. If $\mathfrak{S}$ is any set of maps from X to Y (for instance, if $\mathfrak{S}$ is the set of continuous maps from X to Y) then the *strong map topology* on $\mathfrak{S}$ is the topology induced from the strong map topology on $\mathfrak{F}(X, Y)$; the strong map topology on $\mathfrak{S}$ coincides with the topology induced by the restriction to $\mathfrak{S}$ of the graph map.

A basis of neighborhoods for $f \in \mathfrak{F}(X, Y)$ in the strong map topology is given by:

$$\big\{ \mathcal{G}(U) : U \subset X \times Y \text{ open and } \mathrm{gr}(f) \subset U \big\},$$

where:

$$\mathcal{G}(U) = \{g \in \mathfrak{F}(X,Y) : \mathrm{gr}(g) \subset U\}.$$

Recall that a family $(A_i)_{i\in I}$ of subsets of a topological space X is called *locally finite in* X if every point of X has a neighborhood which intersects A_i for at most a finite number of indices i.

5.2. **Lemma.** *Let $(F_i)_{i\in I}$ be a locally finite family of closed subsets of X and let for each $i \in I$, U_i be an open subset in $F_i \times Y$. Then the set:*

$$\mathcal{G}\big((F_i)_{i\in I}, (U_i)_{i\in I}\big) = \big\{f \in \mathfrak{F}(X,Y) : \mathrm{gr}(f|_{F_i}) \subset U_i, \textit{ for all } i \in I\big\}$$

is open in $\mathfrak{F}(X,Y)$ with respect to the strong map topology.

Proof. Set:

$$V = \big\{(x,y) \in X \times Y : \text{for all } i \in I,\ x \notin F_i \text{ or } (x,y) \in U_i\big\};$$

obviously $\mathcal{G}\big((F_i)_{i\in I}, (U_i)_{i\in I}\big) = \mathcal{G}(V)$, so it suffices to show that V is open in $X \times Y$. The complement of V is $X \times Y$ is given by:

$$\begin{aligned} V^{\mathrm{c}} &= \big\{(x,y) \in X \times Y : \text{for some } i \in I,\ x \in F_i \text{ and } (x,y) \notin U_i\big\} \\ &= \bigcup_{i\in I} \big((F_i \times Y) \setminus U_i\big). \end{aligned}$$

The set $(F_i \times Y) \setminus U_i$ is closed in $F_i \times Y$ and hence closed in $X \times Y$. Moreover, the family $\big((F_i \times Y) \setminus U_i\big)_{i\in I}$ is locally finite in $X \times Y$ because $(F_i \times Y)_{i\in I}$ is locally finite. Since the union of a locally finite family of closed subsets is again closed, the conclusion follows. □

Below we describe the strong map topology of $\mathfrak{F}(X,Y)$ when Y is (at least locally) metrizable.

5.3. **Corollary.** *Let $(F_i)_{i\in I}$ be a locally finite family of closed subsets of X and $(Z_i)_{i\in I}$ an arbitrary family of open subsets of Y. For each $i \in I$ let d_i be a metric for Z_i (compatible with its topology) and choose $k_i \in \,]0, +\infty]$. If $f : X \to Y$ is a continuous map such that $f(F_i) \subset Z_i$ for all $i \in I$ then the set:*

$$\begin{aligned} \mathcal{V}\big(f; (F_i, Z_i, d_i, k_i)_{i\in I}\big) = \big\{g \in \mathfrak{F}(X,Y) : &\textit{ for all } i \in I,\ g(F_i) \subset Z_i \textit{ and } \\ & d_i\big(f(x), g(x)\big) < k_i, \textit{ for all } x \in F_i\big\} \end{aligned}$$

is an open neighborhood of f in $\mathfrak{F}(X,Y)$ with respect to the strong map topology.

Proof. Observe that the set:

$$U_i = \big\{(x,y) \in F_i \times Z_i : d_i\big(f(x), y\big) < k_i\big\}$$

is open in $F_i \times Y$ for all $i \in I$; moreover:

$$\mathcal{V}\big(f; (F_i, Z_i, d_i, k_i)_{i\in I}\big) = \mathcal{G}\big((F_i)_{i\in I}, (U_i)_{i\in I}\big),$$

and the conclusion follows from Lemma 5.2. □

6. The Smallest Locally Arc-Connected Refinement of a Topology

Let X be a topological space. We say that X is *arc-connected* if for every $x, y \in X$ there exists a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. A subset A of X is called *arc-connected* if A is an arc-connected topological space, endowed with the topology induced from X; this is obviously the same as saying that for every $x, y \in A$ there exists a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$ and $\mathrm{Im}(\gamma) \subset A$. For an arbitrary topological space, the relation:

$$x \sim y \Longleftrightarrow \text{there exists a continuous map } \gamma : [0,1] \to X \text{ with } \gamma(0) = x,\ \gamma(1) = y$$

is an equivalence relation on X. For every $p \in X$, the equivalence class C containing p is the largest arc-connected subset of X containing p, i.e., C is arc-connected and C contains any arc-connected subset of X containing p. The equivalence classes are called the *arc-connected components* of the topological space X. If $A \subset X$ is a subset then the *arc-connected components* of A are defined to be the arc-connected components of the topological space A, endowed with the topology induced from X; obviously, the arc-connected component of A containing $p \in A$ is the largest arc-connected subset of X which contains p and is contained in A.

6.1. **Definition.** We say that X is *locally arc-connected* if every point $p \in X$ has a fundamental system of arc-connected neighborhoods, i.e., if every neighborhood of p contains a (not necessarily open) arc-connected neighborhood of p.

Obviously if X is locally arc-connected then every open subset of X is also locally arc-connected, when endowed with the topology induced from X.

6.2. **Lemma.** *If X is locally arc-connected then the arc-connected components of an open subset of X are open.*

Proof. Let $U \subset X$ be open and let C be an arc-connected component of U. Given $p \in C$, we can find an arc-connected neighborhood V of p contained in U. Then $p \in V \subset C$ and thus p is an interior point of C. □

6.3. **Corollary.** *If X is locally arc-connected then every point of X has a fundamental system of* open *arc-connected neighborhoods, i.e., for every $p \in X$ and every neighborhood V of p, there exists an arc-connected open set C with $p \in C \subset V$.*

Proof. Take C to be the arc-connected component of the interior of V containing p. □

Let (X, τ) be a topological space and consider the set $\mathcal{B} \subset \wp(X)$ defined by:

$$\mathcal{B} = \big\{ C \subset X : C \text{ is an arc-connected component of some open subset of } (X, \tau) \big\}.$$

We claim that $\mathcal{B}$ is a basis for a topology on X. To prove that we have to check that:

(i) every point of X belongs to some $C \in \mathcal{B}$;
(ii) given $C_1, C_2 \in \mathcal{B}$ and $p \in C_1 \cap C_2$ there exists $C \in \mathcal{B}$ with $p \in C \subset C_1 \cap C_2$.

To prove (i), observe that the arc-connected components of X are in $\mathcal{B}$ and they obviously form a covering of X. To prove (ii), we argue as follows; let U_i be an open subset of X such that C_i is an arc-connected component of U_i, $i = 1, 2$. Then $p \in U_1 \cap U_2$ and $U_1 \cap U_2$ is open in X, so the arc-connected component C of $U_1 \cap U_2$ containing p is in $\mathcal{B}$. Moreover, $C \subset U_i$ and the arc-connectedness of C imply that $C \subset C_i$, $i = 1, 2$; thus $C \subset C_1 \cap C_2$.

We denote by τ_{ac} the (unique) topology on X having $\mathcal{B}$ as a basis. By definition, the arc-connected components in (X, τ) of an open subset of (X, τ) are open in (X, τ_{ac}). Since every open subset of (X, τ) is the union of its arc-connected components in (X, τ), it follows that:

$$\tau \subset \tau_{\mathrm{ac}},$$

i.e., τ_{ac} is a refinement of τ. We have the following basic lemma.

6.4. **Lemma.** *Let Y be a locally arc-connected topological space and let $f : Y \to X$ be a map. Then $f : Y \to (X, \tau)$ is continuous if and only if $f : Y \to (X, \tau_{\mathrm{ac}})$ is continuous.*

Proof. Obviously the continuity of f with respect to τ_{ac} implies the continuity of f with respect to τ, because τ_{ac} refines τ. Now assume that f is continuous with respect to τ and let us prove that f is continuous with respect to τ_{ac} at an arbitrary point $p \in Y$. Let C be a neighborhood of $f(p)$ in (X, τ_{ac}); we can assume that C is a basic open set, i.e., that C is an arc-connected component in (X, τ) of some open subset U of (X, τ). Then $f^{-1}(U)$ is a neighborhood of p in Y; since Y is locally arc-connected, we can find an arc-connected neighborhood V of p in Y with $V \subset f^{-1}(U)$. Then $f(p) \in f(V) \subset U$ and the continuity of $f : Y \to (X, \tau)$ implies that $f(V)$ is arc-connected in (X, τ). Hence V is a neighborhood of p in Y with $f(V) \subset C$ and f is continuous at the point p. □

6.5. **Corollary.** *A map $\gamma : [0, 1] \to X$ is continuous with respect to τ if and only if it is continuous with respect to τ_{ac}.* □

6.6. **Corollary.** *A subset of X is arc-connected with respect to τ if and only if it is arc-connected with respect to τ_{ac}.* □

6.7. **Corollary.** *The space (X, τ_{ac}) is locally arc-connected.*

Proof. Observe that $\tau_{\rm ac}$ admits a basis of open sets that are arc-connected with respect to τ and thus also arc-connected with respect to $\tau_{\rm ac}$. □

6.8. **Corollary.** *The topology $\tau_{\rm ac}$ is the smallest refinement of τ which is locally arc-connected, i.e., every locally arc-connected topology in X containing τ contains $\tau_{\rm ac}$.*

Proof. Let τ' be a locally arc-connected topology on X that contains τ. Then the identity map $\mathrm{Id} : (X, \tau') \to (X, \tau)$ is continuous and hence also the map $\mathrm{Id} : (X, \tau') \to (X, \tau_{\rm ac})$ is continuous, because (X, τ') is locally arc-connected. □

From now on we refer to $\tau_{\rm ac}$ as the *smallest locally arc-connected refinement* of τ.

6.9. **Lemma.** *Let (X, τ) be a topological space and let $\tau_{\rm ac}$ be the smallest locally arc-connected refinement of τ. Let $U \subset X$ be an open subset of $(X, \tau_{\rm ac})$ and denote by τ_U and $(\tau_{\rm ac})_U$ respectively the topology induced in U by τ and by $\tau_{\rm ac}$. Then $(\tau_{\rm ac})_U$ is the smallest locally arc-connected refinement of τ_U.*

Proof. Denote by $(\tau_U)_{\rm ac}$ the smallest locally arc-connected refinement of τ_U. Since U is open in $(X, \tau_{\rm ac})$, the topology $(\tau_{\rm ac})_U$ is locally arc-connected; since obviously $\tau_U \subset (\tau_{\rm ac})_U$, we have also $(\tau_U)_{\rm ac} \subset (\tau_{\rm ac})_U$. We know that the inclusion map:

$$\big(U, (\tau_U)_{\rm ac}\big) \longrightarrow (X, \tau)$$

is continuous. Since $\big(U, (\tau_U)_{\rm ac}\big)$ is locally arc-connected, also the inclusion map:

$$\big(U, (\tau_U)_{\rm ac}\big) \longrightarrow (X, \tau_{\rm ac})$$

is continuous and hence the identity map:

$$\big(U, (\tau_U)_{\rm ac}\big) \longrightarrow \big(U, (\tau_{\rm ac})_U\big)$$

is continuous, i.e., $(\tau_{\rm ac})_U \subset (\tau_U)_{\rm ac}$. □

In what follows we will use the following simple lemma.

6.10. **Lemma.** *Let M be a topological space and let $M = \bigcup_{i \in I} U_i$ be an open cover of M. Assume that each U_i is endowed with a maximal differentiable atlas $\mathcal{A}_i$, compatible with the topology that U_i inherits from M, such that for all $i, j \in I$, $U_i \cap U_j$ inherits the same maximal differentiable atlas from $(U_i, \mathcal{A}_i)$ and from $(U_j, \mathcal{A}_j)$. Then there exists a unique maximal differentiable atlas $\mathcal{A}$ on M that induces the atlas $\mathcal{A}_i$ on U_i for all $i \in I$.* □

Now we prove the main the result.

6.11. **Proposition.** *Let M be a differentiable manifold and let N be a subset of M. Denote by τ the topology of M, by τ_N the topology induced by τ on N and by $(\tau_N)_{\rm ac}$ the smallest locally arc-connected refinement of τ_N. Assume that every point of N belongs to an open subset of $(\tau_N)_{\rm ac}$ which is* *an embedded submanifold of M. Then there exists a maximal differentiable atlas $\mathcal{A}$ on N, compatible with the topology $(tau_N)_{\rm ac}$, such that the inclusion $(N, \mathcal{A}) \to M$ is an immersion.*

Proof. If $U \subset N$ is an open subset with respect to $(\tau_N)_{\rm ac}$ that is embedded in M, then we may endow U with the maximal differentiable atlas that makes the inclusion $U \to M$ an embedding. Denote by τ_U the topology induced on U by (N, τ_N); observe that τ_U is also the topology induced on U by (M, τ). Since U is open with respect to $(\tau_N)_{\rm ac}$, Lemma 6.9 implies that the topology induced on U by $(\tau_N)_{\rm ac}$ is the smallest locally arc-connected refinement of τ_U. Since (U, τ_U) is already locally arc-connected, we conclude that τ_U is equal to the topology induced on U by $(\tau_N)_{\rm ac}$.

Now let U, V be open subsets of $\big(N, (\tau_N)_{\rm ac}\big)$ that are embedded in M. Then $U \cap V$ is open in $\big(N, (\tau_N)_{\rm ac}\big)$ and contained in U; hence $U \cap V$ is open in (U, τ_U). Similarly, $U \cap V$ is open in (V, τ_V). Then $U \cap V$ inherits the same maximal differentiable atlas from U and from V: that is the unique maximal differentiable atlas for which the inclusion $U \cap V \to M$ is an embedding. Finally, Lemma 6.10 gives us a maximal differentiable atlas $\mathcal{A}$ on N such that every open subset of $\big(N, (\tau_N)_{\rm ac}\big)$ that is embedded on M is an open submanifold of $(N, \mathcal{A})$. This implies that the inclusion map $(N, \mathcal{A}) \to M$ is locally an embedding and hence an immersion. □

7. Shrinking

Recall that a topological space X is called *normal* if given disjoint closed subsets $F_1, F_2 \subset X$ we can find disjoint open sets $U_1, U_2 \subset X$ with $F_i \subset U_i$, $i = 1, 2$. The space X is called *T4* if X is T1 (i.e., the points of X are closed) and X is normal.

A family $(U_i)_{i \in I}$ of subsets of X is called *pointwise finite* if for every $x \in X$ there exists at most a finite number of indices $i \in I$ with $x \in U_i$. We say that $(U_i)_{i \in I}$ is a *covering* of X if $X = \bigcup_{i \in I} U_i$; we say that $(U_i)_{i \in I}$ is an *open covering* of X if in addition each U_i is open in X. Let $(U_i)_{i \in I}$ be an open covering of X. A *shrinking* of $(U_i)_{i \in I}$ is an open covering $(V_i)_{i \in I}$ of X such that $\overline{V_i} \subset U_i$ for all $i \in I$.

Our main result is the following:

7.1. **Lemma.** *A topological space X is normal if and only if every pointwise finite open covering of X has a shrinking.*

Proof. We start with the easier part. Assume that every pointwise finite open cover of X has a shrinking and let us prove that X is normal. Given disjoint closed subsets $F_1, F_2 \subset X$ then their complements $X \setminus F_1$, $X \setminus F_2$ form a (obviously pointwise finite) open cover of X. We can thus find open sets $V_1, V_2 \subset X$ such that $X = V_1 \cup V_2$ and $\overline{V_i} \subset X \setminus F_i$, $i = 1, 2$. Then $X \setminus \overline{V_1}$ and $X \setminus \overline{V_2}$ are disjoint open sets containing F_1 and F_2 respectively.

Now we go for the harder part. Assume that X is normal and let $X = \bigcup_{i \in I} U_i$ be a pointwise finite open cover of X. We will use Zorn's Lemma.

Let $\mathcal{A}$ denote the set of all families of open sets $(V_j)_{j\in J}$, $J \subset I$, such that $\overline{V_j} \subset U_j$ for all $j \in J$ and:

$$X = \Big(\bigcup_{j\in J} V_j\Big) \cup \Big(\bigcup_{i\in I\setminus J} U_i\Big). \tag{7.1}$$

The proof will be completed once we show that there exists a family $(V_j)_{j\in J}$ in $\mathcal{A}$ with $J = I$. We define a partial order $\preceq$ on $\mathcal{A}$ by requiring that $(V_j)_{j\in J} \preceq (V'_j)_{j\in J'}$ if and only if $J \subset J'$ and $V_j = V'_j$ for all $j \in J$. Let $\big\{(V_j^\lambda)_{j\in J_\lambda} : \lambda \in \Lambda\big\}$ be an arbitrary linearly ordered subset of $\mathcal{A}$. We set $J = \bigcup_{\lambda\in\Lambda} J_\lambda \subset I$ and for all $j \in J$, $V_j = V_j^\lambda$, where $\lambda \in \Lambda$ is chosen with $j \in J_\lambda$. We obviously have a well-defined family of open sets $(V_j)_{j\in J}$ and $\overline{V}_j \subset U_j$ for all $j \in J$. We show that (7.1) holds. Let $x \in X$ be fixed and consider the set $F = \big\{i \in I : x \in U_i\big\}$. Since F is finite, there exists $\lambda \in \Lambda$ with $F \cap J \subset J_\lambda$. We know that:

$$X = \Big(\bigcup_{j\in J_\lambda} V_j\Big) \cup \Big(\bigcup_{i\in I\setminus J_\lambda} U_i\Big).$$

If $x \in \bigcup_{j\in J_\lambda} V_j$ then $x \in \bigcup_{j\in J} V_j$ and then (7.1) is proved; otherwise, there exists $i \in I \setminus J_\lambda$ with $x \in U_i$, i.e., there exists $i \in (I \setminus J_\lambda) \cap F$. But $F \cap J \subset J_\lambda$ implies $(I \setminus J_\lambda) \cap F \subset I \setminus J$ and thus again (7.1) is proved.

We are now under the hypothesis of the Zorn Lemma. Let $(V_j)_{j\in J}$ be a maximal element of $\mathcal{A}$ and assume by contradiction that $J \subsetneq I$. Choose $i_0 \in I \setminus J$. We will obtain a contradiction if we can find an open set V with $\overline{V} \subset U_{i_0}$ and:

$$X = \Big(\bigcup_{j\in J} V_j\Big) \cup V \cup \Big(\bigcup_{\substack{i\in I\setminus J\\ i\ne i_0}} U_i\Big).$$

Denote by F the complement of the open set:

$$\Big(\bigcup_{j\in J} V_j\Big) \cup \Big(\bigcup_{\substack{i\in I\setminus J\\ i\ne i_0}} U_i\Big),$$

so that F and $X \setminus U_{i_0}$ are disjoint closed subsets of X. Since X is normal, we can find disjoint open sets V, W containing F and $X \setminus U_{i_0}$ respectively. Then $\overline{V} \subset X \setminus W \subset U_{i_0}$ and $X = (X \setminus F) \cup V$. This concludes the proof. $\square$

7.2. **Corollary.** *Let X be a normal topological space, $F \subset X$ be a closed subset and $(U_i)_{i\in I}$ be a pointwise finite family of open subsets of X such that $F \subset \bigcup_{i\in I} U_i$. Then there exists a family $(V_i)_{i\in I}$ of open subsets of X such that $\overline{V_i} \subset U_i$ for all $i \in I$ and $F \subset \bigcup_{i\in I} V_i$.*

Proof. Take a shrinking of the pointwise finite open cover:

$$X = (X \setminus F) \cup \bigcup_{i\in I} U_i$$

of X; we then obtain a family $(V_i)_{i\in I}$ of open sets and an open set $A \subset X$ with $\overline{A} \subset X \setminus F$, $X = A \cup \bigcup_{i\in I} V_i$ and $\overline{V_i} \subset U_i$ for all $i \in I$. Obviously $F \subset \bigcup_{i\in I} V_i$. □

8. Paracompactness

Let $(U_i)_{i\in I}$ be a family of subsets of a topological space X. We say that $(U_i)_{i\in I}$ is *locally finite* (in X) if every point of x has a neighborhood that intersects U_i for at most a finite number of indices $i \in I$. Given coverings $(U_i)_{i\in I}$, $(V_j)_{j\in J}$ of X, we say that $(V_j)_{j\in J}$ is a *refinement* of $(U_i)_{i\in I}$ if for every $j \in J$ there exists $i \in I$ with $V_j \subset U_i$; we say that $(V_j)_{j\in J}$ is a *strict refinement* of $(U_i)_{i\in I}$ if $J = I$ and $V_i \subset U_i$ for all $i \in I$.

8.1. **Lemma.** *If a family $(U_i)_{i\in I}$ is locally finite in a topological space X then the family $(\overline{U_i})_{i\in I}$ is also locally finite in X.*

Proof. Observe that if V is an open neighborhood of $x \in X$ then V intersects U_i if and only if V intersects $\overline{U_i}$. □

8.2. **Lemma.** *If $(F_i)_{i\in I}$ is a locally finite family of closed subsets of a topological space X then the union $F = \bigcup_{i\in I} F_i$ is closed.*

Proof. Choose $x \in X$ with $x \notin F$. Let V be a neighborhood of x such that the set $J = \big\{i \in I : F_i \cap V \neq \emptyset\big\}$ is finite. Then:

$$W = \Big(\bigcap_{i\in J}(X \setminus F_i)\Big) \cap V$$

is a neighborhood of x that is disjoint from F. □

8.3. **Corollary.** *If X is a topological space and $(U_i)_{i\in I}$ is a locally finite family in X then:*

$$\overline{\bigcup_{i\in I} U_i} = \bigcup_{i\in I} \overline{U_i}.$$

Proof. The inclusion:

$$\overline{\bigcup_{i\in I} U_i} \supset \bigcup_{i\in I} \overline{U_i}$$

holds in general. The reverse inclusion is proven by observing that $\bigcup_{i\in I} \overline{U_i}$ is closed, by Lemmas 8.1 and 8.2. □

A topological space X is called *paracompact* if every open cover of X admits a locally finite open refinement. We say that X is *hereditarily paracompact* if every subspace of X is paracompact. We have the following basic lemmas.

8.4. **Lemma.** *If X is paracompact then every open cover of X admits a* strict *locally finite open refinement.*

Proof. Let $(U_i)_{i\in I}$ be an open cover of X and let $(V_j)_{j\in J}$ be a locally finite open refinement. Define a map $\phi : J \to I$ by choosing for every $j \in J$ an index $\phi(j) \in I$ with $V_j \subset U_{\phi(j)}$. Set:

$$W_i = \bigcup_{\substack{j\in J\\ \phi(j)=i}} V_j,$$

for all $i \in I$. Then it is easy to see that $(W_i)_{i\in I}$ is a strict locally finite open refinement of $(U_i)_{i\in I}$. □

8.5. **Lemma.** *If every open subspace of X is paracompact then X is hereditarily paracompact.*

Proof. Let $Y \subset X$ be an arbitrary subspace and let $(U_i)_{i\in I}$ be an open cover of Y. For each $i \in I$ choose an open subset $V_i \subset X$ with $U_i = V_i \cap Y$. By hypothesis, the open set $V = \bigcup_{i\in I} V_i$ is paracompact; thus, there exists a locally finite open refinement $(W_j)_{j\in J}$ of the open cover $(V_i)_{i\in I}$ of V. Now it is easy to see that $(W_j \cap Y)_{j\in J}$ is a locally finite open refinement of the open cover $(U_i)_{i\in I}$ of Y. □

8.6. **Lemma.** *Let X be a paracompact space and $F \subset X$ be a closed subset. Let $(U_i)_{i\in I}$ be a family of open sets with $F \subset \bigcup_{i\in I} U_i$. Then there exists a family of open sets $(V_i)_{i\in I}$ which is locally finite in X, $V_i \subset U_i$ for all $i \in I$ and $F \subset \bigcup_{i\in I} V_i$.*

Proof. The open cover $X = (X \setminus F) \cup \bigcup_{i\in I} U_i$ of X admits a strict locally finite open refinement, i.e., we can find a family of open sets $(V_i)_{i\in I}$ which is locally finite in X, an open subset $A \subset X$ with $V_i \subset U_i$ for all $i \in I$, $A \subset X \setminus F$ and $X = A \cup \bigcup_{i\in I} V_i$. Obviously $F \subset \bigcup_{i\in I} V_i$. □

8.7. **Corollary.** *If X is paracompact and $F \subset X$ is a closed subspace then F is paracompact.*

Proof. let $(W_i)_{i\in I}$ be an open cover of F and for each $i \in I$ choose an open set $U_i \subset X$ with $W_i = U_i \cap F$. Choose $(V_i)_{i\in I}$ as in Lemma 8.6. Then $(V_i \cap F)_{i\in I}$ is a (strict) locally finite open refinement of the open cover $(W_i)_{i\in I}$ of F. □

8.8. **Lemma.** *Every compact space is paracompact.* □

Recall that a topological space X is called *regular* if given $x \in X$ and a closed subset $F \subset X$ with $x \notin F$ then we can find disjoint open sets $V, W \subset X$ with $x \in V$ and $F \subset W$. The space X is called *T3* if X is T1 and regular. In view of Lemma 8.8, we know that paracompact spaces may not be Hausdorff. On the other hand, Hausdorff paracompact spaces are automatically T3 and T4, as we show in the following:

8.9. **Lemma.** *A paracompact Hausdorff space is T3.*

Proof. Let X be a paracompact Hausdorff space, $x \in X$ be a point and $F \subset X$ be a closed subset with $x \notin F$. For every $y \in F$ we can find an open neighborhood U_y of y with $x \notin \overline{U_y}$. Then $X = (X \setminus F) \cup \bigcup_{y \in F} U_y$ is an open cover of X, from which we can find a strict locally finite open refinement, i.e., a locally finite family of open sets $(V_y)_{y \in F}$ and an open subset $W \subset X \setminus F$ with $V_y \subset U_y$ for all $y \in Y$ and $X = W \cup \bigcup_{y \in F} V_y$. Then $\bigcup_{y \in F} V_y$ is an open set containing F and, by Corollary 8.3:

$$x \notin \bigcup_{y \in F} \overline{V_y} = \overline{\bigcup_{y \in Y} V_y}. \qquad \square$$

8.10. **Lemma.** *A paracompact Hausdorff space is T4.*

Proof. Let X be a paracompact Hausdorff space and let $F_1, F_2 \subset X$ be disjoint closed subspaces. We already know by Lemma 8.9 that X is T3, so for each $x \in F_1$ we can find an open neighborhood U_x of x with:

$$\overline{U_x} \cap F_2 = \emptyset.$$

Then $X = \bigcup_{x \in F_1} U_x \cup (X \setminus F_1)$ is an open cover of X, from which we obtain a strict locally finite open refinement, i.e., a locally finite family of open sets $(V_x)_{x \in F_1}$ and an open set $W \subset X \setminus F_1$ with $V_x \subset U_x$ for all $x \in F_1$ and $X = \bigcup_{x \in X} V_x \cup W$. Then $\bigcup_{x \in F_1} V_x$ is an open set containing F_1 and its closure $\bigcup_{x \in F_1} \overline{V_x}$ (Corollary 8.3) is disjoint from F_2. $\square$

Recall that, given topological spaces X, Y then a map $f : X \to Y$ is called a *local homeomorphism* if for every $x \in X$ there exists an open set U in X such that $x \in U$, $f(U)$ is open in Y and $f|_U : U \to f(U)$ is a homeomorphism. Observe that a local homeomorphism is continuous and open (i.e., takes open sets to open sets); moreover, a bijective local homeomorphism is a homeomorphism. Our main result is the following:

8.11. **Lemma** (the tubular neighborhood trick)**.** *Let X, Y be topological spaces, with Y hereditarily paracompact and Hausdorff. Let $f : X \to Y$ be a local homeomorphism; if $S \subset X$ is a subset such that $f|_S : S \to f(S)$ is a homeomorphism then there exists an open subset $Z \subset X$ containing S such that $f|_Z : Z \to f(Z)$ is a homeomorphism.*

We need a preparatory lemma.

8.12. **Lemma.** *Let X, Y be topological spaces, $f : X \to Y$ be a continuous map and $S \subset X$ be a subset such that $f|_S : S \to f(S)$ is an open map. Given $x \in S$ and an open neighborhood U of x in X then we can find an open neighborhood V of x contained in U such that $f(V \cap S) = f(V) \cap f(S)$.*

Proof. The set $U \cap S$ is open in S and thus $f(U \cap S)$ is open in $f(S)$; let $A \subset Y$ be an open set with $f(U \cap S) = A \cap f(S)$. Then $V = U \cap f^{-1}(A)$ is an open neighborhood of x contained in U. Obviously $f(V \cap S) \subset f(V) \cap f(S)$; moreover:

$$f(V) \cap f(S) \subset A \cap f(S) = f(U \cap S) = f(V \cap S).$$

The last equality above follows by observing that $U \cap S \subset f^{-1}(A)$ and hence $U \cap S = V \cap S$. $\square$

Proof of Lemma 8.11. It suffices to find an open set $Z \subset X$ containing S such that $f|_Z$ is injective. For each $x \in S$ let U'_x be an open neighborhood of x in X such that $f(U'_x) = V'_x$ is open in Y and $f|_{U'_x} : U'_x \to V'_x$ is a homeomorphism. By Lemma 8.12, we can assume that:

$$f(U'_x \cap S) = V'_x \cap f(S). \tag{8.1}$$

The set:

$$Y_0 = \bigcup_{x \in S} V'_x$$

is open in Y and it contains $f(S)$. Moreover, Y_0 is Hausdorff and paracompact; therefore, by Lemma 8.10, Y_0 is also T4. Let $Y_0 = \bigcup_{i \in I} V_i$ be a locally finite open refinement of the open cover $Y_0 = \bigcup_{x \in S} V'_x$ of Y_0 (the family $(V_i)_{i \in I}$ is locally finite *in* Y_0). For each $i \in I$, choose $x \in S$ with $V_i \subset V'_x$ and set:

$$U_i = (f|_{U'_x})^{-1}(V_i).$$

Then $U_i \subset U'_x$ is open in X, $f|_{U_i} : U_i \to V_i$ is a homeomorphism and from (8.1) we get:

$$f(U_i \cap S) = V_i \cap f(S), \tag{8.2}$$

for all $i \in I$. By Lemma 7.1, there exists a shrinking $Y_0 = \bigcup_{i \in I} W_i$ of the open cover $Y_0 = \bigcup_{i \in I} V_i$ of Y_0, i.e., $\overline{W_i} \subset V_i$ for all $i \in I$ (the closure on W_i will always be taken with respect to the space Y_0). For each $i \in I$ set:

$$Z_i = (f|_{U_i})^{-1}(W_i).$$

Then $Z_i \subset U_i$ is open in X, $f|_{Z_i} : Z_i \to W_i$ is a homeomorphism and from (8.2) we get:

$$f(Z_i \cap S) = W_i \cap f(S), \tag{8.3}$$

for all $i \in I$. We claim that:

$$S \subset \bigcup_{i \in I} Z_i. \tag{8.4}$$

Namely, given $x \in S$, there exists $i \in I$ with $f(x) \in W_i$. Then $f(x) \in W_i \cap f(S)$ and therefore, by (8.3), we can find $y \in Z_i \cap S$ with $f(x) = f(y)$. Since $f|_S$ is injective, we obtain $x = y \in Z_i$, proving the claim.

Now for $x \in S$, we set:

$$I_x = \big\{ i \in I : f(x) \in \overline{W_i} \big\};$$

since the cover $Y_0 = \bigcup_{i \in I} \overline{W_i}$ is locally finite, the set I_x is finite and nonempty. Observe that for $i \in I_x$ we have, using (8.2):

$$f(x) \in \overline{W_i} \cap f(S) \subset V_i \cap f(S) = f(U_i \cap S)$$

and thus the injectivity of $f|_S$ implies $x \in U_i$. We have just shown that:

$$x \in \bigcap_{i \in I_x} U_i, \tag{8.5}$$

for all $x \in S$.

Our next goal is to find for each $x \in S$ an open neighborhood G_x of $f(x)$ in Y_0 with the following properties:

(i) for each $i \in I$, G_x intersects W_i if and only if $i \in I_x$;
(ii) $G_x \subset f\big(\bigcap_{i\in I_x} U_i\big)$.

The desired set G_x can be defined by:

$$G_x = f\Big(\bigcap_{i\in I_x} U_i\Big) \cap \Big(Y_0 \setminus \bigcup_{i\in I\setminus I_x} \overline{W_i}\Big).$$

The fact that $f(x) \in G_x$ follows from (8.5) and property (ii) is obvious. For property (i), observe that $i \in I_x$ implies $f(x) \in G_x \cap \overline{W_i}$ and thus $G_x \cap W_i \neq \emptyset$; moreover, for $i \in I \setminus I_x$ we obviously have $G_x \cap W_i = \emptyset$. The fact that G_x is open follows from the fact that f is an open map and from Lemma 8.2.

Now set $G = \bigcup_{x\in S} G_x$ and finally:

$$Z = f^{-1}(G) \cap \bigcup_{i\in I} Z_i.$$

Obviously Z is open in X and $S \subset Z$, by (8.4). We complete the proof by showing that $f|_Z$ is injective. Let $x, y \in Z$ be chosen with $f(x) = f(y)$. We can find $i, j \in I$ with $x \in Z_i$ and $y \in Z_j$. Moreover, $f(x) = f(y) \in G_z$ for some $z \in S$. We have $f(x) \in G_z \cap W_i$ and $f(y) \in G_z \cap W_j$, so that $i, j \in I_z$, by property (i). Now property (ii) implies $G_z \subset f(U_i \cap U_j)$; we can thus find $p \in U_i \cap U_j$ with $f(x) = f(p) = f(y)$. Since f is injective in U_i and in U_j, we conclude that $x = p = y$. □

8.13. *Remark.* In Lemma 8.11, if we add the hypothesis that $f(S)$ be closed in Y then we may replace the hypothesis that Y be hereditarily paracompact by the hypothesis that Y be paracompact. To this aim, the proof of the lemma has to be adapted as follows. When we take the locally finite open refinement $(V_i)_{i\in I}$ of $(V'_x)_{x\in S}$, we use Lemma 8.6 and obtain a family of open sets $(V_i)_{i\in I}$ which is locally finite in Y, $f(S) \subset \bigcup_{i\in I} V_i$ and each V_i contained in some V'_x (actually Lemma 8.6 allows us to take $I = S$ and $V_x \subset V'_x$, but we don't need that). Similarly, when we take the shrinking $(W_i)_{i\in I}$ of $(V_i)_{i\in I}$ we may use Corollary 7.2 to obtain a family of open sets $(W_i)_{i\in I}$ with $\overline{W_i} \subset V_i$ for all $i \in I$ and $f(S) \subset \bigcup_{i\in I} W_i$ (in this case we may even take the closure of W_i in Y, rather than in Y_0).

8.14. **Corollary.** *Let X be a Hausdorff topological space and $D \subset X$ be a discrete subspace. If either X is paracompact and D is closed or X is hereditarily paracompact then for each $p \in D$ we can find an open neighborhood U_p of p in D with $U_p \cap U_q = \emptyset$ for all $p, q \in D$, $p \neq q$.*

Proof. Denote by $\pi : D \times X \to X$ the projection onto the second coordinate and by $\Delta \subset D \times X$ the diagonal of $D \times D$. Then π is a local homeomorphism and $\pi|_\Delta : \Delta \to D$ is a homeomorphism. By Lemma 8.11 (see also

Remark 8.13) there exists an open subset $U \subset D \times X$ containing Δ such that $\pi|_U$ is injective. The proof is now completed by setting:

$$U_p = \big\{x \in X : (p, x) \in U\big\},$$

for all $p \in D$. □

8.15. *Remark.* If X is metrizable then there exists a much simpler proof of Corollary 8.14. Namely, for each $p \in D$ let $r_p > 0$ be such that the open ball $\mathrm{B}(p; r_p)$ intersects D only at p. The desired set U_p can be taken equal to $\mathrm{B}\big(p; \frac{r_p}{2}\big)$.

If X is a topological space then a *sheaf* over X is a pair $(\mathcal{S}, \pi)$, where $\mathcal{S}$ is a topological space and $\pi : \mathcal{S} \to X$ is a local homeomorphism. If $A \subset X$ then a *section* of the sheaf $(\mathcal{S}, \pi)$ over A is a map $s : A \to \mathcal{S}$ with $\pi \circ s = \mathrm{Id}_A$.

8.16. **Corollary.** *Let X be a Hausdorff space and $A \subset X$ a subset. Assume that either X is paracompact and A is closed or that X is hereditarily paracompact. Given a sheaf $(\mathcal{S}, \pi)$ over X then every continuous section $s : A \to \mathcal{S}$ of $(\mathcal{S}, \pi)$ over A extends to a continuous section defined on an open subset $U \subset X$ containing A.*

Proof. We have that $\pi : \mathcal{S} \to X$ is a local homeomorphism (by definition) and that $\pi|_{s(A)} : s(A) \to A$ is a homeomorphism (whose inverse is s). By Lemma 8.11 (see also Remark 8.13), there exists an open subset $Z \subset \mathcal{S}$ containing $s(A)$ such that $\pi|_Z : Z \to \pi(Z)$ is a homeomorphism. Now simply set $U = \pi(Z)$ and observe that $(\pi|_Z)^{-1} : U \to \mathcal{S}$ extends s. □

8.17. **Definition.** A topological space X is called *strongly paracompact* if for every basis $\mathcal{B}$ of open subsets of X there exists a locally finite open cover $(U_i)_{i \in I}$ of X with $U_i \in \mathcal{B}$ for all $i \in I$.

8.18. **Lemma.** *If X is strongly paracompact then for every basis of open sets $\mathcal{B}$ and for every open cover $(U_i)_{i \in I}$ of X, there exists an open locally finite refinement $(V_j)_{j \in J}$ of $(U_i)_{i \in I}$ with $V_j \in \mathcal{B}$ for all $j \in J$. In particular, every strongly paracompact space is paracompact.*

Proof. The set:

$$\mathcal{B}' = \big\{B \in \mathcal{B} : B \subset U_i, \text{ for some } i \in I\big\},$$

is a basis of open sets for X. Then simply take $(V_j)_{j \in J}$ to be a locally finite open cover of X with $V_j \in \mathcal{B}'$ for every $j \in J$. □

Observe that in general one is not supposed to find *strict* locally finite open refinements of an open cover $(U_i)_{i \in I}$ consisting of elements of $\mathcal{B}$.

Recall that a topological space X is called *σ-compact* if X is a countable union of compact subspaces. Every second countable locally compact space is σ-compact.

We have the following important result concerning strong paracompactness:

8.19. **Lemma.** *Every locally compact Hausdorff σ-compact topological space is strongly paracompact.*

Proof. We can write $X = \bigcup_{n=1}^{+\infty} K_n$ as a union of compact subsets $K_n \subset X$ with K_n contained in the interior of K_{n+1} for all n. This is a standard construction whose proof we recall. Write $X = \bigcup_{n=1}^{+\infty} L_n$ as a union of compact subsets L_n. We construct the sequence K_n inductively. Take $K_1 = L_1$. If K_n has been constructed, cover K_n with a finite number of open sets with compact closure; now define K_{n+1} to be equal to the union of L_{n+1} and the closure of the finite union of such open sets. This completes the construction. For the rest of the proof we set $K_n = \emptyset$ for $n \le 0$.

Now let $\mathcal{B}$ be a basis of open subsets of X and let us construct a locally finite open cover of X consisting of elements of $\mathcal{B}$. For each $n \ge 1$ we set:

$$C_n = \overline{K_n \setminus K_{n-1}} = K_n \setminus \operatorname{int}(K_{n-1});$$

observe that $X = \bigcup_{n=1}^{+\infty} C_n$. Let $n \ge 1$ be fixed. For each x in the compact set C_n, we choose $V_x^n \in \mathcal{B}$ with $x \in V_x^n \subset \operatorname{int}(K_{n+1}) \setminus K_{n-2}$. Consider a finite subcover:

$$C_n \subset \bigcup_{j=1}^{r_n} V_{x_j^n}^n,$$

of the open cover $C_n \subset \bigcup_{x \in C_n} V_x^n$. Now the family:

$$\mathcal{V} = \big(V_{x_j^n}^n\big)_{1 \le j \le r_n,\ n \ge 1}$$

is an open cover of X consisting of elements of $\mathcal{B}$. We now show that $\mathcal{V}$ is locally finite. Let $x \in X$ be fixed and let $n \ge 1$ be the smallest integer with $x \in K_n$. Then $A = \operatorname{int}(K_{n+1}) \setminus K_{n-1}$ is an open neighborhood of x. Moreover, A does not intersect $V_{x_j^m}^m$ if $m \le n-2$ or $m \ge n+3$. Thus A intersects at most $\sum_{k=n-1}^{n+2} r_k < +\infty$ elements of $\mathcal{V}$. □

9. Topological Vector Spaces

Let X be a vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and let τ be a topology on X. We say that (X, τ) is a *topological vector space* if the maps:

$$X \times X \ni (x, y) \longmapsto x + y \in X, \quad \mathbb{K} \times X \ni (\lambda, x) \longmapsto \lambda x \in X, \tag{9.1}$$

are continuous, where the products $X \times X$ and $\mathbb{K} \times X$ are endowed with the usual product topologies and $\mathbb{K}$ is endowed with the usual Euclidean topology.

9.1. **Lemma.** *Let (X, τ) be a topological vector space and let $T : \mathbb{K}^n \to X$ be a linear map. If $\mathbb{K}^n$ is endowed with the standard Euclidean topology then T is continuous.*

Proof. Let $(e_i)_{i=1}^n$ denote the canonical basis of $\mathbb{K}^n$ and set $T(e_i) = b_i$, $i = 1, \dots, n$. If $\pi_i : \mathbb{K}^n \to \mathbb{K}$ denotes the projection onto the first coordinate then:

$$T(v) = \sum_{i=1}^{n} \pi_i(v) b_i.$$

Since the projections of $\mathbb{K}^n$ are continuous and the vector space operations of X are continuous, it follows that T is continuous. □

9.2. **Lemma.** *Let X be a topological vector space. If $U \subset X$ is a neighborhood of the origin then there exists an open neighborhood of the origin $V \subset X$ such that every segment with endpoints in V is contained in U, i.e., $(1-t)x + ty \in U$ for all $x, y \in V$, $t \in [0,1]$.*

Proof. We may assume without loss of generality that U is open. The map $S : [0,1] \times X \times X \to X$ defined by $S(t,x,y) = (1-t)x + ty$ is continuous, because the vector space operations of X are continuous. Thus, $S^{-1}(U)$ is an open subset of the product $[0,1] \times X \times X$ containing $[0,1] \times \{0\} \times \{0\}$. Since $[0,1]$ is compact, there exists a neighborhood A of $(0,0)$ in $X \times X$ such that $[0,1] \times A \subset S^{-1}(U)$. We may thus find an open neighborhood V of 0 in X such that $V \times V \subset A$. Hence every segment with endpoints in V is contained in U. □

9.3. **Definition.** A topology in a vector space X is said to be *translation invariant* if for every $v \in X$ the *translation map*:

$$\mathfrak{t}_v : X \ni x \longmapsto x + v \in X$$

is continuous.

Since the inverse of the translation $\mathfrak{t}_v$ is the translation $\mathfrak{t}_{-v}$, it follows that if X is endowed with a translation invariant topology then all translation maps are actually homeomorphisms of X. Obviously the topology of a topological vector space is translation invariant.

9.4. **Lemma.** *Let X, Y be vector spaces over $\mathbb{K}$ endowed with translation invariant topologies (this is the case if X and Y are topological vector spaces). Then a linear map $T : X \to Y$ is continuous if and only if it is continuous at the origin.*

Proof. Obviously if T is continuous then T is continuous at the origin. Assume now that T is continuous at the origin. Since T is linear, for every $v \in X$, we have:

$$T = \mathfrak{t}_{T(v)} \circ T \circ \mathfrak{t}_{-v}. \tag{9.2}$$

Since translations are continuous, the continuity of T at the origin implies the continuity of the righthand side of (9.2) at the point v. Thus, T is continuous at the point v. □

9.5. **Lemma.** *Let X be a topological vector space. If $U \subset X$ is open and $\lambda \in \mathbb{K}$ is not zero then the set:*

$$\lambda U = \big\{\lambda x : x \in U\big\}, \tag{9.3}$$

is open in X.

Proof. Simply observe that the homotety $x \mapsto \lambda x$ is a homeomorphism of X onto itself. □

9.6. **Lemma.** *let τ be a topology on $\mathbb{K}^n$ for which $(\mathbb{K}^n, \tau)$ is a Hausdorff topological vector space. Then τ coincides with the standard Euclidean topology of $\mathbb{K}^n$ (in particular, all norms on a finite-dimensional vector space define the same topology).*

Proof. Denote by τ_{e} the Euclidean topology of $\mathbb{K}^n$. By Lemma 9.1, the identity map:

$$\mathrm{Id} : (\mathbb{K}^n, \tau_{\mathrm{e}}) \longrightarrow (\mathbb{K}^n, \tau) \tag{9.4}$$

is continuous. We will show now that the identity map:

$$\mathrm{Id} : (\mathbb{K}^n, \tau) \longrightarrow (\mathbb{K}^n, \tau_{\mathrm{e}}) \tag{9.5}$$

is also continuous. By Lemma 9.4, it suffices to show that (9.5) is continuous at the origin. We claim that such continuity will follow from the existence of a neighborhood of the origin in $(\mathbb{K}^n, \tau)$ which is bounded with respect to the Euclidean metric. Namely, assume that there exists an open neighborhood V of 0 in $(\mathbb{K}^n, \tau)$ which is bounded with respect to the Euclidean metric. Given an Euclidean ball $\mathrm{B}(0, r)$, $r > 0$, we may then find $\lambda > 0$ such that $\lambda V \subset \mathrm{B}(0, r)$; thus, by Lemma 9.5, λV is a neighborhood of the origin in $(\mathbb{K}^n, \tau)$ which is carried by (9.5) to a subset of $\mathrm{B}(0, r)$. This proves the claim.

Let us now show the existence of a neighborhood V of the origin in $(\mathbb{K}^n, \tau)$ which is bounded in the Euclidean metric. Let S^{n-1} denote the Euclidean unit sphere of $\mathbb{K}^n$. Since S^{n-1} is compact in $(\mathbb{K}^n, \tau_{\mathrm{e}})$ and (9.4) is continuous, it follows that S^{n-1} is also compact in $(\mathbb{K}^n, \tau)$. Since τ is Hausdorff, S^{n-1} is closed (and hence $\mathbb{K}^n \setminus S^{n-1}$ is open) in $(\mathbb{K}^n, \tau)$. By Lemma 9.2, there exists an open neighborhood of the origin V in $(\mathbb{K}^n, \tau)$ such that every segment with endpoints in V is contained in $U = \mathbb{K}^n \setminus S^{n-1}$. But this implies that V is contained in the open unit ball (because a segment with one endpoint outside the open unit ball and the other endpoint at the origin crosses the sphere S^{n-1}). Hence V is bounded. □

We recall a couple of basic definitions and a few facts from general topology.

9.7. **Definition.** Let $\big((\mathcal{Y}_i, \tau_i)\big)_{i \in I}$ be a family of topological spaces, $\mathcal{X}$ be a set and for each $i \in I$ let $f_i : \mathcal{X} \to \mathcal{Y}_i$ be a map; the topology on $\mathcal{X}$ *induced* by the family of maps $(f_i)_{i \in I}$ is the smallest topology on $\mathcal{X}$ for which all the maps f_i are continuous (it is the intersection of all topologies on $\mathcal{X}$ containing the sets $f_i^{-1}(U)$, $U \in \tau_i$, $i \in I$).

A basis of open sets for the topology induced by the maps f_i consists of the sets of the form[2]:

$$f_{i_1}^{-1}(U_1) \cap \ldots \cap f_{i_k}^{-1}(U_k), \tag{9.6}$$

with $i_1, \ldots, i_k \in I$, $U_1 \in \tau_{i_1}, \ldots, U_k \in \tau_{i_k}$. Given a map g defined in some topological space and taking values in $\mathcal{X}$, then g is continuous at a given point with respect to the topology induced by the maps f_i if and only if $f_i \circ g$ is continuous at that point, for all $i \in I$.

9.8. **Definition.** Given a family $(\tau_i)_{i \in I}$ of topologies in a set $\mathcal{X}$ then the *supremum*

$$\sup_{i \in I} \tau_i$$

is the smallest topology in $\mathcal{X}$ that contains τ_i for all $i \in I$, i.e., it is the topology induced by the identity maps:

$$\mathrm{Id} : \mathcal{X} \longrightarrow (\mathcal{X}, \tau_i), \quad i \in I.$$

A basis of open sets for the topology $\sup_{i \in I} \tau_i$ consists of the sets of the form:

$$U_1 \cap \ldots \cap U_k,$$

with $U_1 \in \tau_{i_1}, \ldots, U_k \in \tau_{i_k}$, $i_1, \ldots, i_k \in I$. Given a map g defined in some topological space and taking values in $\mathcal{X}$, then g is continuous at a given point with respect to the topology $\sup_{i \in I} \tau_i$ if and only if g is continuous at that point with respect to τ_i, for all $i \in I$.

9.9. **Lemma.** *Let $(X_i)_{i \in I}$ be a family of topological vector spaces over $\mathbb{K}$, X be a vector space over $\mathbb{K}$ and for each $i \in I$ let $T_i : X \to X_i$ be a linear map. The topology on X induced by the maps T_i makes it into a topological vector space.*

Proof. To prove the continuity of the maps (9.1), we have to prove the continuity of the maps:

$$X \times X \ni (x, y) \longmapsto T_i(x + y) = T_i(x) + T_i(y) \in X_i, \tag{9.7}$$

$$\mathbb{K} \times X \ni (\lambda, x) \longmapsto T_i(\lambda x) = \lambda T_i(x) \in X_i, \tag{9.8}$$

for all $i \in I$. The map (9.7) is the composite of the map:

$$T_i \times T_i : X \times X \ni (x, y) \longmapsto \big(T_i(x), T_i(y)\big) \in X_i \times X_i,$$

with the map:

$$X_i \times X_i \ni (z, w) \longmapsto z + w \in X_i,$$

and therefore it is continuous. Similarly, the map (9.8) is the composite of the map:

$$\mathrm{Id} \times T_i : \mathbb{K} \times X \ni (\lambda, x) \longmapsto \big(\lambda, T_i(x)\big) \in \mathbb{K} \times X_i,$$

[2]When I is empty, the induced topology is the chaotic topology $\{\emptyset, \mathcal{X}\}$. In this case, the only possibility in (9.6) is $k = 0$ and the intersection in (9.6) is understood to be equal to $\mathcal{X}$.

with the map:

$$\mathbb{K} \times X_i \ni (\lambda, z) \longmapsto \lambda z \in X_i,$$

and therefore it is also continuous. □

9.10. **Corollary.** *Let X be a vector space over $\mathbb{K}$ and let $(\tau_i)_{i\in I}$ be a family of topologies on X such that, for all $i \in I$, (X, τ_i) is a topological vector space over $\mathbb{K}$. If $\tau = \sup_{i\in I} \tau_i$ then (X, τ) is a topological vector space over $\mathbb{K}$.*

Proof. In Lemma 9.9 let X_i be X endowed with τ_i and let T_i be the identity map of X. □

9.11. **Lemma.** *Let X be a complex vector space endowed with a topology. If the maps:*

$$\begin{gathered} X \times X \ni (x, y) \longmapsto x + y \in X, \\ \mathbb{R} \times X \ni (\lambda, x) \longmapsto \lambda x \in X, \\ X \ni x \longmapsto ix \in X \end{gathered} \tag{9.9}$$

are continuous then also the map:

$$\mathbb{C} \times X \ni (\lambda, x) \longmapsto \lambda x \in X \tag{9.10}$$

is continuous.

Proof. By identifying $\mathbb{C}$ with $\mathbb{R} \times \mathbb{R}$, the map (9.10) is identified with the map:

$$\mathbb{R} \times \mathbb{R} \times X \ni (a, b, x) \longmapsto ax + bix \in X,$$

which can be easily written as a composition involving the maps (9.9). □

10. Locally convex topologies

Let X be a vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A *semi-norm* on X is a non negative real valued function $p : X \to \mathbb{R}$ satisfying the *triangle inequality*:

$$p(x + y) \leq p(x) + p(y), \quad x, y \in X,$$

and the condition:

$$p(\lambda x) = |\lambda| p(x), \quad \lambda \in \mathbb{K},\ x \in X. \tag{10.1}$$

Condition (10.1) implies $p(0) = 0$ (set $x = 0$ and $\lambda = 0$). Observe that condition (10.1) depends on the field $\mathbb{K}$; we shall sometimes speak of a *real* semi-norm (resp., a *complex* semi-norm) when condition (10.1) is satisfied[3] with $\mathbb{K} = \mathbb{R}$ (resp., with $\mathbb{K} = \mathbb{C}$).

10.1. **Lemma.** *Let X be a vector space over $\mathbb{K}$.*

(a) *given semi-norms p, p' on X then $p + p'$ is a semi-norm on X;*

[3] When we speak just of a "semi-norm" on a vector space X we mean that $\mathbb{K}$ in (10.1) is the scalar field of the space X; however, when X is complex, it is more convenient to speak of a "real semi-norm" on X than to speak of a "semi-norm in the real vector space obtained from X by restricting to $\mathbb{R} \times X$ the operation of multiplication by scalars".

(b) *given a non empty family of semi-norms* $(p_i)_{i\in I}$ *on* X*, if the supremum:*

$$p(x) \overset{def}{=} \sup_{i\in I} p_i(x)$$

is finite for all $x \in X$ *then* p *is a semi-norm on* X *(in particular, if* I *is finite then* $p(x) = \max_{i\in I} p_i(x)$ *defines a semi-norm on* X*);*

(c) *given another vector space* Y *over* $\mathbb{K}$*, a linear map* $T : X \to Y$ *and a semi-norm* p *on* Y *then* $p \circ T$ *is a semi-norm on* X*.*

Proof. Straightforward. □

10.2. **Lemma.** *Let* X *be a vector space endowed with a translation invariant topology (see Definition 9.3; that is the case if* X *is a topological vector space) and let* $p : X \to \mathbb{R}$ *be a semi-norm in* X*. Then* p *is continuous if and only if it is continuous at the origin (where* $\mathbb{R}$ *is endowed with the usual Euclidean topology).*

Proof. Obviously if p is continuous then p is continuous at the origin. Conversely, assume that p is continuous at the origin and let $x \in X$ and $\varepsilon > 0$ be given. We have to show that for y in some neighborhood of x we have $|p(y) - p(x)| < \varepsilon$. The triangle inequality for p easily implies:

$$|p(y) - p(x)| \le p(y - x),$$

so that $|p(y) - p(x)| < \varepsilon$ for all y in the set:

$$\big\{y \in X : p(y - x) < \varepsilon\big\}. \tag{10.2}$$

But the set (10.2) is the image under the translation $\mathfrak{t}_x : y \mapsto y+x$ of the set $p^{-1}\big(\,]-\infty, \varepsilon[\,\big)$, which is a neighborhood of the origin, by the continuity of p at the origin. Since the translation $\mathfrak{t}_x$ is a homeomorphism, the set (10.2) is a neighborhood of x. □

10.3. **Corollary.** *Let* X *be a topological vector space and let* $p : X \to \mathbb{R}$ *be a semi-norm in* X*. Then* p *is continuous if and only if* p *is bounded in some neighborhood of the origin.*

Proof. If p is continuous then $p^{-1}\big(\,]-\infty, 1[\,\big)$ is a neighborhood of the origin in which p is bounded. Conversely, if there exists a neighborhood of the origin V and a constant $c > 0$ such that $p(x) < c$ for all $x \in V$ then for all $\varepsilon > 0$ the set (recall (9.3)):

$$W = \tfrac{\varepsilon}{c} V$$

is a neighborhood of the origin such that $p(y) < \varepsilon$, for all $y \in W$, so that p is continuous at the origin and hence continuous. □

A semi-norm p on X defines a *pseudo-metric*[4]:

$$X \times X \ni (x, y) \longmapsto p(x - y) \in \mathbb{R}$$

[4]A pseudo-metric satisfies the same axioms of a metric, except for the fact that the distance between distinct points is allowed to be zero.

on X which defines a topology $\tau(p)$ on X. By definition, a subset U of X is in $\tau(p)$ if and only if for every $x \in U$ there exists $r > 0$ such that the open ball of center x and radius r:

$$\big\{y \in X : p(y - x) < r\big\} \tag{10.3}$$

is contained in U.

10.4. **Lemma.** *If p is a semi-norm on a vector space X then X endowed with the topology $\tau(p)$ is a topological vector space. Moreover, if X is endowed with $\tau(p)$ then the map $p : X \to \mathbb{R}$ is continuous.*

Proof. We have to prove the continuity of the maps:

$$X \times X \ni (x, y) \longmapsto x + y \in X, \tag{10.4}$$

$$\mathbb{K} \times X \ni (\lambda, x) \longmapsto \lambda x \in X. \tag{10.5}$$

The continuity of the sum map (10.4) follows easily from the inequality:

$$p\big((x' + y') - (x + y)\big) \le p(x' - x) + p(y' - y), \quad x, y, x', y' \in X.$$

More explicitly, given $x, y \in X$ and $\varepsilon > 0$ then it follows from such inequality that:

$$p\big((x' + y') - (x + y)\big) < \varepsilon,$$

for all $x', y' \in X$ with $p(x' - x) < \frac{\varepsilon}{2}$ and $p(y' - y) < \frac{\varepsilon}{2}$. For the continuity of the multiplication map (10.5), observe that for $x, x' \in X$, $\lambda, \lambda' \in \mathbb{K}$, we have:

$$p(\lambda' x' - \lambda x) \le p\big(\lambda'(x' - x)\big) + p\big((\lambda' - \lambda)x\big) = |\lambda'| p(x' - x) + |\lambda' - \lambda| p(x),$$

so that, if $|\lambda - \lambda'| \le 1$, then:

$$p(\lambda' x' - \lambda x) \le \big(|\lambda| + 1\big) p(x' - x) + |\lambda' - \lambda| p(x).$$

Thus, given $x \in X$, $\lambda \in \mathbb{K}$, $\varepsilon > 0$, we have:

$$p(\lambda' x' - \lambda x) < \varepsilon$$

provided that $x' \in X$, $\lambda' \in \mathbb{K}$ satisfy:

$$p(x' - x) < \frac{\varepsilon}{2\big(|\lambda| + 1\big)}, \quad |\lambda' - \lambda| < \min\Big\{\frac{\varepsilon}{2\big(p(x) + 1\big)}, 1\Big\}.$$

Finally, for the continuity of the map p, simply observe that the open ball $p^{-1}\big(\,]-\infty, 1[\,\big)$ is a neighborhood of the origin on which p is bounded, so that p is continuous, by Corollary 10.3. □

10.5. **Lemma.** *Let X be a vector space and p be a semi-norm on X. The topology $\tau(p)$ is the smallest translation invariant topology on X (see Definition 9.3) for which the map $p : X \to \mathbb{R}$ is continuous, i.e., $\tau(p)$ is a translation invariant topology on X for which p is continuous and it is contained in every translation invariant topology τ on X for which p is continuous.*

Proof. It follows from Lemma 10.4 that $\tau(p)$ is a translation invariant topology for which p is continuous. Let τ be a translation invariant topology on X for which p is continuous. The open ball (10.3) is the image under the translation $\mathfrak{t}_x : y \mapsto y + x$ of the set $p^{-1}\big(\left]-\infty, r\right[\big)$ and therefore it belongs to τ. Since every element of $\tau(p)$ is a union of open balls, it follows that $\tau(p) \subset \tau$. □

Let X be a vector space over $\mathbb{K}$ and let $\mathcal{P}$ be a set of semi-norms in X. We set (recall Definition 9.8):

$$\tau(\mathcal{P}) = \sup_{p \in \mathcal{P}} \tau(p).$$

A fundamental system of (open) neighborhoods of the origin with respect to the topology $\tau(\mathcal{P})$ consists of all sets of the form[5]:

$$\big\{x \in X : p_i(x) < \varepsilon,\ i = 1, \ldots, k\big\}, \tag{10.6}$$

with $p_1, \ldots, p_k \in \mathcal{P}$ and $\varepsilon > 0$. A map defined in some topological space and taking values in X is continuous at a given point with respect to $\tau(\mathcal{P})$ if and only if it is continuous at that point with respect to $\tau(p)$, for all $p \in \mathcal{P}$.

10.6. **Lemma.** *Let $\mathcal{P}$ be a set of semi-norms in a vector space X. Then X is a topological vector space endowed with the topology $\tau(\mathcal{P})$. Moreover, every $p \in \mathcal{P}$ is continuous with respect to $\tau(\mathcal{P})$.*

Proof. The fact that X endowed with $\tau(\mathcal{P})$ is a topological vector space follows from Lemma 10.4 and Corollary 9.10. The fact that each $p \in \mathcal{P}$ is continuous with respect to $\tau(\mathcal{P})$ follows from Lemma 10.4 and from the fact that $\tau(p) \subset \tau(\mathcal{P})$. □

10.7. **Lemma.** *Let $\mathcal{P}$ be a set of semi-norms in a vector space X. Then $\tau(\mathcal{P})$ is the smallest translation invariant topology on X that makes each $p \in \mathcal{P}$ continuous.*

Proof. It follows from Lemma 10.6 that $\tau(\mathcal{P})$ is a translation invariant topology for which each $p \in \mathcal{P}$ is continuous. Moreover, if τ is a translation invariant topology on X for which each $p \in \mathcal{P}$ is continuous then, by Lemma 10.5, τ contains $\tau(p)$ for each $p \in \mathcal{P}$ and therefore τ contains $\tau(\mathcal{P}) = \sup_{p \in \mathcal{P}} \tau(p)$. □

10.8. **Corollary.** *Let $\mathcal{P}$ be a set of semi-norms in a vector space X and let $\mathcal{P}_{\max}$ denote the set of all semi-norms in X that are continuous with respect to $\tau(\mathcal{P})$. Then $\tau(\mathcal{P}) = \tau(\mathcal{P}_{\max})$ and every set of semi-norms $\mathcal{P}'$ in X such that $\tau(\mathcal{P}) = \tau(\mathcal{P}')$ is contained in $\mathcal{P}_{\max}$.*

Proof. Since $\mathcal{P} \subset \mathcal{P}_{\max}$, we have $\tau(\mathcal{P}) \subset \tau(\mathcal{P}_{\max})$. Moreover, since $\tau(\mathcal{P})$ is a translation invariant topology that makes each $p \in \mathcal{P}_{\max}$ continuous, it follows that $\tau(\mathcal{P}_{\max}) \subset \tau(\mathcal{P})$. Finally, if $\tau(\mathcal{P}) = \tau(\mathcal{P}')$ then each $p \in \mathcal{P}'$ is continuous with respect to $\tau(\mathcal{P}') = \tau(\mathcal{P})$, so that $\mathcal{P}' \subset \mathcal{P}_{\max}$. □

[5] When $\mathcal{P}$ is empty, $\tau(\mathcal{P})$ is the chaotic topology $\{\emptyset, X\}$. In this case, in (10.6) one must take $k = 0$ and the set (10.6) is X.

10.9. **Lemma.** *Let $\mathcal{Y}$ be a topological space, X be a vector space and $\mathcal{P}$ be a set of semi-norms in X; assume X to be endowed with the topology $\tau(\mathcal{P})$. Let $f : \mathcal{Y} \to X$ be a map and $y \in \mathcal{Y}$ be a point with $f(y) = 0$. Then f is continuous at the point y if and only if $p \circ f$ is continuous at the point y, for all $p \in \mathcal{P}$.*

Proof. If f is continuous at y then $p \circ f$ is continuous at y, for all $p \in \mathcal{P}$, since $p : X \to \mathbb{R}$ is continuous. Conversely, in order to check that $f : \mathcal{Y} \to X$ is continuous at y it suffices to show that, for each $p \in \mathcal{P}$, the map $f : \mathcal{Y} \to X$ is continuous at y when X is endowed with $\tau(p)$. In order to establish such continuity, observe that a neighborhood of $0 = f(y)$ in $\big(X, \tau(p)\big)$ contains a set of the form $p^{-1}\big(\left]-\infty, r\right[\big)$, for some $r > 0$ and that:

$$f^{-1}\Big(p^{-1}\big(\left]-\infty, r\right[\big)\Big) = (p \circ f)^{-1}\big(\left]-\infty, r\right[\big)$$

is a neighborhood of y in $\mathcal{Y}$, by the continuity of $p \circ f$ at y. □

10.10. **Corollary.** *Let X, Y be vector spaces over $\mathbb{K}$. Assume that $\mathcal{P}$ is a set of semi-norms in X, that X is endowed with the topology $\tau(\mathcal{P})$ and that Y is endowed with a translation invariant topology (that is the case if Y is a topological vector space). Given a linear map $T : Y \to X$, then T is continuous if and only if $p \circ T$ is continuous for all $p \in \mathcal{P}$.*

Proof. Since every $p \in \mathcal{P}$ is continuous, the continuity of T implies the continuity of $p \circ T$. Conversely, if $p \circ T$ is continuous for all $p \in \mathcal{P}$ then, since $T(0) = 0$, Lemma 10.9 implies that T is continuous at the origin and then the continuity of T follows from Lemma 9.4. □

10.11. **Lemma.** *Let $(X_i)_{i \in I}$ be a family of vector spaces over $\mathbb{K}$ and let X be a vector space over $\mathbb{K}$. For each $i \in I$ let $\mathcal{P}_i$ be a set of semi-norms in X_i and let $T_i : X \to X_i$ be a linear map. If each X_i is endowed with the topology $\tau(\mathcal{P}_i)$ then the topology τ on X induced by the maps T_i (recall Definition 9.7) coincides with the topology $\tau(\mathcal{P})$, where:*

$$\mathcal{P} = \bigcup_{i \in I} \big\{p \circ T_i : p \in \mathcal{P}_i\big\}.$$

Proof. By Lemma 10.6, each X_i is a topological vector space and therefore, by Lemma 9.9, (X, τ) is a topological vector space; in particular, the topology τ is translation invariant[6]. Since every element of $\mathcal{P}$ is continuous with respect to τ, it follows from Lemma 10.7 that τ contains $\tau(\mathcal{P})$. Moreover, it follows from Corollary 10.10 that all the maps T_i are continuous with respect to $\tau(\mathcal{P})$ and therefore $\tau(\mathcal{P})$ contains τ. □

10.12. **Definition.** Given semi-norms p, q in a vector space X, we say that p is *dominated* by q and we write:

$$p \preccurlyeq q$$

[6]Alternatively, one can check directly that τ is translation invariant by observing that for every $v \in X$ the translation map $\mathfrak{t}_v : x \mapsto x + v$ is continuous with respect to τ. Namely, observe that for every $i \in I$ the map $T_i \circ \mathfrak{t}_v = \mathfrak{t}_{T_i(v)} \circ T_i$ is continuous.

if there exists a non negative constant c such that $p(x) \leq cq(x)$, for all $x \in X$. Given a semi-norm p and a set of semi-norms $\mathcal{P}$ in X, we say that p is *dominated* by $\mathcal{P}$ and we write

$$p \preccurlyeq \mathcal{P}$$

if there exist $p_1, \ldots, p_k \in \mathcal{P}$ with[7]:

$$p \preccurlyeq p_1 + \cdots + p_k. \tag{10.7}$$

Given two sets of semi-norms $\mathcal{P}$, $\mathcal{P}'$ in X, we say that $\mathcal{P}'$ is *dominated* by $\mathcal{P}$ and we write:

$$\mathcal{P}' \preccurlyeq \mathcal{P}$$

if $p \preccurlyeq \mathcal{P}$, for all $p \in \mathcal{P}'$.

Clearly the binary relation $\preccurlyeq$ in the set of semi-norms in X is both reflexive and transitive. Moreover, given semi-norms p, q, p', q' in X then $p \preccurlyeq q$ and $p' \preccurlyeq q'$ imply $p + q \preccurlyeq p' + q'$ and given semi-norms p, q in X and a positive constant $c > 0$ then:

$$p \preccurlyeq q \Longleftrightarrow p \preccurlyeq cq \Longleftrightarrow cp \preccurlyeq q.$$

It is also easy to see that given a semi-norm p in X and sets of semi-norms $\mathcal{P}$, $\mathcal{P}'$ in X, then $p \preccurlyeq \mathcal{P}$ and $\mathcal{P} \preccurlyeq \mathcal{P}'$ imply $p \preccurlyeq \mathcal{P}'$; moreover, the binary relation $\preccurlyeq$ in the set of *sets* of semi-norms in X is also reflexive and transitive. Notice also that given semi-norms p, q in X then:

$$\{p\} \preccurlyeq \{q\} \Longleftrightarrow p \preccurlyeq \{q\} \Longleftrightarrow p \preccurlyeq q.$$

10.13. *Remark.* For $k \geq 1$, given semi-norms $p_1, \ldots, p_k$ in X then:

$$\max\big\{p_1(x), \ldots, p_k(x)\big\} \leq p_1(x) + \cdots + p_k(x) \leq k \max\big\{p_1(x), \ldots, p_k(x)\big\},$$

for all $x \in X$, so that:

$$\max\{p_1, \ldots, p_k\} \preccurlyeq p_1 + \cdots + p_k \preccurlyeq \max\{p_1, \ldots, p_k\}.$$

Thus, for $k \geq 1$, one can replace $p_1 + \cdots + p_k$ with $\max\{p_1, \ldots, p_k\}$ in (10.7).

10.14. **Lemma.** *Let X be a vector space, $\mathcal{P}$ be a set of semi-norms in X and p be a semi-norm in X. Then p is continuous with respect to $\tau(\mathcal{P})$ if and only if $p \preccurlyeq \mathcal{P}$.*

Proof. If $p \preccurlyeq \mathcal{P}$ then there exist $p_1, \ldots, p_k \in \mathcal{P}$ and a constant $c \geq 0$ such that:

$$p(x) \leq c\big(p_1(x) + \cdots + p_k(x)\big),$$

for all $x \in X$. The continuity of the map $x \mapsto c\big(p_1(x) + \cdots + p_k(x)\big)$ with respect to $\tau(\mathcal{P})$ implies that the set:

$$\big\{x \in X : c\big(p_1(x) + \cdots + p_k(x)\big) < 1\big\}$$

is a neighborhood of the origin with respect to $\tau(\mathcal{P})$; since p is bounded in that set, it follows from Corollary 10.3 (and from Lemma 10.6) that p is

[7] If $\mathcal{P}$ is empty then k must be zero and the sum $p_1 + \cdots + p_k$ is understood to be equal to zero. Thus, if $\mathcal{P}$ is empty, $p \preccurlyeq \mathcal{P}$ if and only if p is the zero semi-norm.

continuous with respect to $\tau(\mathcal{P})$. Conversely, assume that p is continuous with respect to $\tau(\mathcal{P})$. Then:

$$\big\{x \in X : p(x) < 1\big\}$$

is a neighborhood of the origin with respect to $\tau(\mathcal{P})$ and therefore it contains a fundamental neighborhood of the form (10.6), with $p_1, \ldots, p_k \in \mathcal{P}$ and $\varepsilon > 0$. We claim that:

$$p(x) \le \tfrac{2}{\varepsilon}\big(p_1(x) + \cdots + p_k(x)\big), \tag{10.8}$$

for all $x \in X$, so that $p \preccurlyeq \mathcal{P}$. Given $x \in X$, if $p_i(x) = 0$ for all $i = 1, \ldots, k$ then tx is in (10.6) for all $t > 0$ and therefore:

$$p(tx) = tp(x) < 1,$$

for all $t > 0$, so that $p(x) = 0$ and (10.8) is satisfied. If, on the other hand, $p_i(x) > 0$ for some i, set:

$$t = \frac{\varepsilon}{2\big(p_1(x) + \cdots + p_k(x)\big)} > 0,$$

so that:

$$p_i(tx) = tp_i(x) = \frac{\varepsilon}{2}\,\frac{p_i(x)}{p_1(x) + \cdots + p_k(x)} < \varepsilon,$$

for all $i = 1, \ldots, k$, which implies tx in (10.6). Thus $p(tx) < 1$ and hence:

$$p(x) < \tfrac{1}{t} = \tfrac{2}{\varepsilon}\big(p_1(x) + \cdots + p_k(x)\big). \qquad \square$$

10.15. **Corollary.** *Let X, Y be vector spaces over $\mathbb{K}$, $\mathcal{P}$ be a set of semi-norms in X, $\mathcal{Q}$ be a set of semi-norms in Y and $T : X \to Y$ be a linear map. If X is endowed with $\tau(\mathcal{P})$ and Y is endowed with $\tau(\mathcal{Q})$ then T is continuous if and only if:*

$$\big\{q \circ T : q \in \mathcal{Q}\big\} \preccurlyeq \mathcal{P}.$$

Proof. Follows directly from Corollary 10.10 and Lemma 10.14. $\square$

10.16. **Corollary.** *Let X be a vector space over $\mathbb{K}$ and $\mathcal{P}$, $\mathcal{P}'$ be sets of semi-norms in X. Then $\tau(\mathcal{P}) \subset \tau(\mathcal{P}')$ if and only if $\mathcal{P} \preccurlyeq \mathcal{P}'$.*

Proof. Apply Corollary 10.15 with T the identity map from $\big(X, \tau(\mathcal{P}')\big)$ to $\big(X, \tau(\mathcal{P})\big)$. $\square$

10.17. **Lemma.** *Let X be a complex vector space and let $p : X \to \mathbb{R}$ be a real semi-norm. Then:*

$$\tilde{p}(x) = \sup\big\{p(\lambda x) : \lambda \in \mathbb{C},\ |\lambda| = 1\big\} \tag{10.9}$$

defines a complex semi-norm $\tilde{p} : X \to \mathbb{R}$ and:

$$p(x) \le \tilde{p}(x) \le p(x) + p(ix), \tag{10.10}$$

for all $x \in X$.

Proof. Since $p(x)$ belongs to the set in the righthand side of (10.9), it follows that $p(x) \le \tilde{p}(x)$, for all $x \in X$. Moreover, given $\lambda = a + bi \in \mathbb{C}$, with $a, b \in \mathbb{R}$, $|\lambda| = 1$, then $|a| \le 1$, $|b| \le 1$ and:

$$p(\lambda x) = p(ax + bix) \le |a|p(x) + |b|p(ix) \le p(x) + p(ix),$$

so that $\tilde{p}(x)$ is finite and inequalities (10.10) hold. Since for each $\lambda \in \mathbb{C}$, the map $x \mapsto \lambda x$ is linear and since p is a real semi-norm, it follows that $x \mapsto p(\lambda x)$ is a real semi-norm and therefore $\tilde{p}$, being the (finite) supremum of a family of real semi-norms, is a real semi-norm (see Lemma 10.1). It follows directly from the definition of $\tilde{p}$ that:

$$\tilde{p}(\mu x) = \tilde{p}(x),$$

for all $x \in X$ and and all $\mu \in \mathbb{C}$ with $|\mu| = 1$. Therefore, for any non zero complex number λ, we have:

$$\tilde{p}(\lambda x) = \tilde{p}\big(|\lambda|\mu x) = |\lambda|\tilde{p}(\mu x) = |\lambda|\tilde{p}(x),$$

where $\mu = \frac{\lambda}{|\lambda|}$. Hence $\tilde{p}$ is a complex semi-norm. □

10.18. **Corollary.** *Let X be a complex vector space and let $\mathcal{P}$ be a set of real semi-norms in X. Assume that the complex structure $x \mapsto ix$ of X is continuous with respect to the topology $\tau(\mathcal{P})$. Then:*

$$\widetilde{\mathcal{P}} = \big\{\tilde{p} : p \in \mathcal{P}\big\},$$

with $\tilde{p}$ defined as in (10.9) *is a set of complex semi-norms in X such that:*

$$\tau(\mathcal{P}) = \tau(\widetilde{\mathcal{P}}).$$

Proof. The first inequality in (10.10) implies that $\mathcal{P} \preccurlyeq \widetilde{\mathcal{P}}$. The continuity of the map $x \mapsto ix$ with respect to $\tau(\mathcal{P})$ implies that for all $p \in \mathcal{P}$ the semi-norm $x \mapsto p(x) + p(ix)$ is continuous with respect to $\tau(\mathcal{P})$ and thus, by Lemma 10.14, it is dominated by $\mathcal{P}$. It then follows from the second inequality in (10.10) that $\widetilde{\mathcal{P}} \preccurlyeq \mathcal{P}$. Hence, by Corollary 10.16, $\tau(\mathcal{P}) = \tau(\widetilde{\mathcal{P}})$. □

10.19. **Definition.** Let X be a vector space and let V be a subset of X. Given a point $x \in X$, we say that V *absorbs* x if there exists $\alpha > 0$ such that $\alpha x \in V$. We say that V is *absorbent* if V absorbs every x in X.

Clearly, V absorbs the origin if and only if the origin is in V; thus, every absorbent set contains the origin.

10.20. **Lemma.** *Given vector spaces X, Y, if $T : X \to Y$ is a surjective linear map and V is an absorbent subset of X then $T(V)$ is an absorbent subset of Y. In particular, if V is an absorbent subset of X and λ is a nonzero scalar then $\lambda V = \big\{\lambda x : x \in V\big\}$ is an absorbent subset of X.*

Proof. Given $y \in Y$ then $y = T(x)$ for some $x \in X$; then $\alpha x \in V$ for some $\alpha > 0$ and therefore $\alpha y = T(\alpha x)$ is in $T(V)$. □

10.21. **Lemma.** *If X is a topological vector space then every neighborhood V of the origin is absorbent.*

Proof. Since the map $(\alpha, x) \mapsto \alpha x$ is continuous and it carries $(0, x)$ to 0, it follows that αx is in V for all α in some neighborhood of zero; in particular, αx is in V for some $\alpha > 0$. □

10.22. **Definition.** Given an absorbent subset V of a vector space X, we set:

$$p_V(x) = \inf\big\{\alpha > 0 : \tfrac{x}{\alpha} \in V\big\}. \tag{10.11}$$

The map p_V is called the *gauge* of V.

Notice that the hypothesis that V be absorbent guarantees that the set in (10.11) is nonempty; clearly such set is bounded from below by zero, so that $p_V(x) \ge 0$, for all $x \in X$.

10.23. **Lemma.** *Let V be an absorbent subset of a vector space X. Then:*

$$p_V(\lambda x) = \lambda p_V(x), \tag{10.12}$$

for every $x \in X$ and every non negative scalar $\lambda \in \mathbb{R}$.

Proof. If $x = 0$ then $\frac{x}{\alpha} = 0 \in V$ for all $\alpha > 0$, so that $p_V(x) = 0$ and (10.12) holds with $\lambda = 0$. If $\lambda > 0$ then, setting:

$$A_x = \big\{\alpha > 0 : \tfrac{x}{\alpha} \in V\big\}, \tag{10.13}$$

it is easy to see that:

$$A_{\lambda x} = \lambda A_x \stackrel{\text{def}}{=} \big\{\lambda\alpha : \alpha \in A_x\big\},$$

so that:

$$p_V(\lambda x) = \inf A_{\lambda x} = \lambda \inf A_x = \lambda p_V(x).$$

□

Recall that a subset V of a vector space X is called *convex* if for every $x, y \in V$, the line segment $[x, y] = \big\{(1-t)x + ty : t \in [0, 1]\big\}$ is contained in the set V. The direct and the inverse image of a convex set under a linear map is again convex and the intersection of a family of convex sets is convex.

10.24. **Lemma.** *Let V be a convex absorbent subset of a vector space X. Given $x \in X$ then $\frac{x}{\alpha} \in V$ for all $\alpha > p_V(x)$.*

Proof. Since $\alpha > p_V(x)$ there exists $\beta \in \,]0, \alpha[$ with $\frac{x}{\beta} \in V$. But $\frac{x}{\alpha} = \frac{\beta}{\alpha}\frac{x}{\beta}$ and $0 < \frac{\beta}{\alpha} < 1$, so that $\frac{x}{\alpha}$ is in the line segment connecting $\frac{x}{\beta} \in V$ and the origin (which is in V). □

10.25. **Corollary.** *If V is a convex absorbent subset of a vector space X then:*

$$p_V^{-1}\big(\,]-\infty, 1[\,\big) \subset V \subset p_V^{-1}\big(\,]-\infty, 1]\big). \tag{10.14}$$

Proof. If $p_V(x) < 1$ then, by Lemma 10.24, $x = \frac{x}{1}$ is in V. Moreover, it is obvious that if x is in V then $p_V(x) \le 1$. □

10.26. **Corollary.** *If V, W are convex absorbent subsets of a vector space X then $V \cap W$ is also (convex and) absorbent.*

Proof. Given $x \in X$ then, by Lemma 10.24, we have $\frac{x}{\alpha} \in V \cap W$ if α is bigger than both $p_V(x)$ and $p_W(x)$. □

10.27. **Lemma.** *If a subset V of a vector space X is absorbent and convex then its gauge p_V satisfies the triangle inequality, i.e.:*

$$p_V(x+y) \le p_V(x) + p_V(y),$$

for all $x, y \in X$.

Proof. Let $\varepsilon > 0$ be given. By Lemma 10.24, we have:

$$\frac{x}{p_V(x)+\varepsilon} \in V, \quad \frac{y}{p_V(y)+\varepsilon} \in V,$$

so that, by the convexity of V:

$$\frac{x+y}{p_V(x)+p_V(y)+2\varepsilon} = \frac{p_V(x)+\varepsilon}{p_V(x)+p_V(y)+2\varepsilon}\,\frac{x}{p_V(x)+\varepsilon} + \frac{p_V(y)+\varepsilon}{p_V(x)+p_V(y)+2\varepsilon}\,\frac{y}{p_V(y)+\varepsilon}$$

is in V. This implies:

$$p_V(x+y) \le p_V(x) + p_V(y) + 2\varepsilon,$$

and since $\varepsilon > 0$ is arbitrary the conclusion follows. □

10.28. **Definition.** A subset V of a vector space X over $\mathbb{K}$ is called *balanced* (with respect to the field $\mathbb{K}$) if $\lambda x \in V$ for all $x \in V$ and all $\lambda \in \mathbb{K}$ with $|\lambda| \le 1$.

10.29. **Lemma.** *Let V be a convex subset of a vector space X over $\mathbb{K}$ containing the origin (this is the case if V is convex and absorbent). Then V is balanced if and only if λx is in V for all $x \in V$ and all $\lambda \in \mathbb{K}$ with $|\lambda| = 1$. In particular, if $\mathbb{K} = \mathbb{R}$, then a convex subset V of X containing the origin is balanced if and only if $-x$ is in V for all $x \in V$.*

Proof. Obviously if V is balanced then $\lambda x \in V$ for all $x \in V$ and all $\lambda \in \mathbb{K}$ with $|\lambda| = 1$. Conversely, assume that $\lambda x \in V$ for all $x \in V$ and all $\lambda \in \mathbb{K}$ with $|\lambda| = 1$ and let $x \in V$ and $\lambda \in \mathbb{K}$ with $|\lambda| \le 1$ be given. If $\lambda = 0$ then $\lambda x = 0$ is in V. If $\lambda \neq 0$, we set:

$$\mu = \frac{\lambda}{|\lambda|},$$

so that $|\mu| = 1$ and μx is in V. Then $\lambda x = |\lambda|\mu x$ is in the line segment connecting μx and the origin and therefore it is also in V. Hence V is balanced. □

10.30. **Corollary.** *If V is a convex absorbent subset of a* real *vector space X then:*

$$V \cap (-V)$$

is convex, absorbent and balanced, where $-V = \big\{-x : x \in V\big\}$.

Proof. The set $-V$ is convex and it is absorbent by Lemma 10.20. Therefore $V \cap (-V)$ is convex and absorbent, by Corollary 10.26. Since:

$$-\big(V \cap (-V)\big) = V \cap (-V),$$

and the field of scalars is $\mathbb{R}$, it follows from Lemma 10.29 that $V \cap (-V)$ is balanced. □

10.31. **Lemma.** *If V is an absorbent balanced subset of a vector space X over $\mathbb{K}$ then its gauge p_V satisfies:*

$$p_V(\lambda x) = |\lambda| p_V(x), \tag{10.15}$$

for all $x \in X$, $\lambda \in \mathbb{K}$.

Proof. Since the set of those $\lambda \in \mathbb{K}$ for which (10.15) holds for all $x \in X$ is closed under multiplication, it follows from Lemma 10.23 that we only have to check (10.15) in case $|\lambda| = 1$. Defining A_x as in (10.13), it is readily checked using the fact that V is balanced that:

$$A_x = A_{\lambda x},$$

if $|\lambda| = 1$. Thus:

$$p_V(\lambda x) = \inf A_{\lambda x} = \inf A_x = p_V(x) = |\lambda| p_V(x).$$ □

10.32. **Corollary.** *If V is an absorbent, balanced, convex subset of a vector space X over $\mathbb{K}$ then its gauge p_V is a semi-norm in X.*

Proof. It follows directly from Lemmas 10.27 and 10.31. □

10.33. **Definition.** A topology τ on a vector space X over $\mathbb{K}$ is called *locally convex* (with respect to the field $\mathbb{K}$) if it turns X into a topological vector space over $\mathbb{K}$ and if the origin of X has a fundamental system of convex neighborhoods (i.e., every neighborhood of the origin contains a convex neighborhood of the origin). If X is endowed with a locally convex topology τ then we say that (X, τ) is a *locally convex topological vector space.*

Observe that, since the topology of a topological vector space is translation invariant, it follows that every point of a locally convex topological vector space has a fundamental system of convex neighborhoods.

10.34. **Lemma.** *Let X be a vector space over $\mathbb{K}$. A topology τ for X is locally convex if and only if there exists a set of semi-norms $\mathcal{P}$ in X such that $\tau = \tau(\mathcal{P})$.*

Proof. If $\tau = \tau(\mathcal{P})$ for some set of semi-norms $\mathcal{P}$ then (X, τ) is a topological vector space (Lemma 10.6) and, since the fundamental neighborhood (10.6) of the origin is convex, it follows that τ is a locally convex topology. Conversely, assume that the topology τ is locally convex. First, we observe that it suffices to consider the case $\mathbb{K} = \mathbb{R}$. Namely, assume that the lemma has been proven for $\mathbb{K} = \mathbb{R}$. If (X, τ) is a *complex* locally convex topological vector space, we restrict the operation of multiplication by scalars of X to $\mathbb{R} \times X$, obtaining a *real* locally convex topological vector space; then $\tau = \tau(\mathcal{P})$ for some set of *real* semi-norms $\mathcal{P}$ in X. Since (X, τ) is a complex topological vector space, the map $x \mapsto ix$ is continuous with respect to $\tau = \tau(\mathcal{P})$ and then it follows from Corollary 10.18 that there exists a set $\widetilde{\mathcal{P}}$ of *complex* semi-norms in X such that $\tau(\mathcal{P}) = \tau(\widetilde{\mathcal{P}})$.

Now let us prove the lemma for $\mathbb{K} = \mathbb{R}$. Let $\mathcal{P}$ denote the set of all semi-norms $p : X \to \mathbb{R}$ that are continuous with respect to τ. Let us show that $\tau = \tau(\mathcal{P})$. Since τ is a translation invariant topology and each $p \in \mathcal{P}$ is continuous with respect to τ, it follows from Lemma 10.7 that τ contains $\tau(\mathcal{P})$. Proving that $\tau(\mathcal{P})$ contains τ is the same as proving the continuity of the identity map from $\big(X, \tau(\mathcal{P})\big)$ to (X, τ) and, by Lemma 9.4, it suffices to establish such continuity at the origin. We have thus to show that every neighborhood U of the origin with respect to τ is a neighborhood of the origin with respect to $\tau(\mathcal{P})$. Since τ is locally convex we can assume without loss of generality that U is convex; U is also absorbent, by Lemma 10.21. Setting $-U = \big\{-x : x \in U\big\}$ and:

$$V = U \cap (-U)$$

then, by Corollary 10.30, V is convex, absorbent and balanced (here we use the fact that $\mathbb{K} = \mathbb{R}$). Since the map $x \mapsto -x$ is a homeomorphism with respect to τ it follows that $-U$ (and thus V) is a neighborhood of the origin with respect to τ. By Corollary 10.32, the gauge p_V is a semi-norm. Since p_V is bounded in V (recall (10.14)) and V is a neighborhood of the origin with respect to τ, it follows from Corollary 10.3 that p_V is continuous with respect to τ, i.e., $p_V \in \mathcal{P}$. Hence the open ball $p_V^{-1}\big(\,]-\infty, 1[\,\big)$ is a neighborhood of the origin with respect to $\tau(\mathcal{P})$ and it follows from (10.14) that V (and hence U) is also a neighborhood of the origin with respect to $\tau(\mathcal{P})$. □

10.1. **The locally convex topology co-induced by a family of maps.**

10.35. **Lemma.** *Let $(X_i)_{i \in I}$ be a family of locally convex topological vector spaces over $\mathbb{K}$, X be a vector space over $\mathbb{K}$ and for each $i \in I$ let $T_i : X_i \to X$ be a linear map. There exists a unique topology τ on X such that:*

(i) *(X, τ) is a locally convex topological vector space over $\mathbb{K}$;*
(ii) *the map $T_i : X_i \to (X, \tau)$ is continuous, for all $i \in I$;*
(iii) *given a locally convex topological vector space Y over $\mathbb{K}$ and a linear map $S : X \to Y$ such that $S \circ T_i$ is continuous for all $i \in I$ then $S : (X, \tau) \to Y$ is continuous.*

The topology τ coincides with the topology $\tau(\mathcal{P})$, where $\mathcal{P}$ is the set of all semi-norms $p : X \to \mathbb{R}$ such that $p \circ T_i$ is continuous, for all $i \in I$.

Proof. First, let us show the uniqueness of τ. Let τ, τ' be topologies in X satisfying (i), (ii) and (iii). If S is the identity map from (X, τ) to (X, τ') then $S \circ T_i$ is continuous for all $i \in I$ (because τ' satisfies (ii)) and therefore (since τ satisfies (iii) and τ' satisfies (i)), S is continuous, i.e., $\tau' \subset \tau$. An analogous argument shows that $\tau \subset \tau'$. In order to complete the proof, it suffices to show that the topology $\tau = \tau(\mathcal{P})$ satisfies (i), (ii) and (iii). Clearly $\tau = \tau(\mathcal{P})$ satisfies (i) (Lemma 10.34). Since $p \circ T_i$ is continuous for all $i \in I$ and all $p \in \mathcal{P}$, it follows from Corollary 10.10 that $\tau = \tau(\mathcal{P})$ satisfies (ii). Let Y be a locally convex topological vector space over $\mathbb{K}$ and let $S : X \to Y$ be a linear map such that $S \circ T_i$ is continuous, for all $i \in I$. Let $\mathcal{Q}$ be a set of semi-norms in Y such that the topology of Y is $\tau(\mathcal{Q})$. For every $q \in \mathcal{Q}$, the semi-norm $q \circ S$ is in $\mathcal{P}$; namely, for all $i \in I$ the map $(q \circ S) \circ T_i = q \circ (S \circ T_i)$ is continuous. Thus, $q \circ S$ is continuous with respect to $\tau(\mathcal{P})$ for all $q \in \mathcal{Q}$ and it follows from Corollary 10.10 that T is continuous, proving that $\tau = \tau(\mathcal{P})$ satisfies (iii). □

10.36. **Definition.** The topology on X whose existence and uniqueness is guaranteed by Lemma 10.35 is called the *locally convex topology on X co-induced by the family of linear maps $(T_i)_{i \in I}$*.

10.37. **Corollary** (of Lemma 10.35)**.** *Let $(X_i)_{i \in I}$ be a family of locally convex topological vector spaces over $\mathbb{K}$, X be a vector space over $\mathbb{K}$ and for each $i \in I$ let $T_i : X_i \to X$ be a linear map. If X is endowed with the locally convex topology co-induced by the family of maps $(T_i)_{i \in I}$ then a semi-norm $p : X \to \mathbb{R}$ is continuous if and only if $p \circ T_i$ is continuous, for all $i \in I$.*

Proof. If p is continuous then $p \circ T_i$ is continuous for all $i \in I$, because the topology of X satisfies property (ii) in the statement of the lemma. Conversely, if $p \circ T_i$ is continuous for all $i \in I$ then p is in the set $\mathcal{P}$ defined in the statement of the lemma and since the topology of X is $\tau(\mathcal{P})$, it follows that p is continuous. □

Given a complex vector space X, we denote by $X^{\mathbb{R}}$ its *realification*, i.e., the real vector space obtained from X by restricting to $\mathbb{R} \times X$ its operation of multiplication by scalars. We now analyze the relationship between the locally convex topology co-induced by a family of maps and the operation of realification.

10.38. **Lemma.** *Let $(X_i)_{i \in I}$ be a family of complex locally convex topological vector spaces, X be a complex vector space and for each $i \in I$ let $T_i : X_i \to X$ be a complex linear map; denote by $\tau^{\mathbb{C}}$ the topology on X co-induced by the family of linear maps $T_i : X_i \to X$, $i \in I$. Now, consider the realification $X^{\mathbb{R}}$ of the vector space X and for each $i \in I$ consider the realification $X_i^{\mathbb{R}}$ of the vector space X_i; denote by $\tau^{\mathbb{R}}$ the topology on X co-induced by the family of linear maps $T_i : X_i^{\mathbb{R}} \to X^{\mathbb{R}}$. Then $\tau^{\mathbb{R}} = \tau^{\mathbb{C}}$.*

Proof. The topology $\tau^{\mathbb{C}}$ is characterized by the following properties:

(i) $(X, \tau^{\mathbb{C}})$ is a complex locally convex topological vector space;
(ii) the map $T_i : X_i \to (X, \tau^{\mathbb{C}})$ is continuous, for all $i \in I$;
(iii) given a complex locally convex topological vector space Y and a complex linear map $S : X \to Y$ such that $S \circ T_i$ is continuous for all $i \in I$ then $S : (X, \tau^{\mathbb{C}}) \to Y$ is continuous;

the topology $\tau^{\mathbb{R}}$ is characterized by the following properties:

(i') $(X, \tau^{\mathbb{R}})$ is a real locally convex topological vector space;
(ii') the map $T_i : X_i \to (X, \tau^{\mathbb{R}})$ is continuous, for all $i \in I$;
(iii') given a real locally convex topological vector space Y and a real linear map $S : X \to Y$ such that $S \circ T_i$ is continuous for all $i \in I$ then $S : (X, \tau^{\mathbb{R}}) \to Y$ is continuous.

We prove that the topology $\tau^{\mathbb{R}}$ satisfies properties (i), (ii) and (iii); since such properties characterize $\tau^{\mathbb{C}}$, it will follow that $\tau^{\mathbb{R}} = \tau^{\mathbb{C}}$. Property (ii) is the same as (ii') and thus $\tau^{\mathbb{R}}$ satisfies (ii). To prove that $\tau^{\mathbb{R}}$ satisfies property (iii), let Y be a complex locally convex topological vector space and let $S : X \to Y$ be a complex linear map such that $S \circ T_i$ is continuous for all $i \in I$. Then the realification $Y^{\mathbb{R}}$ is a real locally convex topological vector space and, since $S \circ T_i$ is continuous for all $i \in I$, it follows from property (iii') that S is continuous with respect to $\tau^{\mathbb{R}}$. Let us check now that $\tau^{\mathbb{R}}$ satisfies property (i). Denote by $J : X \to X$ the complex structure of X (i.e., $J(x)$ is the product of x by the imaginary unit) and by $J_i : X_i \to X_i$ the complex structure of X_i, for all $i \in I$. By Lemma 9.11 and property (i'), in order to prove that $(X, \tau^{\mathbb{R}})$ is a complex locally convex topological vector space, it suffices to prove that the map J is continuous with respect to $\tau^{\mathbb{R}}$. Since $\tau^{\mathbb{R}}$ satisfies (iii'), we have to prove that $J \circ T_i : X_i \to (X, \tau^{\mathbb{R}})$ is continuous, for all $i \in I$; but, since each T_i is complex linear, we have $J \circ T_i = T_i \circ J_i$. Since X_i is a complex topological vector space, the map J_i is continuous and the map $T_i : X_i \to (X, \tau^{\mathbb{R}})$ is continuous by (ii'). This concludes the proof. □

10.39. **Lemma.** *Let $(X_i)_{i \in I}$ be a family of locally convex topological vector spaces over $\mathbb{K}$, X be a vector space over $\mathbb{K}$ and for each $i \in I$ let $T_i : X_i \to X$ be a linear map. Assume that X is endowed with the locally convex topology co-induced by the family of maps $(T_i)_{i \in I}$. If V is an absorbent convex subset of X such that $T_i^{-1}(V)$ is a neighborhood of the origin in X_i for all $i \in I$ then V is a neighborhood of the origin in X.*

Proof. By Lemma 10.38, we can replace the spaces X_i and X by their realifications (so that the co-induced topology on X does not change), and therefore we can assume without loss of generality that $\mathbb{K} = \mathbb{R}$. Since V is absorbent and convex (and $\mathbb{K} = \mathbb{R}$), by Corollary 10.30, the set:

$$W = V \cap (-V),$$

is absorbent, convex and balanced, where $-V = \big\{-x : x \in V\big\}$. Clearly:

$$T_i^{-1}(W) = T_i^{-1}(V) \cap T_i^{-1}(-V) = T_i^{-1}(V) \cap \big(-T_i^{-1}(V)\big),$$

for all $i \in I$; since the map $x \mapsto -x$ is a homeomorphism of X_i and $T_i^{-1}(V)$ is a neighborhood of the origin in X_i, it follows that also $T_i^{-1}(W)$ is a neighborhood of the origin in X_i. By Corollary 10.32, the gauge p_W of W is a semi-norm in X and by (10.14) p_W is bounded in W; thus $p_W \circ T_i$ is a semi-norm in X_i that is bounded in $T_i^{-1}(W)$ and since $T_i^{-1}(W)$ is a neighborhood of the origin in X_i it follows from Corollary 10.3 that $p_W \circ T_i$ is continuous. Now, by Corollary 10.37, p_W is continuous and therefore $p_W^{-1}\big(\left]-\infty, 1\right[\big)$ is a neighborhood of the origin in X; from (10.14), W (and hence V) is a neighborhood of the origin in X. □

10.40. **Corollary.** *Let $(X_i)_{i\in I}$ be a family of locally convex topological vector spaces over $\mathbb{K}$, X be a vector space over $\mathbb{K}$ and for each $i \in I$ let $T_i : X_i \to X$ be a linear map. Assume that X is endowed with the locally convex topology co-induced by the family of maps $(T_i)_{i\in I}$ and that:*

$$X = \bigcup_{i\in I} T_i(X_i).$$

If V is a convex subset of X such that $T_i^{-1}(V)$ is a neighborhood of the origin in X_i for all $i \in I$ then V is a neighborhood of the origin in X.

Proof. By Lemma 10.39, it suffices to check that V is absorbent. Given $x \in X$ then, by our assumptions, there exists $i \in I$ and $x_0 \in X_i$ such that $T_i(x_0) = x$. Since $T_i^{-1}(V)$ is a neighborhood of the origin in X_i, it follows from Lemma 10.21 that $T_i^{-1}(V)$ is absorbent and therefore there exists $\alpha > 0$ such that $\alpha x_0 \in T_i^{-1}(V)$. Hence $\alpha x = T_i(\alpha x_0)$ is in V. □

10.41. *Remark.* If X is endowed with the locally convex topology co-induced by a family of maps $T_i : X_i \to X$ then every neighborhood V of the origin in X contains a convex (and, by Lemma 10.21, automatically absorbent) neighborhood of the origin V_0; since each T_i is continuous, it follows that $T_i^{-1}(V_0)$ is a neighborhood of the origin in X_i, for all $i \in I$. Hence, the neighborhoods of the origin in X given in the statement of Lemma 10.39 (or in the statement of Corollary 10.40) actually constitute a *fundamental* system of neighborhoods of the origin in X.

11. Normed Spaces

11.1. **Lemma.** *Let X be a separable normed vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{A}$ denote the σ-algebra induced by all continuous linear functionals $\lambda : X \to \mathbb{K}$, i.e., the smallest σ-algebra which contains the sets $\lambda^{-1}(B)$ for every continuous linear functional $\lambda : X \to \mathbb{K}$ and every Borel set $B \subset \mathbb{K}$. Then $\mathcal{A}$ is the Borelian σ-algebra of X, i.e., $\mathcal{A}$ is the smallest σ-algebra containing the topology of X.*

Proof. Obviously the Borelian σ-algebra contains $\mathcal{A}$. It suffices then to show that $\mathcal{A}$ contains all open subsets of X. We observe first that $\mathcal{A}$ is invariant by translations, i.e., if $v \in X$ then the translation map $\mathfrak{t}_v : X \ni x \mapsto x + v \in X$ is measurable; namely, for every continuous linear functional $\lambda : X \to \mathbb{K}$, $\lambda \circ \mathfrak{t}_v = \mathfrak{t}_{\lambda(v)} \circ \lambda$ is measurable, where $\mathfrak{t}_{\lambda(v)} : \mathbb{K} \to \mathbb{K}$ denotes translation by $\lambda(v)$ in $\mathbb{K}$. This implies that $\mathfrak{t}_v$ is indeed measurable.

Since X is separable, every open subset is a countable union of open balls and thus all we need to show is that open balls are in $\mathcal{A}$; the invariance by translations of $\mathcal{A}$ implies that it is sufficient to show that open balls centered at the origin are in $\mathcal{A}$. The proof of the latter statement will be accomplished by showing that the norm function $\|\cdot\| : X \to \mathbb{R}$ is measurable. Let $\{x_n : n \in \mathbb{N}\}$ be a countable dense subset of X and choose (by Hahn–Banach's theorem) a continuous linear functional $\lambda_n : X \to \mathbb{K}$ with $\|\lambda_n\| = 1$ and $\lambda_n(x_n) = \|x_n\|$. Set:

$$p(x) = \sup_{n \in \mathbb{N}} \big|\lambda_n(x)\big|, \quad x \in X.$$

Obviously p is measurable. We will show that $p(x) = \|x\|$ for all x. Since $\|\lambda_n\| = 1$ for all n we have $p(x) \le \|x\|$ for all x. Moreover, the equality $p(x) = \|x\|$ obviously holds if x is in the dense set $\{x_n : n \in \mathbb{N}\}$. The conclusion will follow once we show that p is continuous. To this aim, observe first that p is a semi-norm in X and therefore:

$$\big|p(x) - p(y)\big| \le p(x - y),$$

for all $x, y \in X$. Moreover, $p(x-y) \le \|x-y\|$, which implies that $p : X \to \mathbb{R}$ is Lipschitz. This concludes the proof. □

11.2. *Remark.* The result proved in Lemma 11.3 below also follows from Corollary 10.18.

11.3. **Lemma.** *Let $\big(X, \|\cdot\|\big)$ be a real normed space and let $J : X \to X$ be a continuous complex structure. Then there exists a norm $\|\cdot\|'$ in X, equivalent to $\|\cdot\|$, which is compatible with J, i.e., $\|\cdot\|'$ is a norm on the complex vector space (X, J).*

Proof. Let $\mathrm{Lin}(X, \mathbb{C})$ denote the space of all continuous $\mathbb{R}$-linear maps $\alpha : X \to \mathbb{C}$ endowed with the standard norm:

$$\|\alpha\| = \sup_{\|x\| \le 1} \big|\alpha(x)\big|.$$

Let Y denote the subspace of $\mathrm{Lin}(X, \mathbb{C})$ consisting of all linear functionals $\alpha : (X, J) \to \mathbb{C}$ that are $\mathbb{C}$-linear. We regard Y as a complex vector space by setting $(i\alpha)(x) = i\big(\alpha(x)\big) = \alpha\big(J(x)\big)$, $\alpha \in Y$, $x \in X$. Observe that the norm of Y is compatible with its complex structure, so that Y is indeed a complex normed vector space. We denote by Y^* the (complex) dual of Y, consisting of complex continuous linear functionals on Y. Then Y^* is also a complex normed vector space in the usual way. Consider the linear map $\phi : X \to Y^*$ defined by $\phi(x) = \hat{x}$, where $\hat{x}(\alpha) = \alpha(x)$, $\alpha \in Y$. The map ϕ is injective since given $x \in X$, $x \neq 0$, we can always find a continuous $\mathbb{R}$-linear functional $\alpha_0 : X \to \mathbb{R}$ with $\alpha_0(x) \neq 0$ and α_0 is the real part of $\alpha = \alpha_0 - i(\alpha_0 \circ J) \in Y$ (here we use that J *is continuous*!). Since $\phi : (X, J) \to Y^*$ is $\mathbb{C}$-linear, the norm $\|x\|' = \|\hat{x}\|$ on X induced by the norm of Y^* is compatible with J. We now show that $\|\cdot\|'$ is equivalent to $\|\cdot\|$. To this aim, observe first that, for every $x \in X$:

$$\|x\|' = \|\hat{x}\| = \sup_{\|\alpha\| \le 1} \big|\hat{x}(\alpha)\big| = \sup_{\|\alpha\| \le 1} \big|\alpha(x)\big| \le \|x\|.$$

We will now obtain $k > 0$ such that $\|x\|' \ge k\|x\|$, for all $x \in X$. Let then $x \in X$, $x \neq 0$, be fixed. Choose an $\mathbb{R}$-linear functional $\alpha_0 : X \to \mathbb{R}$ with $\|\alpha_0\| = 1$ and $\alpha_0(x) = \|x\|$; then, as before, $\alpha = \alpha_0 - i(\alpha_0 \circ J)$ is in Y and:

$$\big|\alpha(v)\big| \le \big|\alpha_0(v)\big| + \big|\alpha_0\big(J(v)\big)\big| \le \big(1 + \|J\|\big)\|v\|, \quad v \in X,$$

so that $\|\alpha\| \le 1 + \|J\|$. We have also $\alpha \neq 0$, since $\alpha(x) \neq 0$, and therefore:

$$\|x\|' = \|\hat{x}\| \ge \Big|\hat{x}\Big(\frac{\alpha}{\|\alpha\|}\Big)\Big| = \frac{\big|\alpha(x)\big|}{\|\alpha\|} \ge \frac{\alpha_0(x)}{1 + \|J\|} = \frac{1}{1 + \|J\|}\|x\|,$$

which concludes the proof. □

12. Riesz Representation Theorem

12.1. **Convention.** In this section, X will always denote a *locally compact Hausdorff space*, $C_c(X)$ will denote the real vector space of all continuous maps $f : X \to \mathbb{R}$ having compact support and λ will denote a *positive linear functional* on $C_c(X)$, i.e., a linear map $\lambda : C_c(X) \to \mathbb{R}$ such that $\lambda(f) \ge 0$, for any nonnegative map $f \in C_c(X)$.

12.2. **Lemma.** *Given a compact set $K \subset X$ and a closed set $F \subset X$ with $K \cap F = \emptyset$ then there exists a continuous map $f : X \to [0, 1]$ with $f|_K \equiv 1$ and $f|_F \equiv 0$.*

Proof. Let $\widehat{X} = X \cup \{\omega\}$ denote the one-point compactification of X, i.e., the open subsets of $\widehat{X}$ are the open subsets of X and the complements in $\widehat{X}$ of the compact subsets of X. Since X is locally compact and Hausdorff, it follows that $\widehat{X}$ is (compact and) Hausdorff. In particular, $\widehat{X}$ is normal, so that the Urisohn's Lemma applies. Since $\widehat{X}$ is Hausdorff, the compact set K is closed in $\widehat{X}$. If $\overline{F}$ denotes the closure of F in $\widehat{X}$ then, since F is closed in X, $\overline{F} \cap X = F$ and thus $\overline{F} \cap K = \emptyset$. The conclusion is now obtained by applying Urisohn's Lemma to K and $\overline{F}$ in $\widehat{X}$ and by taking the restriction to X of the continuous map obtained. □

12.3. **Corollary.** *Given a compact subset $K \subset X$ contained in an open subset $U \subset X$ then there exists $f \in C_c(X)$ such that $f|_K \equiv 1$, $0 \le f \le 1$ and such that the support of f is contained in U.*

Proof. By Lemma 17.1, there exists a compact subset L of U whose interior contains K. The map f is obtained by applying Lemma 12.2 to K and the complement of the interior of L. □

In what follows, if f and g are real valued maps defined on X, we write $f \le g$ if $f(x) \le g(x)$, for all $x \in X$. Clearly the positive linear functional λ is *monotone*, i.e., $\lambda(f) \le \lambda(g)$ if $f, g \in C_{\mathrm{c}}(X)$ and $f \le g$. We denote by $\mathfrak{F}(X)$ the set of all maps $f : X \to [0, +\infty[$ having the following property:

$$f = \sup \big\{ \phi \in C_{\mathrm{c}}(X) : 0 \le \phi \le f \big\}.$$

Obviously if $f \in C_{\mathrm{c}}(X)$ and $f \ge 0$ then $f \in \mathfrak{F}(X)$; moreover:

$$\lambda(f) = \sup \big\{ \lambda(\phi) : \phi \in C_{\mathrm{c}}(X),\ 0 \le \phi \le f \big\}.$$

We thus consider an extension of λ to $\mathfrak{F}(X)$ (also denoted by λ) by setting:

$$\lambda(f) = \sup \big\{ \lambda(\phi) : \phi \in C_{\mathrm{c}}(X),\ 0 \le \phi \le f \big\} \in [0, +\infty],$$

for all $f \in \mathfrak{F}(X)$. Clearly this extension of λ is again monotone.

In what follows, if f is a real valued map on X and $a \in \mathbb{R}$, we denote by $[f > a]$ the set $\big\{ x \in X : f(x) > a \big\}$; the sets $[f \ge a]$, $[f < a]$ and $[f \le a]$ are defined analogously.

12.4. **Lemma.** *If $f \in \mathfrak{F}(X)$ then the set $[f > a]$ is open in X, for all $a \in \mathbb{R}$.*

Proof. If $f \in \mathfrak{F}(X)$ then $[f > a]$ is equal to the union of all sets of the form $[\phi > a]$, with $\phi \in C_{\mathrm{c}}(X)$ and $0 \le \phi \le f$; since each ϕ is continuous, the set $[\phi > a]$ is open. The conclusion follows. □

12.5. **Lemma.** *For every open subset $U \subset X$, the characteristic function $\chi_U : X \to \mathbb{R}$ of U is in $\mathfrak{F}(X)$.*

Proof. Set:

$$f = \sup \big\{ \phi \in C_{\mathrm{c}}(X) : 0 \le \phi \le \chi_U \big\}.$$

Clearly $0 \le f \le \chi_U$. It is therefore sufficient to show that $f(x) \ge 1$, for all $x \in U$. Given $x \in U$ then, by Corollary 12.3, there exists a map $\phi \in C_{\mathrm{c}}(X)$ with $0 \le \phi \le 1$, $\phi(x) = 1$ and $\phi(y) = 0$, for all $y \in U^{\mathrm{c}}$. Then $0 \le \phi \le \chi_U$ and hence $f(x) \ge \phi(x) = 1$. □

12.6. **Lemma.** *If $f, g \in \mathfrak{F}(X)$ and $c \ge 0$ then $f + g \in \mathfrak{F}(X)$ and $cf \in \mathfrak{F}(X)$.*

Proof. We have:

$$\begin{aligned} f + g &= \sup \big\{ \phi \in C_{\mathrm{c}}(X) : 0 \le \phi \le f \big\} + \sup \big\{ \psi \in C_{\mathrm{c}}(X) : 0 \le \psi \le g \big\} \\ &= \sup \big\{ \phi + \psi : \phi, \psi \in C_{\mathrm{c}}(X),\ 0 \le \phi \le f,\ 0 \le \psi \le g \big\} \\ &\le \sup \big\{ \xi \in C_{\mathrm{c}}(X) : 0 \le \xi \le f + g \big\} \le f + g, \end{aligned}$$

proving that $f + g \in \mathfrak{F}(X)$. Moreover:

$$\begin{aligned} cf = c \sup \big\{ \phi \in C_{\mathrm{c}}(X) : 0 \le \phi \le f \big\} &= \sup \big\{ c\phi : \phi \in C_{\mathrm{c}}(X),\ 0 \le \phi \le f \big\} \\ &= \sup \big\{ \xi \in C_{\mathrm{c}}(X) : 0 \le \xi \le cf \big\}, \end{aligned}$$

proving that $cf \in \mathfrak{F}(X)$. □

In what follows, given maps $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$, we set:

$$f \vee g = \max\{f, g\}, \quad f \wedge g = \min\{f, g\}.$$

Clearly if $f, g \in C_c(X)$ then $f \vee g \in C_c(X)$ and $f \wedge g \in C_c(X)$.

12.7. **Lemma.** *Given $f \in \mathfrak{F}(X)$ and $g \in C_c(X)$ with $f \geq g$ then $f - g \in \mathfrak{F}(X)$.*

Proof. If $\phi \in C_c(X)$ and $0 \leq \phi \leq f$ then $\psi = (\phi \wedge g) - g \in C_c(X)$, $0 \leq \psi \leq f - g$ and $\psi \geq \phi - g$; it follows that:

$$\begin{aligned} f - g &\geq \sup\big\{\psi \in C_c(X) : 0 \leq \psi \leq f - g\big\} \\ &\geq \sup\big\{\phi - g : \phi \in C_c(X),\ 0 \leq \phi \leq f\big\} = f - g. \end{aligned}$$ □

A set $\mathcal{S}$ of real valued maps on X is said to be *directed* if for all $f, g \in \mathcal{S}$, there exists $h \in \mathcal{S}$ with $h \geq f \vee g$. We write:

$$\sup \mathcal{S} = \sup_{\phi \in \mathcal{S}} \phi.$$

12.8. **Lemma.** *Given a directed set $\mathcal{S} \subset \mathfrak{F}(X)$ and $f \in \mathfrak{F}(X)$ with $f = \sup \mathcal{S}$ then:*

$$\lambda(f) = \sup_{g \in \mathcal{S}} \lambda(g).$$

Proof. Obviously $g \leq f$ for all $g \in \mathcal{S}$ and thus:

$$\sup_{g \in \mathcal{S}} \lambda(g) \leq \lambda(f).$$

We divide the proof of the reverse inequality into three steps.

Step 1. *The result holds if $\mathcal{S} \subset C_c(X)$ and $f \in C_c(X)$.*

Given $\varepsilon > 0$, we will find $g \in \mathcal{S}$ with $\lambda(g) \geq \lambda(f) - \varepsilon$. By Corollary 12.3, there exists a nonnegative map $f_0 \in C_c(X)$ that equals 1 on the support of f. Choose $\varepsilon' > 0$ with $\varepsilon'\lambda(f_0) \leq \varepsilon$. For each $g \in \mathcal{S}$, we have $f - g \in C_c(X)$ and thus the set $[f - g \geq \varepsilon']$ is compact. Clearly:

$$\bigcap_{g \in \mathcal{S}} [f - g \geq \varepsilon'] = \emptyset;$$

since X is Hausdorff, there exists $g_1, \ldots, g_n \in \mathcal{S}$ with:

$$\bigcap_{i=1}^{n} [f - g_i \geq \varepsilon'] = \emptyset.$$

Moreover, since $\mathcal{S}$ is directed, there exists $g \in \mathcal{S}$ with $g \geq g_1 \vee \cdots \vee g_n$ and thus $[f - g \geq \varepsilon'] = \emptyset$, i.e., $f - g < \varepsilon'$. Since the support of $f - g$ is contained in the support of f, we have $f - g \leq \varepsilon' f_0$ and hence:

$$\lambda(f) - \lambda(g) = \lambda(f - g) \leq \varepsilon'\lambda(f_0) \leq \varepsilon,$$

proving that $\lambda(g) \geq \lambda(f) - \varepsilon$.

Step 2. *The result holds if* $\mathcal{S} \subset C_c(X)$.

Let $\phi \in C_c(X)$ with $0 \le \phi \le f$ be fixed. It suffices to show that $\sup_{g\in\mathcal{S}} \lambda(g) \ge \lambda(\phi)$. The set:

$$\mathcal{S}_\phi = \big\{ g \wedge \phi : g \in \mathcal{S} \big\} \subset C_c(X)$$

is directed and $\sup \mathcal{S}_\phi = \phi$. By step 1, we have:

$$\sup_{h\in\mathcal{S}_\phi} \lambda(h) = \lambda(\phi).$$

Since $\lambda(g) \ge \lambda(g \wedge \phi)$, for all $g \in \mathcal{S}$, we obtain:

$$\sup_{g\in\mathcal{S}} \lambda(g) \ge \sup_{g\in\mathcal{S}} \lambda(g \wedge \phi) = \sup_{h\in\mathcal{S}_\phi} \lambda(h) = \lambda(\phi),$$

completing the proof of Step 2.

Step 3. *The result holds in general.*

Clearly, the set:

$$\mathcal{S}' = \bigcup_{g\in\mathcal{S}} \big\{ \phi \in C_c(X) : 0 \le \phi \le g \big\} \subset C_c(X)$$

is directed and $\sup \mathcal{S}' = f$. Thus, by step 2, we have:

$$\sup_{\phi\in\mathcal{S}'} \lambda(\phi) = \lambda(f).$$

The definition of λ gives:

$$\lambda(g) = \sup \big\{ \lambda(\phi) : \phi \in C_c(X),\ 0 \le \phi \le g \big\},$$

for all $g \in \mathcal{S}$; hence:

$$\sup_{g\in\mathcal{S}} \lambda(g) = \sup_{\phi\in\mathcal{S}'} \lambda(\phi) = \lambda(f). \qquad \square$$

If $(f_n)_{n\ge1}$ is a sequence of real valued maps on X with $f_n \le f_{n+1}$ for all $n \ge 1$ and with $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$, we write $f_n \nearrow f$ and we say that $(f_n)_{n\ge1}$ *converges monotonically* to the map $f : X \to \mathbb{R}$.

12.9. **Corollary.** *Let* $(f_n)_{n\ge1}$ *be a sequence in* $\mathfrak{F}(X)$ *and assume that* $f_n \nearrow f$, *with* $f \in \mathfrak{F}(X)$. *Then:*

$$\lim_{n\to\infty} \lambda(f_n) = \sup_{n\ge1} \lambda(f_n) = \lambda(f).$$

Proof. Simply apply Lemma 12.8 to the directed set $\mathcal{S} = \big\{ f_n : n \ge 1 \big\}$. $\square$

12.10. **Corollary.** *Given* $f, g \in \mathfrak{F}(X)$ *and* $c \ge 0$ *then* $\lambda(f+g) = \lambda(f) + \lambda(g)$ *and* $\lambda(cf) = c\lambda(f)$ *(see also Lemma 12.6).*

Proof. The set:

$$\mathcal{S} = \big\{ \phi + \psi : \phi, \psi \in C_c(X),\ 0 \le \phi \le f,\ 0 \le \psi \le g \big\}$$

is directed and $\sup \mathcal{S} = f + g$. It follows from Lemma 12.8 that:

$$\begin{aligned}\lambda(f+g) = \sup_{\xi\in\mathcal{S}} \lambda(\xi) &= \sup\big\{\lambda(\phi) : \phi \in C_{\rm c}(X),\ 0 \le \phi \le f\big\}\\ &\quad + \sup\big\{\lambda(\psi) : \psi \in C_{\rm c}(X),\ 0 \le \psi \le g\big\}\\ &= \lambda(f) + \lambda(g).\end{aligned}$$

The equality $\lambda(cf) = c\lambda(f)$ follows simply by observing that:

$$\big\{c\phi : \phi \in C_{\rm c}(X),\ 0 \le \phi \le f\big\} = \big\{\xi \in C_{\rm c}(X) : 0 \le \xi \le cf\big\}. \qquad \square$$

For each open subset U of X we define $\mu(U) \in [0, +\infty]$ by setting (recall Lemma 12.5):

$$\mu(U) = \lambda(\chi_U).$$

We have the following:

12.11. **Lemma.** *The map μ has the following properties:*

(a) $\mu(\emptyset) = 0$*;*
(b) *given open subsets $U, V \subset X$ then $\mu(U\cup V)+\mu(U\cap V) = \mu(U)+\mu(V)$ and, in particular, $\mu(U \cup V) \le \mu(U) + \mu(V)$;*
(c) *μ is monotone, i.e., given open subsets $U, V \subset X$ with $U \subset V$ then $\mu(U) \le \mu(V)$;*
(d) *if $(U_n)_{n\ge1}$ is a sequence of open subsets of X with $U_n \subset U_{n+1}$ for all $n \ge 1$ then $\mu\big(\bigcup_{n=1}^{\infty} U_n\big) = \lim_{n\to\infty} \mu(U_n)$.*

Proof. Item (a) follows by observing that $\lambda(0) = 0$. Item (b) follows from Corollary 12.10 by observing that $\chi_U + \chi_V = \chi_{U\cup V} + \chi_{U\cap V}$. Item (c) follows from the monotonicity of λ, observing that $\chi_U \le \chi_V$ if $U \subset V$. Finally, item (d) follows from Corollary 12.9 by observing that $\chi_{U_n} \nearrow \chi_U$, where $U = \bigcup_{n=1}^{\infty} U_n$. $\square$

We now define a map $\mu^* : \wp(X) \to [0, +\infty]$ on the set $\wp(X)$ of all subsets of X by setting:

$$\mu^*(A) = \inf\big\{\mu(U) : U \supset A,\ U \text{ open in } X\big\}, \tag{12.1}$$

for all $A \subset X$. Clearly the monotonicity of μ implies that $\mu^*(U) = \mu(U)$, for every open subset $U \subset X$.

12.12. **Lemma.** *The map μ^* is an* outer measure *on $\wp(X)$, i.e., it satisfies the following conditions:*

- $\mu^*(\emptyset) = 0$*;*
- *$\mu^*(A) \le \mu^*(B)$, for all $A, B \subset X$ with $A \subset B$ (monotonicity);*
- *$\mu^*\big(\bigcup_{n=1}^{\infty} A_n\big) \le \sum_{n=1}^{\infty} \mu^*(A_n)$, for any sequence $(A_n)_{n\ge1}$ of subsets of X (σ-subadditivity).*

Proof. The fact that $\mu^*(\emptyset) = 0$ follows from item (a) of Lemma 12.11 and from the observation that μ^* extends μ. The monotonicity of μ^* is trivial. Finally, let us prove the σ-subadditivity of μ^*. Let $(A_n)_{n\ge1}$ be a sequence of subsets of X and let $A = \bigcup_{n=1}^{\infty} A_n$. If $\mu^*(A_n) = +\infty$ for some $n \ge 1$ then the inequality $\mu^*(A) \le \sum_{n=1}^{\infty} \mu^*(A_n)$ is trivial. Suppose that $\mu^*(A_n) < +\infty$

for all $n \geq 1$. Given $\varepsilon > 0$ then for each $n \geq 1$ we can find an open subset U_n of X containing A_n with $\mu(U_n) < \mu^*(A_n) + \frac{\varepsilon}{2^n}$. By item (b) of Lemma 12.11 we get:

$$\mu\Big(\bigcup_{n=1}^{k} U_n\Big) \leq \sum_{n=1}^{k} \mu(U_n);$$

taking the limit as $k \to \infty$ and using item (d) of Lemma 12.11 we obtain:

$$\mu\Big(\bigcup_{n=1}^{\infty} U_n\Big) \leq \sum_{n=1}^{\infty} \mu(U_n).$$

Since $A \subset \bigcup_{n=1}^{\infty} U_n$ we have:

$$\mu^*(A) \leq \mu\Big(\bigcup_{n=1}^{\infty} U_n\Big) \leq \sum_{n=1}^{\infty} \mu(U_n) < \sum_{n=1}^{\infty} \big(\mu^*(A_n) + \tfrac{\varepsilon}{2^n}\big) = \Big(\sum_{n=1}^{\infty} \mu^*(A_n)\Big) + \varepsilon.$$

The conclusion follows by taking the limit as $\varepsilon \to 0$. □

12.13. **Lemma.** *Given an arbitrary subset $A \subset X$ and a map $f \in \mathfrak{F}(X)$ with $f \geq \chi_A$ then $\lambda(f) \geq \mu^*(A)$.*

Proof. Given $a \in \,]0, 1[$ then $A \subset [f > a]$ and thus:

$$\mu^*(A) \leq \mu^*\big([f > a]\big).$$

Since $U = [f > a]$ is open (see Lemma 12.4), we have $\mu^*(U) = \mu(U) = \lambda(\chi_U)$; thus:

$$\mu^*(A) \leq \lambda(\chi_U).$$

Moreover, we have $f \geq a\chi_U$ and therefore (see Corollary 12.10):

$$\lambda(f) \geq a\lambda(\chi_U) \geq a\mu^*(A).$$

The conclusion is obtained by taking the limit as $a \to 1$. □

12.14. **Corollary.** *The outer measure μ^* is finite on compact subsets of X.*

Proof. Let $K \subset X$ be a compact subset and let $f \in C_{\mathrm{c}}(X)$ be a nonnegative map such that $f|_K \equiv 1$ (see Corollary 12.3). We have $f \geq \chi_K$ and thus Lemma 12.13 gives:

$$+\infty > \lambda(f) \geq \mu^*(K).$$ □

12.15. **Corollary.** *Given open subsets $U, V \subset X$ with $U \subset V$ then:*

$$\mu(V) \geq \mu(U) + \mu^*(V \cap U^{\mathrm{c}}).$$

Proof. Let $\phi \in C_{\mathrm{c}}(X)$ with $0 \leq \phi \leq \chi_U$ be fixed. Since $\chi_V \in \mathfrak{F}(X)$ and since $\phi \leq \chi_V$, by Lemma 12.7, we have $\chi_V - \phi \in \mathfrak{F}(X)$; moreover (see Corollary 12.10):

$$\mu(V) = \lambda(\chi_V) = \lambda(\chi_V - \phi) + \lambda(\phi),$$

so that:

$$\lambda(\chi_V - \phi) = \mu(V) - \lambda(\phi). \tag{12.2}$$

Keeping in mind that $\chi_V - \phi \geq \chi_V - \chi_U = \chi_{V \cap U^{\mathrm{c}}}$, Lemma 12.13 gives us:

$$\lambda(\chi_V - \phi) \geq \mu^*(V \cap U^{\mathrm{c}}). \tag{12.3}$$

From (12.2) and (12.3), we obtain:

$$\mu(V) \geq \mu^*(V \cap U^{\mathrm{c}}) + \lambda(\phi);$$

hence:

$$\begin{aligned}\mu(V) \geq \sup\big\{\mu^*(V \cap U^{\mathrm{c}}) + \lambda(\phi) : \phi \in C_{\mathrm{c}}(X),\ 0 \leq \phi \leq \chi_U\big\}\\ = \mu^*(V \cap U^{\mathrm{c}}) + \lambda(\chi_U) = \mu^*(V \cap U^{\mathrm{c}}) + \mu(U). \quad \Box\end{aligned}$$

Recall that the elements of the collection:

$$\mathfrak{M} = \big\{E \in \wp(X) : \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^{\mathrm{c}}),\ \text{for all } A \in \wp(X)\big\}$$

are called *μ^*-measurable sets*. It is well-known that $\mathfrak{M}$ is a σ-algebra of subsets of X and that the restriction of μ^* to $\mathfrak{M}$ is a (σ-additive) measure.

12.16. **Lemma.** *Every open subset of X is in $\mathfrak{M}$.*

Proof. Let $U \subset X$ be an open subset and let $A \subset X$ be an arbitrary subset. We have to show that:

$$\mu^*(A) = \mu^*(A \cap U) + \mu^*(A \cap U^{\mathrm{c}}).$$

By the subadditivity of μ^*, it suffices to show the inequality:

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \cap U^{\mathrm{c}}).$$

In order to prove the inequality above, we show that:

$$\mu(V) \geq \mu^*(A \cap U) + \mu^*(A \cap U^{\mathrm{c}}),$$

for every open subset V of X containing A. By applying Corollary 12.15 to the open sets V and $U_0 = V \cap U$, we obtain:

$$\mu(V) \geq \mu(U_0) + \mu^*(V \cap U_0^{\mathrm{c}}) = \mu(V \cap U) + \mu^*(V \cap U^{\mathrm{c}}) \geq \mu^*(A \cap U) + \mu^*(A \cap U^{\mathrm{c}}).$$

This concludes the proof. $\Box$

12.17. **Corollary.** *All Borel subsets of X are in $\mathfrak{M}$.* $\Box$

In view of Corollary 12.17, the outer measure μ^* restricts to a Borel measure on X; we denote such restriction just by μ. The Borel measure μ and the positive functional λ are related by the following:

12.18. **Lemma.** *Every $f \in \mathfrak{F}(X)$ is Borel measurable and $\int_X f \, \mathrm{d}\mu = \lambda(f)$.*

Proof. Let $f \in \mathfrak{F}(X)$ be fixed. The Borel measurability of f follows from Lemma 12.4. For each $n \geq 1$, we write:

$$f_n = n\, \chi_{[f > n]} + \sum_{k=1}^{n2^n} \frac{k-1}{2^n}\, \chi_{[\frac{k-1}{2^n} < f \leq \frac{k}{2^n}]};$$

this is just the standard way of providing a sequence of simple measurable nonnegative maps $f_n : X \to \mathbb{R}$ with $f_n \nearrow f$. It is easy to check that:

$$f_n = \frac{1}{2^n} \sum_{k=1}^{n2^n} \chi_{[f > \frac{k}{2^n}]},$$

for all $n \geq 1$. By Lemma 12.4, the sets $\big[f > \frac{k}{2^n}\big]$ are open and thus by Lemmas 12.5 and 12.6, we have $f_n \in \mathfrak{F}(X)$; moreover, using Corollary 12.10, we get:

$$\lambda(f_n) = \frac{1}{2^n} \sum_{k=1}^{n2^n} \mu\big(\big[f > \tfrac{k}{2^n}\big]\big) = \int_X f_n \,\mathrm{d}\mu,$$

for all $n \geq 1$. Hence, using the Monotone Convergence Theorem and Corollary 12.9, we obtain:

$$\lambda(f) = \lim_{n\to\infty} \lambda(f_n) = \lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu,$$

concluding the proof. □

12.19. **Corollary.** *Given $f \in C_{\mathrm{c}}(X)$ then f is μ-integrable and:*

$$\int_X f \,\mathrm{d}\mu = \lambda(f).$$

Proof. Write $f = f^+ - f^-$, with $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$, so that $f^+, f^- \in \mathfrak{F}(X)$; by Lemma 12.18, we have:

$$\int_X f^+ \,\mathrm{d}\mu = \lambda(f^+), \qquad \int_X f^- \,\mathrm{d}\mu = \lambda(f^-).$$

Note that $\lambda(f^+)$ and $\lambda(f^-)$ are finite, since $f^+, f^- \in C_{\mathrm{c}}(X)$. Hence f is μ-integrable and:

$$\lambda(f) = \lambda(f^+) - \lambda(f^-) = \int_X f^+ \,\mathrm{d}\mu - \int_X f^- \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu. \qquad \square$$

12.20. **Lemma.** *For every open set $U \subset X$ we have:*

$$\mu(U) = \sup\big\{\mu(K) : K \subset U \text{ compact}\big\}.$$

Proof. By the monotonicity of the measure μ it is sufficient to show that:

$$\mu(U) \leq \sup\big\{\mu(K) : K \subset U \text{ compact}\big\}.$$

Let $\mathcal{S}$ denote the collection of all maps $f \in C_{\mathrm{c}}(X)$ whose support is contained in U and that satisfy $0 \leq f \leq 1$. Clearly $\mathcal{S}$ is a directed subset of $\mathfrak{F}(X)$ and it follows easily from Corollary 12.3 that $\sup \mathcal{S} = \chi_U$. Thus, by Lemma 12.8, we have:

$$\sup_{f\in\mathcal{S}} \lambda(f) = \lambda(\chi_U) = \mu(U).$$

Given $f \in \mathcal{S}$ then, denoting by K the support of f, we have that K is a compact subset of U and, using Lemma 12.18 we get:

$$\lambda(f) = \int_X f \, \mathrm{d}\mu \leq \mu(K).$$

Hence:

$$\mu(U) = \sup_{f \in \mathcal{S}} \lambda(f) \leq \sup \big\{ \mu(K) : K \subset U \text{ compact} \big\}. \qquad \square$$

Finally, we have the following:

12.21. **Theorem** (Riesz representation)**.** *Let X be a locally compact Hausdorff topological space and let λ be a positive linear functional on $C_{\mathrm{c}}(X)$. Then there exists a unique Borel measure μ on X satisfying the following conditions:*

(a) $\mu(K) < +\infty$*, for every compact subset* $K \subset X$*;*
(b) $\mu(A) = \inf \big\{ \mu(U) : U \supset A \text{ open in } X \big\}$*, for any Borel subset* $A \subset X$*;*
(c) $\mu(U) = \sup \big\{ \mu(K) : K \subset U \text{ compact} \big\}$*, for any open subset* $U \subset X$*;*
(d) $\int_X f \, \mathrm{d}\mu = \lambda(f)$*, for every* $f \in C_{\mathrm{c}}(X)$*.*

Proof. Let μ be the Borel measure on X that we have constructed from λ. Property (a) follows from Corollary 12.14. Property (b) follows from the definition of μ^* and from the fact that μ is just a restriction of μ^* (recall (12.1)). Property (c) follows from Lemma 12.20 and property (d) follows from Corollary 12.19. Now, we just have to prove the uniqueness of μ. Let μ' be a Borel measure on X satisfying properties (a), (b), (c) and (d). We show that $\mu = \mu'$. Since both μ and μ' satisfy property (b), it suffices to show that μ and μ' agree on open subsets of X. Let $U \subset X$ be open and let $\mathcal{S} \subset C_{\mathrm{c}}(X)$ denote the set of all maps $f \in C_{\mathrm{c}}(X)$ whose support is contained in U and that satisfy $0 \leq f \leq 1$. We will show that:

$$\mu'(U) = \sup_{f \in \mathcal{S}} \lambda(f). \tag{12.4}$$

Once (12.4) is proven, it will follow that $\mu = \mu'$; namely, (12.4) clearly holds with μ' replaced by μ and thus $\mu(U) = \mu'(U)$ for every open subset U of X. Let us prove (12.4). Given $f \in \mathcal{S}$ then $f \leq \chi_U$ and thus, using (d):

$$\lambda(f) = \int_X f \, \mathrm{d}\mu' \leq \mu'(U);$$

this proves:

$$\sup_{f \in \mathcal{S}} \lambda(f) \leq \mu'(U).$$

Now, given a compact subset K of U, there exists $f \in \mathcal{S}$ with $f|_K \equiv 1$ (see Corollary 12.3); thus $f \geq \chi_K$ and:

$$\lambda(f) = \int_X f \, \mathrm{d}\mu' \geq \mu'(K).$$

Hence, by (c):

$$\sup_{f\in\mathcal{S}}\lambda(f) \ge \sup\big\{\mu'(K) : K \subset U \text{ compact}\big\} = \mu'(U). \qquad \square$$

13. Quotients of Manifolds

In this section, a *smooth manifold* means a set M endowed with a smooth maximal atlas; no assumptions are made on the topology induced on M by such atlas. Our goal is to prove the following:

13.1. **Theorem.** *Let M be a smooth manifold and let $R \subset M \times M$ be an equivalence relation on M. The following conditions are equivalent:*

- (a) *there exists a smooth maximal atlas on the quotient set M/R such that the quotient map $q : M \to M/R$ is a smooth submersion;*
- (b) *R is a smooth submanifold of $M \times M$ and the restriction to R of the first projection $\pi_1 : M \times M \to M$ is a submersion.*

13.2. *Remark.* Since R is symmetric, condition (b) implies that the restriction of the second projection $\pi_2 : M \times M \to M$ to R is also a submersion. Namely, the diffeomorphism $\sigma(x, y) = (y, x)$ of $M \times M$ maps R onto R and $\pi_2 = \pi_1 \circ \sigma$.

13.3. *Remark.* Assume that G is a group acting on M by smooth maps and that the equivalence relation R is the equivalence relation determined by such action, i.e., $(x, y) \in R$ if and only if $y = g \cdot x$, for some $g \in G$. Observe that if R is a smooth submanifold of $M \times M$ then the restriction to R of the first projection $\pi_1 : M \times M \to M$ is automatically a submersion. Namely, given $(x, y) \in R$, let $g \in G$ be such that $y = g \cdot x$ and consider the smooth map $s : M \to R$ defined by $s(z) = (z, g \cdot z)$, for all $z \in M$. Clearly $s(x) = (x, y)$ and s is a right inverse for $\pi_1|_R : R \to M$, so that $\mathrm{d}s(x)$ is a right inverse for $\mathrm{d}\pi_1(x, y)|_{T_{(x,y)}R} : T_{(x,y)}R \to T_xM$, thus proving that $\mathrm{d}\pi_1(x, y)|_{T_{(x,y)}R}$ is surjective.

Let us first prove that condition (a) implies condition (b). Since the quotient map q is a submersion, the map:

$$q \times q : M \times M \ni (x, y) \longmapsto \big(q(x), q(y)\big) \in (M/R) \times (M/R) \tag{13.1}$$

is also a submersion. In particular, $q \times q$ is transverse to the diagonal submanifold:

$$\Delta_{M/R} = \big\{(x, x) : x \in M/R\big\} \subset (M/R) \times (M/R).$$

Thus $R = (q \times q)^{-1}(\Delta_{M/R})$ is a smooth submanifold of $M \times M$ and its tangent space at a point $(x, y) \in R$ is given by:

$$\begin{aligned} T_{(x,y)}R &= \big(\mathrm{d}(q \times q)_{(x,y)}\big)^{-1}\big(T_{(q(x),q(y))}\Delta_{M/R}\big) \\ &= \big\{(v, w) \in T_xM \oplus T_yM : \mathrm{d}q_x(v) = \mathrm{d}q_y(w)\big\}. \end{aligned}$$

To complete the proof of (b) we have to show that the restriction to $T_{(x,y)}R$ of the projection $T_xM \oplus T_yM \to T_xM$ is surjective. Let $v \in T_xM$ be fixed;

since $\mathrm{d}q_x(v) \in T_{q(x)}(M/R) = T_{q(y)}(M/R)$ and since $\mathrm{d}q_y$ is onto $T_{q(y)}(M/R)$, there exists $w \in T_yM$ such that $\mathrm{d}q_x(v) = \mathrm{d}q_y(w)$. Hence $(v, w) \in T_{(x,y)}R$ and condition (b) follows.

Let us now prove that condition (b) implies condition (a). For each $x \in M$ consider the map $i_x : M \to M \times M$ defined by:

$$i_x(y) = (x, y), \quad y \in M.$$

We claim that i_x is transversal to R. Namely, for $y \in M$, the image of $\mathrm{d}(i_x)_y$ equals $\{0\} \oplus T_yM$; we have to prove that:

$$T_{(x,y)}R + \big(\{0\} \oplus T_yM\big) = T_xM \oplus T_yM. \tag{13.2}$$

Since $\pi_1|_R : R \to M$ is a submersion, the restriction to $T_{(x,y)}R$ of the projection $T_xM \oplus T_yM \to T_xM$ is surjective. This implies equality (13.2) and proves the claim. Now denote by $C_x \subset M$ the equivalence class of x with respect to the equivalence relation R; we have:

$$C_x = i_x^{-1}(R),$$

and since i_x is transversal to R, it follows that C_x is a smooth submanifold of M and that its tangent space at a point $y \in C_x$ is given by:

$$T_yC_x = \big(\mathrm{d}(i_x)_y\big)^{-1}(T_{(x,y)}R) = \big\{v \in T_yM : (0, v) \in T_{(x,y)}R\big\}. \tag{13.3}$$

Set:

$$\mathcal{D}_x = T_xC_x = \big\{v \in T_xM : (0, v) \in T_{(x,x)}R\big\}, \tag{13.4}$$

for each $x \in M$. We have the following:

13.4. **Lemma.** *The set $\mathcal{D} = \bigcup_{x \in M} \mathcal{D}_x \subset TM$ is a smooth distribution on M, i.e., a smooth vector subbundle of the tangent bundle TM of M.*

Proof. Consider the diagonal map $d : M \to M \times M$ defined by $d(x) = (x, x)$, for all $x \in M$. The pull-back $d^*\big(T(M \times M)\big)$ of the tangent bundle of $M \times M$ by the map d equals the vector bundle $TM \oplus TM$ over M. Since R is a smooth submanifold of $M \times M$ containing the image of d, the pull-back:

$$d^*(TR) = \bigcup_{x \in M} T_{(x,x)}R$$

is a smooth subbundle of $TM \oplus TM$. Denote by $P : d^*(TR) \to TM$ the restriction to $d^*(TR)$ of the first projection $TM \oplus TM \to TM$. Then P is a smooth vector bundle morphism; moreover, P is surjective, because $\pi_1|_R : R \to M$ is a submersion. By (13.4), we have:

$$\mathrm{Ker}(P) = \bigcup_{x \in M} \big(\{0\} \oplus \mathcal{D}_x\big).$$

But the kernel of a smooth surjective vector bundle morphism is a smooth vector subbundle of its domain. The conclusion follows. □

We will say that a smooth submanifold $S \subset M$ is *complementary to $\mathcal{D}$* if $T_xM = T_xS \oplus \mathcal{D}_x$, for all $x \in S$.

13.5. **Corollary.** *Let $S \subset M$ be a smooth submanifold of M and assume that $T_xM = T_xS \oplus \mathcal{D}_x$ for some $x \in S$. Then x has an open neighborhood in S that is complementary to $\mathcal{D}$.*

Proof. Since S is a smooth submanifold of M and, by Lemma 13.4, $\mathcal{D}$ is a smooth distribution on M, the set:

$$\big\{y \in S : T_yM = T_yS \oplus \mathcal{D}_y\big\}$$

is open in S. The conclusion follows. □

13.6. **Corollary.** *Every point of M belongs to a smooth submanifold $S \subset M$ that is complementary to $\mathcal{D}$.*

Proof. Given $x \in M$ we can obviously find a smooth submanifold $S \subset M$ containing x such that $T_xM = T_xS \oplus \mathcal{D}_x$. The conclusion follows from Corollary 13.5. □

13.7. **Lemma.** *If $S \subset M$ is a smooth submanifold that is complementary to $\mathcal{D}$ then the inclusion map of $M \times S$ into $M \times M$ is transversal to R.*

Proof. Let $(x, y) \in (M \times S) \cap R$ be fixed. We have to show that:

$$T_xM \oplus T_yM = (T_xM \oplus T_yS) + T_{(x,y)}R.$$

Given $v \in T_xM$, $w \in T_yM$, since $T_yM = T_yS \oplus \mathcal{D}_y$, we can find $w' \in T_yS$ with $w - w' \in \mathcal{D}_y$. Since $(x, y) \in R$ we have $C_x = C_y$ and thus:

$$\mathcal{D}_y = T_yC_y = T_yC_x;$$

by (13.3), $w - w' \in \mathcal{D}_y = T_yC_x$ implies $(0, w - w') \in T_{(x,y)}R$. Hence:

$$(v, w) = (v, w') + (0, w - w') \in (T_xM \oplus T_yS) + T_{(x,y)}R,$$

and the proof is completed. □

13.8. **Corollary.** *If $S \subset M$ is a smooth submanifold that is complementary to $\mathcal{D}$ then $(M \times S) \cap R$ is a smooth submanifold of $M \times M$.* □

13.9. **Lemma.** *Let $S \subset M$ be a smooth submanifold that is complementary to $\mathcal{D}$. Then the restriction to $(M \times S) \cap R$ of the first projection $M \times M \to M$ is a local diffeomorphism (notice that $(M \times S) \cap R$ is indeed a smooth manifold, by Corollary 13.8).*

Proof. Let $(x, y) \in (M \times S) \cap R$ be fixed and denote by P the restriction to:

$$T_{(x,y)}\big((M \times S) \cap R\big) = (T_xM \oplus T_yS) \cap T_{(x,y)}R$$

of the first projection $T_xM \oplus T_yM \to T_xM$. By the Inverse Function Theorem, it suffices to prove that P is an isomorphism. Observe that an element in the kernel of P is of the form $(0, v)$, with $v \in T_yS$ and $(0, v) \in T_{(x,y)}R$. But (13.3) implies $v \in T_yC_x = T_yC_y = \mathcal{D}_y$; since $T_yS \cap \mathcal{D}_y = \{0\}$, we get $v = 0$ and thus P is injective. Let us prove that P is surjective. Let $v \in T_xM$ be fixed. Since the restriction to $T_{(x,y)}R$ of the projection $T_xM \oplus T_yM \to T_xM$ is surjective, there exists $w \in T_yM$ with $(v, w) \in T_{(x,y)}R$. Since $T_yM =$

$T_yS \oplus \mathcal{D}_y$, we can find $w' \in T_yS$ with $w - w' \in \mathcal{D}_y = T_yC_y = T_yC_x$. By (13.3) we have:

$$(0, w - w') \in T_{(x,y)}R$$

and thus:

$$(v, w') = (v, w) + (0, w' - w) \in T_{(x,y)}R.$$

Hence (v, w') is in the domain of P and we are done. □

13.10. **Lemma.** *Let $S \subset M$ be a smooth submanifold that is complementary to $\mathcal{D}$. Then the restriction to $(M \times S) \cap R$ of the second projection $M \times S \to S$ is a submersion (notice that $(M \times S) \cap R$ is indeed a smooth manifold, by Corollary 13.8).*

Proof. Let $(x, y) \in (M \times S) \cap R$ be fixed. We have to prove that the restriction to:

$$T_{(x,y)}\big((M \times S) \cap R\big) = (T_xM \oplus T_yS) \cap T_{(x,y)}R$$

of the second projection $T_xM \oplus T_yS \to T_yS$ is surjective. But this follows directly from the fact that the restriction to $T_{(x,y)}R$ of the second projection $T_xM \oplus T_yM \to T_yM$ is surjective (see Remark 13.2). □

We will say that a smooth submanifold $S \subset M$ is *fundamental* if S is complementary to $\mathcal{D}$ and, in addition, S intercepts each equivalence class determined by R at most once; more explicitly, S is fundamental if S is complementary to $\mathcal{D}$ and $(x, y) \in R$ implies $x = y$, for all $x, y \in S$. We have the following:

13.11. **Lemma.** *If $S \subset M$ is a smooth submanifold that is complementary to $\mathcal{D}$ then any $x \in S$ has an open neighborhood S' in S that is fundamental.*

Proof. Since $(x, x) \in (M \times S) \cap R$, Lemma 13.9 implies that (x, x) has an open neighborhood in $(M \times S) \cap R$ on which the first projection $M \times M \to M$ is injective. Such neighborhood can be chosen in the form $(U \times V) \cap R$, where U is an open neighborhood of x in M and V is an open neighborhood of x in S. Set $S' = U \cap V$. Then S' is an open neighborhood of x in S. Being an open submanifold of S, the submanifold S' is complementary to $\mathcal{D}$. Let us prove that S' is fundamental. Choose $x, y \in S'$ with $(x, y) \in R$. Then (x, y) and (x, x) are both in $(U \times V) \cap R$ and have the same image under the first projection; hence $(x, y) = (x, x)$ and we are done. □

13.12. **Corollary.** *Every point of M belongs to a fundamental smooth submanifold of M.*

Proof. Follows from Corollary 13.6 and from Lemma 13.11. □

13.13. **Lemma.** *Let $S \subset M$ be a fundamental smooth submanifold of M. Then the set:*

$$A = \big\{x \in M : \textit{there exists } y \in S \textit{ with } (x, y) \in R\big\}$$

is open in M. Moreover, for each $x \in M$ there exists precisely one $y \in S$ with $(x, y) \in R$ and the map $p : A \ni x \mapsto y \in S$ is a smooth submersion.

Proof. Denote by P the restriction to $(M \times S) \cap R$ of the first projection $M \times M \to M$. By Lemma 13.9, P is a local diffeomorphism. The set A is precisely the image of P and thus A is indeed open in M. Moreover, P is injective; namely, if (x, y) and (x, y') are in $(M \times S) \cap R$ then y and y' are in S and $(y, y') \in R$. Being an injective local diffeomorphism, P is actually a global diffeomorphism onto the open set A. The fact that the map p is a smooth submersion follows by observing that p is equal to the composite of the diffeomorphism $P^{-1} : A \to (M \times S) \cap R$ with the restriction to $(M \times S) \cap R$ of the second projection $M \times S \to S$ (recall Lemma 13.10). □

13.14. **Lemma.** *Let $S_1, S_2 \subset M$ be fundamental smooth submanifolds. Consider the sets:*

$$S_1' = \big\{x \in S_1 : \textit{there exists } y \in S_2 \textit{ with } (x, y) \in R\big\},$$
$$S_2' = \big\{y \in S_2 : \textit{there exists } x \in S_1 \textit{ with } (x, y) \in R\big\}.$$

Then S_1' is open in S_1 and S_2' is open in S_2. Moreover, for each $x \in S_1'$ there exists precisely one point $y = \alpha(x) \in S_2'$ with $(x, y) \in R$ and the map $\alpha : S_1' \to S_2'$ is a smooth diffeomorphism.

Proof. For $i = 1, 2$, set:

$$A_i = \big\{x \in M : \text{there exists } y \in S_i \text{ with } (x, y) \in R\big\},$$

and denote by $p_i : A_i \to S_i$ the map that carries each $x \in A_i$ to the unique $y \in S_i$ such that $(x, y) \in R$. By Lemma 13.13, the set A_i is open in M and the map p_i is smooth. To conclude the proof, simply observe that $S_1' = A_2 \cap S_1$, $S_2' = A_1 \cap S_2$, $\alpha = p_2|_{S_1'}$ and $\alpha^{-1} = p_1|_{S_2'}$. □

In order to define a smooth maximal atlas on M/R we make use of the following elementary result.

13.15. **Lemma.** *Let N be a set and let $(\phi_i : U_i \to N_i)_{i \in I}$ be a family of bijective maps, where each U_i is a subset of N and each N_i is a smooth manifold. Assume that $N = \bigcup_{i \in I} U_i$ and that for any $i, j \in I$ the maps ϕ_i and ϕ_j are* smoothly compatible, *i.e., the sets $\phi_i(U_i \cap U_j)$ and $\phi_j(U_i \cap U_j)$ are open respectively in N_i and in N_j and the* transition map*:*

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$$

is a smooth diffeomorphism. Then there exists a unique smooth maximal atlas on N such that all U_i are open in N with respect to the topology induced by such atlas and such that all maps ϕ_i are smooth diffeomorphisms.

Proof. A smooth atlas in N is defined by considering compositions of the maps ϕ_i with local charts in the manifolds N_i. Details are left to the reader. □

If $S \subset M$ is a fundamental smooth submanifold then the restriction to S of the quotient map q is injective. Thus $(q|_S)^{-1} : q(S) \to S$ is a bijection defined in the subset $q(S)$ of M/R, taking values in the smooth manifold S.

By Corollary 13.12, when S runs over all fundamental smooth submanifolds of M, the sets $q(S)$ cover M/R. Moreover, Lemma 13.14 says that the bijections $(q|_S)^{-1} : q(S) \to S$ are pairwise smoothly compatible. Thus, Lemma 13.15 gives us a smooth maximal atlas on M/R such that, for every smooth fundamental submanifold $S \subset M$, the set $q(S)$ is open in M/R and the map $q|_S : S \to q(S)$ is a smooth diffeomorphism. In order to complete the proof of Theorem 13.1, we show that $q : M \to M/R$ is a smooth submersion. Let $x \in M$ be fixed and let $S \subset M$ be a fundamental smooth submanifold of M with $x \in S$ (see Corollary 13.12). Define A and p as in the statement of Lemma 13.13. Then A is an open neighborhood of x and:

$$q|_A = (q|_S) \circ p.$$

Since p is a smooth submersion and $q|_S : S \to q(S)$ is a smooth diffeomorphism, it follows that $q|_A$ is a smooth submersion. This concludes the proof of Theorem 13.1.

Now we assume that M/R is endowed with a smooth maximal atlas such that the quotient map $q : M \to M/R$ is a smooth submersion and let us study the topology of M/R. Since q is a submersion, it follows that q is an open mapping; being continuous, surjective and open, the map q is a quotient map in the topological sense. Moreover, we have the following.

13.16. **Lemma.** *If M is second countable then M/R is also second countable.*

Proof. Simply observe that, since q is continuous, open and surjective, it maps a basis of open subsets of M onto a basis of open subsets of M/R. □

13.17. **Lemma.** *The quotient M/R is Hausdorff if and only if R is closed in $M \times M$.*

Proof. Recall that M/R is Hausdorff if and only if the diagonal $\Delta_{M/R}$ is closed in $(M/R) \times (M/R)$. The map $q \times q$ (recall (13.1)) is a surjective submersion and thus it is continuous, open and surjective. It follows that $q \times q$ is a quotient map in the topological sense. Since $R = (q \times q)^{-1}(\Delta_{M/R})$, we have that $\Delta_{M/R}$ is closed in $(M/R) \times (M/R)$ if and only if R is closed in $M \times M$. □

14. The Frobenius Theorem

Recall that a *smooth distribution* $\mathcal{D}$ on a smooth manifold M is a smooth vector subbundle of the tangent bundle TM. For $x \in M$ we set $\mathcal{D}_x = T_xM \cap \mathcal{D}$, i.e., $\mathcal{D}_x$ is the fiber of the vector bundle $\mathcal{D}$ over x. A vector field X on M is called *horizontal* with respect to a distribution $\mathcal{D}$ (or simply *$\mathcal{D}$-horizontal*) if X takes values in $\mathcal{D}$, i.e., if $X(x) \in \mathcal{D}_x$ for all $x \in M$.

14.1. **Definition.** Let M be a smooth manifold and let $\mathcal{D}$ be a distribution on M. The *Levi form* of $\mathcal{D}$ at a point $x \in M$ is the bilinear map:

$$\mathfrak{L}_x^{\mathcal{D}} : \mathcal{D}_x \times \mathcal{D}_x \longrightarrow T_xM/\mathcal{D}_x$$

defined by $\mathfrak{L}^{\mathcal{D}}_x(v,w) = [X,Y](x) + \mathcal{D}_x \in T_xM/\mathcal{D}_x$, where X and Y are $\mathcal{D}$-horizontal smooth vector fields defined in an open neighborhood of x in M with $X(x) = v$ and $Y(x) = w$. By $[X,Y]$ we denote the Lie bracket of the vector fields X and Y.

Below we show that the Levi form is well-defined, i.e., $[X,Y](x) + \mathcal{D}_x$ does not depend on the choice of the $\mathcal{D}$-horizontal vector fields X and Y with $X(x) = v$, $Y(x) = w$. Let θ be a smooth $\mathbb{R}^k$-valued 1-form on an open neighborhood U of x such that $\mathrm{Ker}(\theta_x) = \mathcal{D}_x$ for all $x \in U$. If X and Y are vector fields on an open neighborhood of x then Cartan's formula for exterior differentiation gives:

$$\mathrm{d}\theta(X,Y) = X\big(\theta(Y)\big) - Y\big(\theta(X)\big) - \theta\big([X,Y]\big).$$

If X and Y are $\mathcal{D}$-horizontal then the equality above reduces to:

$$\mathrm{d}\theta(X,Y) = -\theta\big([X,Y]\big).$$

The formula above implies that if X', Y' are $\mathcal{D}$-horizontal vector fields with $X'(x) = X(x)$ and $Y'(x) = Y(x)$ then $\theta\big([X,Y] - [X',Y']\big)(x) = 0$, i.e., $[X,Y](x) - [X',Y'](x) \in \mathcal{D}_x$. Hence the Levi form is well-defined. Setting $X(x) = v$ and $Y(x) = w$ we obtain the following formula:

$$\bar\theta_x\big(\mathfrak{L}^{\mathcal{D}}_x(v,w)\big) = -\mathrm{d}\theta(v,w), \quad v,w \in \mathcal{D}_x, \tag{14.1}$$

where $\bar\theta_x : T_xM/\mathcal{D}_x \to \mathbb{R}^k$ denotes the linear map induced by θ_x in the quotient space.

14.2. **Lemma.** *Let M be a smooth manifold, $\mathcal{D}$ be a smooth distribution on M and let*

$$U \ni (t,s) \longmapsto H(t,s) \in M$$

be a smooth map defined on an open subset $U \subset \mathbb{R}^2$. Let $I \subset \mathbb{R}$ be an interval and let $s_0 \in \mathbb{R}$ be such that $I \times \{s_0\} \subset U$ and $\mathfrak{L}^{\mathcal{D}}_{H(t,s_0)} = 0$ for all $t \in I$. Assume that $\frac{\partial H}{\partial t}(t,s) \in \mathcal{D}$ for all $(t,s) \in U$. If $\frac{\partial H}{\partial s}(t_0,s_0) \in \mathcal{D}$ for some $t_0 \in I$ then $\frac{\partial H}{\partial s}(t,s_0) \in \mathcal{D}$ for all $t \in I$.

Proof. The set:

$$I' = \big\{t \in I : \tfrac{\partial H}{\partial s}(t,s_0) \in \mathcal{D}\big\}$$

is obviously closed in I because the map $I \ni t \mapsto \frac{\partial H}{\partial s}(t,s_0) \in TM$ is continuous and $\mathcal{D}$ is a closed subset of TM. Since I is connected and $t_0 \in I'$, the proof will be complete once we show that I' is open in I. Let $t_1 \in I'$ be fixed. Let θ be an $\mathbb{R}^k$-valued smooth 1-form defined in an open neighborhood V of $H(t_1,s_0)$ in M such that the linear map $\theta_x : T_xM \to \mathbb{R}^k$ is surjective and $\mathrm{Ker}(\theta_x) = \mathcal{D}_x$ for all $x \in V$. Choose a distribution $\mathcal{D}'$ on V such that $T_xM = \mathcal{D}_x \oplus \mathcal{D}'_x$ for all $x \in V$. Then, for each $x \in V$, θ_x restricts to an isomorphism from $\mathcal{D}'_x$ onto $\mathbb{R}^k$. Let J be a connected neighborhood of t_1 in I such that $H(t,s_0) \in V$ for all $t \in J$. We will show below that the map:

$$J \ni t \longmapsto \theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial s}(t,s_0)\big) \in \mathbb{R}^k \tag{14.2}$$

is a solution of a homogeneous linear ODE; since $\theta_{H(t_1,s_0)}\big(\frac{\partial H}{\partial s}(t_1,s_0)\big)=0$, it will follow that $\theta_{H(t,s_0)}\big(\frac{\partial H}{\partial s}(t,s_0)\big)=0$ for all $t\in J$, i.e., $J\subset I'$.

We denote by $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ the canonical basis of $\mathbb{R}^2$ and we apply Cartan's formula for exterior differentiation to the 1-form $H^*\theta$ obtaining:

$$\mathrm{d}(H^*\theta)\big(\tfrac{\partial}{\partial t},\tfrac{\partial}{\partial s}\big)=\tfrac{\partial}{\partial t}\Big((H^*\theta)\big(\tfrac{\partial}{\partial s}\big)\Big)-\tfrac{\partial}{\partial s}\Big((H^*\theta)\big(\tfrac{\partial}{\partial t}\big)\Big)-(H^*\theta)\big(\big[\tfrac{\partial}{\partial t},\tfrac{\partial}{\partial s}\big]\big).$$

Since $\mathrm{d}(H^*\theta)=H^*(\mathrm{d}\theta)$ and $\big[\frac{\partial}{\partial t},\frac{\partial}{\partial s}\big]=0$ we get:

$$\text{(14.3)}\quad \mathrm{d}\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial t}(t,s_0),\tfrac{\partial H}{\partial s}(t,s_0)\big)=\tfrac{\partial}{\partial t}\Big(\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial s}(t,s_0)\big)\Big)\\ -\tfrac{\partial}{\partial s}\big|_{s=s_0}\Big(\theta_{H(t,s)}\big(\tfrac{\partial H}{\partial t}(t,s)\big)\Big),\quad t\in J.$$

Observe that, since $\frac{\partial H}{\partial t}(t,s)$ is in $\mathcal{D}$, the last term on the righthand side of (14.3) vanishes. We can write $\frac{\partial H}{\partial s}(t,s_0)=u_1(t)+u_2(t)$ with $u_1(t)\in\mathcal{D}$ and $u_2(t)\in\mathcal{D}'$. Since the Levi form of $\mathcal{D}$ vanishes at points of the form $H(t,s_0)$, equation (14.1) implies that $\mathrm{d}\theta_{H(t,s_0)}(v,w)=0$ for all $v,w\in\mathcal{D}_{H(t,s_0)}$. We may thus replace $\frac{\partial H}{\partial s}(t,s_0)$ by $u_2(t)$ in the lefthand side of (14.3). For $t\in J$ we consider the linear map $L(t):\mathbb{R}^k\to\mathbb{R}^k$ defined by:

$$L(t)\cdot z=\mathrm{d}\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial t}(t,s_0),\sigma_{H(t,s_0)}(z)\big),\quad z\in\mathbb{R}^k,$$

where, for $x\in V$, $\sigma_x:\mathbb{R}^k\to\mathcal{D}'_x$ denotes the inverse of the isomorphism $\theta_x|_{\mathcal{D}'_x}:\mathcal{D}'_x\to\mathbb{R}^k$. Observe that:

$$\begin{aligned}\mathrm{d}\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial t}(t,s_0),\tfrac{\partial H}{\partial s}(t,s_0)\big)&=\mathrm{d}\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial t}(t,s_0),u_2(t)\big)\\&=L(t)\cdot\theta_{H(t,s_0)}\big(u_2(t)\big)\\&=L(t)\cdot\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial s}(t,s_0)\big).\end{aligned}$$

Equation (14.3) can now be rewritten as:

$$\tfrac{\partial}{\partial t}\Big(\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial s}(t,s_0)\big)\Big)=L(t)\cdot\theta_{H(t,s_0)}\big(\tfrac{\partial H}{\partial s}(t,s_0)\big),\quad t\in J.$$

Hence the map (14.2) is a solution of a homogeneous linear ODE and we are done. □

Given smooth manifolds M and N we denote by $\mathrm{Lin}(TM,TN)$ the vector bundle over $M\times N$ whose fiber at a point $(x,y)\in M\times N$ is the space $\mathrm{Lin}(T_xM,T_yN)$ of linear maps from T_xM to T_yN.

14.3. **Theorem.** *Let M, N be smooth manifolds and F be a smooth section of the vector bundle $\mathrm{Lin}(TM,TN)$ defined in an open subset $A\subset M\times N$. Consider the distribution $\mathrm{gr}(F)$ on A whose fiber at a point $(x,y)\in A$ is the graph of the linear map $F(x,y):T_xM\to T_yN$. Let*

$$\mathbb{R}\times\Lambda\supset Z\ni(t,\lambda)\longmapsto\phi(t,\lambda)\in M,\quad \mathbb{R}\times\Lambda\supset Z\ni(t,\lambda)\longmapsto\psi(t,\lambda)\in N,$$

be smooth maps, where Λ is a smooth manifold and $Z\subset\mathbb{R}\times\Lambda$ is an open subset. Let $\alpha:V\to Z$ be a smooth map defined in an open subset $V\subset M$ such that $\phi\big(\alpha(x)\big)=x$ for all $x\in V$. Assume that:

(1) $\big(\phi(t,\lambda),\psi(t,\lambda)\big) \in A$ *and the Levi form of* $\mathrm{gr}(F)$ *vanishes at the point* $\big(\phi(t,\lambda),\psi(t,\lambda)\big)$ *for all* $(t,\lambda)\in Z$*;*
(2) $\frac{\partial\psi}{\partial t}(t,\lambda) = F\big(\phi(t,\lambda),\psi(t,\lambda)\big)\cdot\frac{\partial\phi}{\partial t}(t,\lambda)$*, for all* $(t,\lambda)\in Z$*;*
(3) *for every* $(t_1,\lambda)\in Z$ *there exists* $t_0\in\mathbb{R}$ *such that* $I\times\{\lambda\}\subset Z$ *and* $\frac{\partial\psi}{\partial\lambda}(t_0,\lambda) = F\big(\phi(t_0,\lambda),\psi(t_0,\lambda)\big)\circ\frac{\partial\phi}{\partial\lambda}(t_0,\lambda)$*, where* $I\subset\mathbb{R}$ *denotes the closed interval whose endpoints are* t_0 *and* t_1*.*

Then the map $f=\psi\circ\alpha : V\to N$ *satisfies the PDE:*

$$\mathrm{d}f(x) = F\big(x,f(x)\big),\quad x\in V.$$

Proof. We start by proving that:

$$\frac{\partial\psi}{\partial\lambda}(t_1,\lambda)\cdot v = \Big[F\big(\phi(t_1,\lambda),\psi(t_1,\lambda)\big)\circ\frac{\partial\phi}{\partial\lambda}(t_1,\lambda)\Big]\cdot v, \tag{14.4}$$

for all $(t_1,\lambda)\in Z$ and all $v\in T_\lambda\Lambda$. Let $(t_1,\lambda)\in Z$ and $v\in T_\lambda\Lambda$ be fixed; let $t_0\in\mathbb{R}$ be as in condition (3) above. Let $\gamma : \left]-\varepsilon,\varepsilon\right[\to\Lambda$ be a smooth curve with $\gamma(0)=\lambda$ and $\gamma'(0)=v$. We define a map $H : U\to A\subset M\times N$ by setting $H(t,s)=\big(\phi(t,\gamma(s)),\psi(t,\gamma(s))\big)$, where $U\subset\mathbb{R}^2$ is the open set:

$$U=\big\{(t,s)\in\mathbb{R}\times\left]-\varepsilon,\varepsilon\right[: \big(t,\gamma(s)\big)\in Z\big\}.$$

Obviously $I\times\{0\}\subset U$. Condition (2) above implies that $\frac{\partial H}{\partial t}(t,s)\in\mathrm{gr}(F)$ for all $(t,s)\in U$; moreover, since

$$\frac{\partial H}{\partial s}(t_0,0)=\big(\tfrac{\partial\phi}{\partial\lambda}(t_0,\lambda)\cdot v,\tfrac{\partial\psi}{\partial\lambda}(t_0,\lambda)\cdot v\big),$$

condition (3) above implies that $\frac{\partial H}{\partial s}(t_0,0)\in\mathrm{gr}(F)$. Since the Levi form of $\mathrm{gr}(F)$ vanishes at $H(t,0)$ for all $t\in I$, Lemma 14.2 gives $\frac{\partial H}{\partial s}(t,0)\in\mathrm{gr}(F)$ for all $t\in I$. In particular $\frac{\partial H}{\partial s}(t_1,0)\in\mathrm{gr}(F)$, i.e., (14.4) holds.

Condition (2) above and identity (14.4) imply that:

$$\mathrm{d}\psi(t_1,\lambda)=F\big(\phi(t_1,\lambda),\psi(t_1,\lambda)\big)\circ\mathrm{d}\phi(t_1,\lambda),$$

for all $(t_1,\lambda)\in Z$. Using this equality and recalling that $\phi\big(\alpha(x)\big)=x$, for all $x\in V$, we compute:

$$\begin{aligned}\mathrm{d}f(x)=\mathrm{d}\psi\big(\alpha(x)\big)\circ\mathrm{d}\alpha(x)&=F\big(\phi(\alpha(x)),\psi(\alpha(x))\big)\circ\mathrm{d}\phi\big(\alpha(x)\big)\circ\mathrm{d}\alpha(x)\\&=F\big(x,f(x)\big)\circ\mathrm{d}(\phi\circ\alpha)(x)=F\big(x,f(x)\big).\end{aligned}$$

This concludes the proof. □

14.4. **Corollary.** *Let* $A\ni(x,y)\mapsto F(x,y)\in\mathrm{Lin}(\mathbb{R}^m,\mathbb{R}^n)$ *be a smooth map defined in an open subset* $A\subset\mathbb{R}^m\times\mathbb{R}^n$*. Assume that:*

$$\frac{\partial F_{kj}}{\partial x_i}(x,y)+\sum_{r=1}^{n}F_{ri}(x,y)\frac{\partial F_{kj}}{\partial y_r}(x,y)=\frac{\partial F_{ki}}{\partial x_j}(x,y)+\sum_{r=1}^{n}F_{rj}(x,y)\frac{\partial F_{ki}}{\partial y_r}(x,y), \tag{14.5}$$

for all $(x,y)\in A$*,* $i,j=1,\ldots,m$*,* $k=1,\ldots,n$*. Then, for every* $(x_0,y_0)\in A$ *there exists a smooth map* $f : V\to\mathbb{R}^n$ *defined in an open neighborhood* V

of x_0 in $\mathbb{R}^m$ such that $f(x_0) = y_0$, $\big(x, f(x)\big) \in A$ for all $x \in V$ and f is a solution of the PDE:

$$\mathrm{d}f(x) = F\big(x, f(x)\big), \quad x \in V.$$

14.5. **Corollary.** *Let M, N be smooth manifolds, E be a finite-dimensional real vector space and ω^M, ω^N be E-valued smooth 1-forms respectively on M and on N. Assume that $\omega^N(y) : T_yN \to E$ is an isomorphism for all $y \in N$. Let*

$$\mathbb{R} \times \Lambda \supset Z \ni (t, \lambda) \longmapsto \phi(t, \lambda) \in M, \quad \mathbb{R} \times \Lambda \supset Z \ni (t, \lambda) \longmapsto \psi(t, \lambda) \in N,$$

be smooth maps, where Λ is a smooth manifold and $Z \subset \mathbb{R} \times \Lambda$ is an open subset. Let $\alpha : V \to Z$ be a smooth map defined in an open subset $V \subset M$ such that $\phi\big(\alpha(x)\big) = x$ for all $x \in V$. Assume that:

(1) *for every $(t, \lambda) \in Z$,*

$$F\big(\phi(t, \lambda), \psi(t, \lambda)\big)^*\big[\mathrm{d}\omega^N\big(\psi(t, \lambda)\big)\big] = \mathrm{d}\omega^M\big(\phi(t, \lambda)\big),$$

where $F(x, y) : T_xM \to T_yN$ denotes the linear map defined by:

$$F(x, y) = \omega^N(y)^{-1} \circ \omega^M(x), \tag{14.6}$$

for all $x \in M$, $y \in N$;

(2) *for every $(t, \lambda) \in Z$,*

$$\omega^N_{\psi(t,\lambda)}\big(\tfrac{\partial \psi}{\partial t}(t, \lambda)\big) = \omega^M_{\phi(t,\lambda)}\big(\tfrac{\partial \phi}{\partial t}(t, \lambda)\big);$$

(3) *for every $(t_1, \lambda) \in Z$ there exists $t_0 \in \mathbb{R}$ such that $I \times \{\lambda\} \subset Z$ and*

$$\omega^N_{\psi(t_0,\lambda)} \circ \tfrac{\partial \psi}{\partial \lambda}(t_0, \lambda) = \omega^M_{\phi(t_0,\lambda)} \circ \tfrac{\partial \phi}{\partial \lambda}(t_0, \lambda),$$

where I denotes the closed interval whose endpoints are t_0 and t_1.

Then the map $f = \psi \circ \alpha : V \to N$ satisfies $f^*\omega_N = \omega_M|_V$.

Proof. We just have to apply Theorem 14.3 to the section F of $\mathrm{Lin}(TM, TN)$ defined by (14.6). Observe indeed that the condition $f^*\omega^N = \omega^M|_V$ is equivalent to the PDE $\mathrm{d}f(x) = F\big(x, f(x)\big)$; moreover, it is clear that conditions (2) and (3) in the statement of this Corollary are equivalent respectively to conditions (2) and (3) in the statement of Theorem 14.3. To complete the proof, we will show that the Levi form of $\mathrm{gr}(F)$ vanishes at a point $(x, y) \in M \times N$ if and only if $F(x, y)^*\big[\mathrm{d}\omega^N(y)\big] = \mathrm{d}\omega^M(x)$. Considere the E-valued 1-form θ on $M \times N$ defined by $\theta = \pi_2^*\omega^N - \pi_1^*\omega^M$, where π_1 and π_2 denote the projections of the product $M \times N$. We have:

$$\theta_{(x,y)}(v, w) = \omega^N_y(w) - \omega^M_x(v),$$

for all $(x, y) \in M \times N$, $(v, w) \in T_xM \oplus T_yN$; it is thus easy to see that the linear map $\theta_{(x,y)} : T_xM \oplus T_yN \to E$ is surjective and that its kernel equals the graph of $F(x, y)$, for all $(x, y) \in M \times N$. In particular, the linear map

$$\bar{\theta}_{(x,y)} : (T_xM \oplus T_yN)/\mathrm{gr}(F)_{(x,y)} \longrightarrow E$$

induced on the quotient space is an isomorphism. It follows from (14.1) that the Levi form of $\mathrm{gr}(F)$ vanishes at (x,y) if and only if $\mathrm{d}\theta_{(x,y)}$ annihilates $\mathrm{gr}(F)$. We have $\mathrm{d}\theta = \pi_2^*(\mathrm{d}\omega^N) - \pi_1^*(\mathrm{d}\omega^M)$, i.e.:

$$\mathrm{d}\theta_{(x,y)}\big((v_1,w_1),(v_2,w_2)\big) = \mathrm{d}\omega^N(w_1,w_2) - \mathrm{d}\omega^M(v_1,v_2),$$

for all $(x,y)\in M\times N$, $(v_1,w_1),(v_2,w_2)\in T_xM\oplus T_yN$. It follows that $\mathrm{d}\theta_{(x,y)}$ annihilates $\mathrm{gr}(F)$ if and only if $F(x,y)^*\big[\mathrm{d}\omega^N(y)\big] = \mathrm{d}\omega^M(x)$. □

15. Global Frobenius

In what follows M denotes a smooth manifold (i.e., a set endowed with a smooth maximal atlas; no topological assumptions are made) and $\mathcal{D}$ denotes a smooth distribution on M (i.e., a smooth vector subbundle of the tangent bundle TM). By an *integral submanifold* of $\mathcal{D}$ we mean an immersed smooth submanifold S of M such that $T_xS = \mathcal{D}_x$, for all $x\in S$. We have the following:

15.1. **Lemma.** *Let S be an integral submanifold of $\mathcal{D}$ and let $f : N \to M$ be a smooth map defined in a smooth manifold N. Assume that the image of $\mathrm{d}f_p$ is contained in $\mathcal{D}_{f(p)}$, for all $p\in N$. Then $f^{-1}(S)$ is an open subset of N.*

Proof. Let $p\in f^{-1}(S)$ be fixed. Let $\varphi : U \to \widetilde{U}$ be a local chart in M such that $f(p)\in U$, $\widetilde{U} = \widetilde{U}_1\times\widetilde{U}_2$, with $\widetilde{U}_1$ an open subset of $\mathbb{R}^{n_1}$, $\widetilde{U}_2$ an open subset of $\mathbb{R}^{n_2}$ and such that $\mathrm{d}\varphi_{f(p)}(\mathcal{D}_{f(p)}) = \mathbb{R}^{n_1}\times\{0\}^{n_2}$. We write $\varphi = (\varphi^1,\varphi^2)$, where $\varphi^i : U\to\widetilde{U}_i$, $i = 1,2$. Since $\mathrm{d}\varphi^1_{f(p)}$ maps $\mathcal{D}_{f(p)}$ isomorphically onto $\mathbb{R}^{n_1}$, we may assume (possibly taking a smaller U) that $\mathrm{d}\varphi^1_x$ maps $\mathcal{D}_x$ isomorphically onto $\mathbb{R}^{n_1}$, for all $x\in U$. Consider the smooth map $F : \widetilde{U}_1\times\widetilde{U}_2 \to \mathrm{Lin}(\mathbb{R}^{n_1},\mathbb{R}^{n_2})$ defined by:

$$F(u_1,u_2) = \mathrm{d}\varphi^2_x\circ(\mathrm{d}\varphi^1_x|_{\mathcal{D}_x})^{-1},$$

for all $u_1\in\widetilde{U}_1$, $u_2\in\widetilde{U}_2$, where $x = \varphi^{-1}(u_1,u_2)$.

Since $\mathrm{d}(\varphi^1|_{S\cap U})_{f(p)} = \big(\mathrm{d}\varphi^1_{f(p)}\big)\big|_{\mathcal{D}_{f(p)}}$ is an isomorphism, the Inverse Function Theorem gives us an open neighborhood S_0 of $f(p)$ in $S\cap U$ that is mapped diffeomorphically by φ^1 onto an open subset A of $\mathbb{R}^{n_1}$ contained in $\widetilde{U}_1$. The set $f^{-1}\big((\varphi^1)^{-1}(A)\big)$ is an open neighborhood of p in N; let V be an open neighborhood of p in N contained in $f^{-1}\big((\varphi^1)^{-1}(A)\big)$ that is diffeomorphic to some open ball in Euclidean space. We will now complete the proof by showing that V is contained in $f^{-1}(S)$.

Let $p'\in V$ be fixed. Since V is diffeomorphic to an open ball, there exists a smooth curve $\lambda : [0,1]\to V$ with $\lambda(0) = p$ and $\lambda(1) = p'$. Set $\gamma = \varphi^1\circ f\circ\lambda$ and $\mu = \varphi^2\circ f\circ\lambda$. For every $t\in[0,1]$, we have:

$$(f\circ\lambda)'(t) = \mathrm{d}f_{\lambda(t)}\big(\lambda'(t)\big)\in\mathcal{D}_{f(\lambda(t))};$$

since $\mu'(t) = \mathrm{d}\varphi^2_{f(\lambda(t))}\big((f\circ\lambda)'(t)\big)$ and $\gamma'(t) = \mathrm{d}\varphi^1_{f(\lambda(t))}\big((f\circ\lambda)'(t)\big)$, we have:

$$\mu'(t) = \Big(\mathrm{d}\varphi^2_{f(\lambda(t))} \circ \big(\mathrm{d}\varphi^1_{f(\lambda(t))}\big|_{\mathcal{D}_{f(\lambda(t))}}\big)^{-1}\Big)\big(\gamma'(t)\big).$$

By the definition of F, this means:

$$\mu'(t) = F\big(\gamma(t), \mu(t)\big)\cdot\gamma'(t), \tag{15.1}$$

for all $t \in [0,1]$. Since the image of γ is contained in A and $\varphi^1|_{S_0} : S_0 \to A$ is a diffeomorphism, we can consider the curve $\sigma : [0,1] \to S_0 \subset U$ defined by $\sigma = (\varphi^1|_{S_0})^{-1}\circ\gamma$; set $\bar\mu = \varphi^2\circ\sigma$. For $t\in[0,1]$, we have:

$$\sigma'(t) = \big(\mathrm{d}\varphi^1_{\sigma(t)}\big|_{T_{\sigma(t)}S_0}\big)^{-1}\big(\gamma'(t)\big) = \big(\mathrm{d}\varphi^1_{\sigma(t)}\big|_{\mathcal{D}_{\sigma(t)}}\big)^{-1}\big(\gamma'(t)\big);$$

thus:

$$\bar\mu'(t) = \Big(\mathrm{d}\varphi^2_{\sigma(t)} \circ \big(\mathrm{d}\varphi^1_{\sigma(t)}\big|_{\mathcal{D}_{\sigma(t)}}\big)^{-1}\Big)\big(\gamma'(t)\big).$$

By the definition of F, this means:

$$\bar\mu'(t) = F\big(\gamma(t), \bar\mu(t)\big)\cdot\gamma'(t), \tag{15.2}$$

for all $t \in [0,1]$. Now (15.1) and (15.2) imply that $\mu : [0,1] \to \mathbb{R}^{n_2}$ and $\bar\mu : [0,1] \to \mathbb{R}^{n_2}$ are both solutions of the same ODE. Since $\mu(0) = \bar\mu(0)$, we conclude that $\mu = \bar\mu$. Thus:

$$\varphi\circ f\circ\lambda = (\gamma,\mu) = (\gamma,\bar\mu) = \varphi\circ\sigma;$$

since φ is injective, we get $f\circ\lambda = \sigma$. In particular:

$$f(p') = f\big(\lambda(1)\big) = \sigma(1) \in S_0 \subset S.$$

This concludes the proof. □

15.2. **Corollary.** *Under the hypotheses of Lemma 15.1, assume in addition that $f(N) \subset S$. Then the map $f : N \to S$ is smooth.*

Proof. Since S is an immersed submanifold of M, it suffices to prove that the map $f : N \to S$ is continuous when S is endowed with the topology induced by its own atlas (which maybe finer than the topology induced by M on S). If S' is an open subset of S then S' is also an integral submanifold of $\mathcal{D}$; thus Lemma 15.1 implies that $f^{-1}(S')$ is open in N. The conclusion follows. □

15.3. **Corollary.** *Let $S_1, S_2 \subset M$ be integral submanifolds of $\mathcal{D}$. Then $S_1\cap S_2$ is open in S_1 and in S_2 (relative to the topology induced by the atlases of S_1 and S_2). Moreover, $S_1\cap S_2$ inherits the same manifold structure from S_1 and from S_2.*

Proof. Let $i_1 : S_1 \to M$, $i_2 : S_2 \to M$ denote the inclusion maps. It follows from Lemma 15.1 that $i_1^{-1}(S_2) = S_1\cap S_2$ is open in S_1 and that $i_2^{-1}(S_1) = S_1\cap S_2$ is open in S_2. Let $\mathcal{A}_i$ denote the manifold structure induced on $S_1\cap S_2$ by S_i, $i = 1,2$; it follows from Corollary 15.2 that the identity map $\mathrm{Id} : (S_1\cap S_2, \mathcal{A}_1) \to (S_1\cap S_2, \mathcal{A}_2)$ is a smooth diffeomorphism. Hence $\mathcal{A}_1 = \mathcal{A}_2$ and the proof is completed. □

Let $S_{\max} \subset M$ denote the union of all integral submanifolds of $\mathcal{D}$. We have the following:

15.4. **Lemma.** *The set $S_{\max}$ admits a unique maximal smooth atlas such that every integral submanifold of $\mathcal{D}$ is an open submanifold of $S_{\max}$.*

Proof. This is an application of Lemma 13.15. Let $(S_i)_{i\in I}$ denote the family of all integral submanifolds of $\mathcal{D}$. By definition, we have $S_{\max} = \bigcup_{i\in I} S_i$. Moreover, for each $i \in I$, the identity map $\mathrm{Id} : S_i \to S_i$ is a bijection between the subset S_i of $S_{\max}$ and the smooth manifold S_i. To complete the proof, observe that the smooth compatibility requirement of Lemma 13.15 is a consequence of Corollary 15.3. □

The manifold $S_{\max}$ is called the *global integral submanifold* of $\mathcal{D}$. We already know that every integral submanifold of $\mathcal{D}$ is an open submanifold of $S_{\max}$; moreover, we have the following:

15.5. **Lemma.** *The manifold $S_{\max}$ is an integral submanifold of $\mathcal{D}$.*

Proof. Denote by $i : S_{\max} \to M$ the inclusion map. We first prove that i is a smooth immersion, so that $S_{\max}$ is a smooth immersed submanifold of M. Let $x \in S_{\max}$ be fixed. Since $S_{\max}$ is the union of the integral submanifolds of $\mathcal{D}$, there exists an integral submanifold S of $\mathcal{D}$ with $x \in S \subset S_{\max}$. Since S is an open submanifold of $S_{\max}$, $i|_S : S \to M$ is a smooth immersion and $x \in S_{\max}$ is arbitrary, it follows that i is a smooth immersion. Moreover, we have:

$$T_x S_{\max} = T_x S = \mathcal{D}_x.$$

Hence $S_{\max}$ is indeed an integral submanifold of $\mathcal{D}$. □

Since the topology of $S_{\max}$ is in general finer than the topology induced by M, it follows that if M is Hausdorff then also $S_{\max}$ is Hausdorff. It is not in general true that $S_{\max}$ is second countable, even if M is. For instance, if $\mathcal{D}$ is integrable then $S_{\max}$ equals M as a set (but not as a manifold); the connected components of $S_{\max}$ are the *maximal connected integral submanifolds* of $\mathcal{D}$. In the integrable case it is known that, if M is second countable, then each connected component of $S_{\max}$ is second countable; however, if the rank of $\mathcal{D}$ is smaller than the dimension of M then $S_{\max}$ has an uncountable number of connected components.

16. Compact-Open Topology

Let X, Y be topological spaces. Denote by $\mathfrak{C}(X,Y)$ the set of continuous maps $f : X \to Y$. Given a compact subset $K \subset X$ and an open subset $U \subset Y$ we denote by $\mathcal{V}(K;U)$ the subset of $\mathfrak{C}(X,Y)$ consisting of those maps $f : X \to Y$ such that $f(K) \subset U$. The smallest topology on $\mathfrak{C}(X,Y)$ containing the sets $\mathcal{V}(K;U)$ is called the *compact-open topology* on $\mathfrak{C}(X,Y)$.

The sets $\mathcal{V}(K;U)$ with $K \subset X$ compact and $U \subset Y$ open form a subbasis for the compact-open topology and the finite intersections:

$$\bigcap_{i=1}^{r} \mathcal{V}(K_i;U_i),$$

with $K_1, \ldots, K_r \subset X$ compact, $U_1, \ldots, U_r \subset Y$ open and r a positive integer, form a basis for the compact-open topology on $\mathfrak{C}(X,Y)$ (observe that if $K = U = \emptyset$ then $\mathcal{V}(K;U) = \mathfrak{C}(X,Y)$).

16.1. **Lemma.** *Let Λ, X, Y be topological spaces and let $f : \Lambda \times X \to Y$ be a continuous map. Then, if $\mathfrak{C}(X,Y)$ is endowed with the compact-open topology, the map:*

$$\tilde{f} : \Lambda \longrightarrow \mathfrak{C}(X,Y),$$

defined by $\tilde{f}(\lambda)(x) = f(\lambda,x)$, $\lambda \in \Lambda$, $x \in X$, is continuous.

Proof. It is sufficient to prove that $\tilde{f}^{-1}\big(\mathcal{V}(K;U)\big)$ is open in Λ for every compact set $K \subset X$ and every open set $U \subset Y$. Let $\lambda \in \tilde{f}^{-1}\big(\mathcal{V}(K;U)\big)$ be fixed. The set $f^{-1}(U)$ is open in the product $\Lambda \times X$ and it contains $\{\lambda\} \times K$; since K is compact, $f^{-1}(U)$ also contains $V \times K$ for some neighborhood V of λ in Λ. Hence $V \subset \tilde{f}^{-1}\big(\mathcal{V}(K;U)\big)$ and we are done. □

16.2. **Lemma.** *Let Λ, X, Y be topological spaces and let $\tilde{f} : \Lambda \to \mathfrak{C}(X,Y)$ be a continuous map, where $\mathfrak{C}(X,Y)$ is endowed with the compact-open topology. Assume that X is locally compact*[8]*. Then the map $f : \Lambda \times X \to Y$ defined by $f(\lambda,x) = \tilde{f}(\lambda)(x)$, $\lambda \in \Lambda$, $x \in X$ is continuous.*

Proof. Let $\lambda \in \Lambda$, $x \in X$ be fixed and let U be an open neighborhood of $f(\lambda,x)$ in Y. Since the map $\tilde{f}(\lambda) : X \to Y$ is continuous, the set $\tilde{f}(\lambda)^{-1}(U)$ is an open neighborhood of x in X. Let K be a compact neighborhood of x contained in $\tilde{f}(\lambda)^{-1}(U)$. Then $\tilde{f}(\lambda)$ is in $\mathcal{V}(K;U)$ and therefore we can find a neighborhood V of λ in Λ with $\tilde{f}(V) \subset \mathcal{V}(K;U)$. Hence $V \times K$ is a neighborhood of (λ,x) in $\Lambda \times X$ and $f(V \times K) \subset U$. □

We now focus on the space $\mathfrak{C}\big([a,b],X\big)$ of continuous curves $\gamma : [a,b] \to X$ on a fixed topological space X. By a *partition* of the interval $[a,b]$ we mean a finite subset P of $[a,b]$ containing a and b; we write $P = \{t_0, \ldots, t_r\}$ meaning that $a = t_0 < t_1 < \cdots < t_r = b$. Given a partition $P = \{t_0, \ldots, t_r\}$ of $[a,b]$ and a sequence U_1, U_2, $\ldots$, U_r of open subsets of X, we write:

(16.1)
$$\mathfrak{V}(P;U_1,\ldots,U_r) = \big\{\gamma \in \mathfrak{C}\big([a,b],X\big) : \gamma\big([t_{i-1},t_i]\big) \subset U_i,\ i = 1,\ldots,r\big\}.$$

Obviously $\mathfrak{V}(P;U_1,\ldots,U_r)$ is an open subset of $\mathfrak{C}\big([a,b],X\big)$ with respect to the compact-open topology. Moreover, we have the following:

[8] This means that any neighborhood of an arbitrary point $x \in X$ contains a compact neighborhood of x.

16.3. **Lemma.** *Let X be a topological space and $\mathcal{B}$ be a basis of open subsets for X. The sets $\mathfrak{V}(P; U_1, \ldots, U_r)$, where P runs over the partitions of $[a, b]$ and $U_1, \ldots, U_r \in \mathcal{B}$, form a basis of open subsets for the compact-open topology on $\mathfrak{C}\big([a, b], X\big)$.*

Proof. Let $\mathcal{Z}$ be an open subset of $\mathfrak{C}\big([a, b], X\big)$ with respect to the compact-open topology and let $\gamma \in \mathcal{Z}$ be fixed. We'll find a partition $P = \{t_0, \ldots, t_r\}$ of $[a, b]$ and basic open sets $U_1, \ldots, U_r \in \mathcal{B}$ such that:

$$\gamma \in \mathfrak{V}(P; U_1, \ldots, U_r) \subset \mathcal{Z}. \tag{16.2}$$

By the definition of the compact-open topology, we can find compact subsets $K_1, \ldots, K_s \subset [a, b]$ and open subsets $V_1, \ldots, V_s \subset X$ such that:

$$\gamma \in \bigcap_{i=1}^{s} \mathcal{V}(K_i; V_i) \subset \mathcal{Z}.$$

Let $u \in [a, b]$ be fixed. The set:

$$\bigcap_{\substack{i=1,\ldots,s \\ u \in K_i}} V_i$$

is an open neighborhood of $\gamma(u)$ in X and therefore it contains a basic open set $B_u \in \mathcal{B}$ such that $\gamma(u) \in B_u$. Set:

$$I_u = \gamma^{-1}(B_u) \cap \bigcap_{\substack{i=1,\ldots,s \\ u \notin K_i}} \big([a, b] \setminus K_i\big). \tag{16.3}$$

Then $u \in I_u$ and I_u is open in $[a, b]$. Let $\delta > 0$ be a Lebesgue number for the open cover $\bigcup_{u \in [a,b]} I_u$ of the compact metric space $[a, b]$; this means that every subset of $[a, b]$ having diameter less than δ is contained in some I_u. Let $P = \{t_0, \ldots, t_r\}$ be a partition of $[a, b]$ with $t_j - t_{j-1} < \delta$, for $j = 1, \ldots, r$. For each $j = 1, \ldots, r$ we can find $u_j \in [a, b]$ with $[t_{j-1}, t_j] \subset I_{u_j}$; set $U_j = B_{u_j}$. We claim that (16.2) holds.

Since for $j = 1, \ldots, r$, $[t_{j-1}, t_j] \subset I_{u_j}$ and $\gamma(I_{u_j}) \subset B_{u_j} = U_j$, we have $\gamma \in \mathfrak{V}(P; U_1, \ldots, U_r)$. To complete the proof, choose $\mu \in \mathfrak{V}(P; U_1, \ldots, U_r)$ and let us prove that $\mu \in \bigcap_{i=1}^{s} \mathcal{V}(K_i; V_i)$. Let $i = 1, \ldots, s$ and $t \in K_i$ be fixed. We have $t \in [t_{j-1}, t_j]$, for some $j = 1, \ldots, r$. We claim that $u_j \in K_i$; namely, otherwise I_{u_j} would be contained in $[a, b] \setminus K_i$ (recall (16.3)), but t is in $I_{u_j} \cap K_i$. But $u_j \in K_i$ implies $U_j = B_{u_j} \subset V_i$. Finally, since $\mu \in \mathfrak{V}(P; U_1, \ldots, U_r)$, we have $\mu(t) \in U_j \subset V_i$. This proves that $\mu(K_i) \subset V_i$ for $i = 1, \ldots, s$ and completes the prove of the lemma. □

17. The Group of Homeomorphisms of a Topological Space

Recall that given topological spaces X, Y then $\mathfrak{C}(X, Y)$ denotes the set of all continuous maps $f : X \to Y$; in what follows, we consider $\mathfrak{C}(X, Y)$ to be endowed with the compact-open topology (recall Section 16). Given a topological space X, denote by $\mathrm{Homeo}(X)$ the subset of $\mathfrak{C}(X, X)$ consisting of all homeomorphisms $f : X \to X$. Obviously $\mathrm{Homeo}(X)$ is a group under composition. In what follows, we investigate under which conditions $\mathrm{Homeo}(X)$ is a topological group (when endowed with the topology inherited from the compact-open topology of $\mathfrak{C}(X, X)$).

17.1. **Lemma.** *Let Y be a locally compact topological space. Given a compact subset K of Y and an open subset U of Y containing K then there exists a compact subset L of U whose interior contains K.*

Proof. For each $x \in K$, there exists a compact neighborhood L_x of x contained in U. The interiors $\mathrm{int}(L_x)$ of the sets L_x constitute an open cover of the compact set K, from which we can extract a finite subcover $K \subset \bigcup_{i=1}^{n} \mathrm{int}(L_{x_i})$. Now define L by setting $L = \bigcup_{i=1}^{n} L_{x_i}$. □

17.2. **Lemma.** *Let X, Y, Z be topological spaces with Y locally compact. Then the* composition map*:*

$$\mathfrak{C}(X, Y) \times \mathfrak{C}(Y, Z) \ni (f, g) \longmapsto g \circ f \in \mathfrak{C}(X, Z) \tag{17.1}$$

is continuous.

Proof. Let $f_0 \in \mathfrak{C}(X, Y)$, $g_0 \in \mathfrak{C}(Y, Z)$ be fixed. It suffices to show that given a compact subset K of X and an open subset U of Z with $(g_0 \circ f_0)(K) \subset U$ then there exists a neighborhood of (f_0, g_0) in $\mathfrak{C}(X, Y) \times \mathfrak{C}(Y, Z)$ that is mapped by (17.1) into $\mathcal{V}(K, U)$. We have $f_0(K) \subset g_0^{-1}(U)$, with $f_0(K)$ a compact subset of Y and $g_0^{-1}(U)$ an open subset of Y; by Lemma 17.1, there exists a compact subset L of $g_0^{-1}(U)$ whose interior $\mathrm{int}(L)$ contains $f_0(K)$. Clearly $\mathcal{V}\big(K, \mathrm{int}(L)\big) \times \mathcal{V}(L, U)$ is a neighborhood of (f_0, g_0) having the desired property. □

17.3. **Lemma.** *Let G be a group endowed with a topology for which the multiplication map $G \times G \to G$ is continuous. Then the inversion map $G \ni x \mapsto x^{-1} \in G$ is continuous if and only if it is continuous at the identity element $1 \in G$.*

Proof. For each $g \in G$, denote by $\mathfrak{l}_g : G \to G$ and $\mathfrak{r}_g : G \to G$ the maps defined by $\mathfrak{l}_g(x) = gx$ and $\mathfrak{r}_g(x) = xg$ respectively. Clearly, $\mathfrak{l}_g$ and $\mathfrak{r}_g$ are continuous for all $g \in G$. The conclusion follows from the commutativity of the diagram below:

$$\begin{array}{ccc} G & \xrightarrow{\text{inversion}} & G \\ {\scriptstyle \mathfrak{l}_{g^{-1}}}\big\downarrow & & \big\uparrow{\scriptstyle \mathfrak{r}_{g^{-1}}} \\ G & \xrightarrow[\text{inversion}]{} & G \end{array}$$

□

17.4. **Lemma.** *Let X be a Hausdorff, locally compact and locally connected topological space. Given a compact subset K contained in an open subset U of X then there exists a finite sequence $(K_i)_{i=1}^{n}$ of compact subsets of X and a finite sequence $(U_i)_{i=1}^{n}$ of open subsets of X such that:*

- $K_i \subset U_i$, *for all* $i = 1, \ldots, n$;
- *the interior of* K_i *is nonempty, for all* $i = 1, \ldots, n$;
- K_i *is connected and* $\overline{U_i}$ *is compact, for all* $i = 1, \ldots, n$;
- $\bigcap_{i=1}^n \mathcal{V}(K_i, U_i) \subset \mathcal{V}(K, U)$.

Proof. For each $x \in K$, let V_x be a compact neighborhood of x contained in U, V'_x be a compact neighborhood of x contained in the interior of V_x and let W_x be a connected neighborhood of x contained in V'_x. The open cover $\bigcup_{x \in K} \mathrm{int}(W_x)$ of K has a finite subcover $\bigcup_{i=1}^n \mathrm{int}(W_{x_i})$. Now take K_i to be the closure of W_{x_i} and U_i to be the interior of V_{x_i}. Since $W_{x_i} \subset V'_{x_i} \subset U_i$ and V'_{x_i} is closed, it follows that $K_i \subset U_i$. Clearly, x_i belongs to the interior of K_i, so that $\mathrm{int}(K_i) \neq \emptyset$. The connectedness of K_i follows from the connectedness of W_{x_i} and the compactness of $\overline{U_i}$ follows from the compactness of V_{x_i}. Finally, the inclusion $\bigcap_{i=1}^n \mathcal{V}(K_i, U_i) \subset \mathcal{V}(K, U)$ follows by observing that $K \subset \bigcup_{i=1}^n K_i$ and $\bigcup_{i=1}^n U_i \subset U$. □

17.5. **Lemma.** *Let* X *be a Hausdorff, locally compact and locally connected topological space. Then* Homeo(X) *is a* topological group, *i.e., both its multiplication map and its inversion map are continuous.*

Proof. By Lemma 17.2, the multiplication map of the group Homeo(X) is continuous and by Lemma 17.3, in order to complete the proof, it suffices to show that the inversion map of Homeo(X) is continuous at the identity map $\mathrm{Id} : X \to X$. The continuity of the inversion map of Homeo(X) at the point Id is equivalent to the following condition: given a compact subset K of X contained in an open subset U of X then for every f in some neighborhood of Id in Homeo(X) we have $f^{-1}(K) \subset U$. By Lemma 17.4, in order to prove this condition, it suffices to consider the case in which K is connected, $\mathrm{int}(K) \neq \emptyset$ and $\overline{U}$ is compact. Let x be an arbitrary point in the interior of K. Then:

$$\mathcal{V}(\partial U, K^{\mathrm{c}}) \cap \mathcal{V}\big(\{x\}, \mathrm{int}(K)\big) \cap \mathrm{Homeo}(X) \tag{17.2}$$

is a neighborhood of the point Id in Homeo(X), where ∂U denotes the boundary of U. Given f in (17.2), we prove that $f^{-1}(K) \subset U$. Since $f : X \to X$ is a homeomorphism, we have that $f(U)$ is open in X and $f(\overline{U})$ is closed in X; moreover, since $f(\partial U) \cap K = \emptyset$, we have:

$$f(\overline{U}) \cap K = \big(f(U) \cup f(\partial U)\big) \cap K = f(U) \cap K.$$

It follows that $f(U) \cap K$ is both open and closed relatively to K. Moreover, $f(x) \in f(U) \cap K$ and, since K is connected, we get $f(U) \cap K = K$. Hence $K \subset f(U)$ and $f^{-1}(K) \subset U$. □

17.6. **Example.** Let X be the subspace of the real line $\mathbb{R}$ defined by:

$$X = \{0, 1, 2, 3, \ldots\} \cup \big\{\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\big\}.$$

Then X is locally compact (obviously Hausdorff), but *not* locally connected. It follows from Lemma 17.2 that the multiplication map of the group Homeo(X) is continuous. We claim that the inversion map of Homeo(X) is *not* continuous. For each $n \geq 2$, let $f_n : X \to X$ be the homeomorphism defined by:

- $f_n(0) = 0$;
- $f_n\big(\frac{1}{k}\big) = \frac{1}{k+1}$, for $k \geq n$;
- $f_n\big(\frac{1}{k}\big) = \frac{1}{k}$ and $f_n(k) = k$, for $k = 1, \ldots, n-1$;
- $f_n(k) = k - 1$, for $k > n$;
- $f_n(n) = \frac{1}{n}$.

It is easy to see that $(f_n)_{n\geq 2}$ converges to the identity of X in $\mathrm{Homeo}(X)$ but $(f_n^{-1})_{n\geq 2}$ does not.

18. Liftings

We start with the following:

18.1. **Lemma.** *Let X, $\widetilde{X}$ and Y be topological spaces, with $\widetilde{X}$ Hausdorff and Y connected. Let $\pi : \widetilde{X} \to X$ be a locally injective map (i.e., every point of $\widetilde{X}$ has a neighborhood in which π is injective) and let $\tilde{f}_1 : Y \to \widetilde{X}$, $\tilde{f}_2 : Y \to \widetilde{X}$ be continuous maps with $\pi \circ \tilde{f}_1 = \pi \circ \tilde{f}_2$. If $\tilde{f}_1$ and $\tilde{f}_2$ agree on some point of Y then $\tilde{f}_1 = \tilde{f}_2$.*

Proof. Since $\widetilde{X}$ is Hausdorff, the set:

$$\big\{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\big\} \tag{18.1}$$

is closed in Y. It is also nonempty, by our hypotheses. We claim that (18.1) is open in Y. Namely, Let $y \in Y$ be fixed with $\tilde{f}_1(y) = \tilde{f}_2(y)$. If A is an open neighborhood of $\tilde{f}_1(y)$ in $\widetilde{X}$ such that $\pi|_A$ in injective then $\tilde{f}_1^{-1}(A) \cap \tilde{f}_2^{-1}(A)$ is an open neighborhood of y in Y contained in (18.1). This proves the claim and concludes the proof. □

In what follows $\widetilde{X}$ and X are topological spaces and $\pi : \widetilde{X} \to X$ is a local homeomorphism, i.e., for every $\tilde{x} \in \widetilde{X}$ there exists an open subset $A \subset \widetilde{X}$ such that $\tilde{x} \in A$, $\pi(A)$ is open in X and $\pi|_A : A \to \pi(A)$ is a homeomorphism. By a *lifting* of a continuous map $f : Y \to X$ defined in a topological space Y we mean a continuous map $\tilde{f} : Y \to \widetilde{X}$ such that $\pi \circ \tilde{f} = f$. Lemma 18.1 implies that, if $\widetilde{X}$ is Hausdorff and Y is connected, a continuous map $f : Y \to X$ admits *at most* one lifting $\tilde{f} : Y \to \widetilde{X}$ satisfying a prescribed condition of the form $f(y_0) = \tilde{x}_0$.

We are now concerned with liftings of curves $\gamma : [a, b] \to X$. Let $\mathcal{L}$ denote the subset of $\widetilde{X} \times \mathfrak{C}\big([a, b], X\big)$ consisting of pairs $(\tilde{x}_0, \gamma)$ such that γ admits a unique lifting $\tilde{\gamma} : [a, b] \to \widetilde{X}$ satisfying the *initial condition* $\tilde{\gamma}(a) = \tilde{x}_0$. We endow the sets $\mathfrak{C}\big([a, b], X\big)$ and $\mathfrak{C}\big([a, b], \widetilde{X}\big)$ with the compact-open topology (see Section 16). Obviously if $(\tilde{x}_0, \gamma)$ is in $\mathcal{L}$ then $\gamma(a) = \pi(\tilde{x}_0)$; observe also that if $\widetilde{X}$ is Hausdorff then the uniqueness of the lifting of γ satisfying the prescribed initial condition $\tilde{\gamma}(a) = \tilde{x}_0$ is automatic; its existence, however, is not. Consider the map:

$$L : \mathcal{L} \longrightarrow \mathfrak{C}\big([a,b], \widetilde{X}\big)$$

defined by $L(\tilde{x}_0, \gamma) = \tilde{\gamma}$, where $\tilde{\gamma} : [a,b] \to \widetilde{X}$ is the unique lifting of γ such that $\tilde{\gamma}(a) = \tilde{x}_0$. We have the following:

18.2. **Lemma.** *The map L is continuous.*

Proof. Let $\mathcal{B}$ denote the collection of all open subsets A of $\widetilde{X}$ such that $\pi(A)$ is open in X and $\pi|_A : A \to \pi(A)$ is a homeomorphism. Since π is a local homeomorphism, the set $\mathcal{B}$ is a basis of open subsets of $\widetilde{X}$. Let $(\tilde{x}_0, \gamma) \in \mathcal{L}$ be fixed and set $\tilde{\gamma} = L(\tilde{x}_0, \gamma)$. Let $P = \{t_0, \ldots, t_r\}$ be a partition of $[a,b]$ and let $A_1, \ldots, A_r \in \mathcal{B}$ be such that (recall (16.1)):

$$\tilde{\gamma} \in \mathfrak{V}(P; A_1, \ldots, A_r).$$

By Lemma 16.3, in order to complete the proof, it suffices to find a neighborhood of $(\tilde{x}_0, \gamma)$ in $\mathcal{L}$ that is mapped by L into $\mathfrak{V}(P; A_1, \ldots, A_r)$. Let $\mathcal{Z}$ denote the set of pairs $(\tilde{y}_0, \mu)$ in $\mathcal{L}$ such that:

- $\tilde{y}_0 \in A_1$;
- $\mu\big([t_{i-1}, t_i]\big) \subset \pi(A_i)$, for $i = 1, \ldots, r$;
- $\mu(t_i) \in \pi(A_i \cap A_{i+1})$, for $i = 1, \ldots, r-1$.

Keeping in mind the definition of the compact-open topology in $\mathfrak{C}\big([a,b], X\big)$, it is immediate that $\mathcal{Z}$ is open in $\mathcal{L}$. Moreover, $(\tilde{x}_0, \gamma)$ is in $\mathcal{Z}$. We will show that $L(\mathcal{Z}) \subset \mathfrak{V}(P; A_1, \ldots, A_r)$. Let $(\tilde{y}_0, \mu) \in \mathcal{Z}$ be fixed. For $i = 1, \ldots, r$, we consider the continuous curve $\tilde{\mu}_i : [t_{i-1}, t_i] \to A_i \subset \widetilde{X}$ defined by:

$$\tilde{\mu}_i = (\pi|_{A_i})^{-1} \circ \mu|_{[t_{i-1}, t_i]}.$$

We claim that $\tilde{\mu}_i(t_i) = \tilde{\mu}_{i+1}(t_i)$, for $i = 1, \ldots, r-1$. Namely, since $\mu(t_i)$ is in $\pi(A_i \cap A_{i+1})$, there exists $p \in A_i \cap A_{i+1}$ with $\mu(t_i) = \pi(p)$. Since $\pi|_{A_i}$ is injective, $\tilde{\mu}_i(t_i)$ and p are in A_i and $\pi\big(\tilde{\mu}_i(t_i)\big) = \mu(t_i) = \pi(p)$, it follows that $\tilde{\mu}_i(t_i) = p$. Similarly, since $\pi|_{A_{i+1}}$ is injective, $\tilde{\mu}_{i+1}(t_i)$ and p are in A_{i+1} and $\pi\big(\tilde{\mu}_{i+1}(t_i)\big) = \mu(t_i) = \pi(p)$, it follows that $\tilde{\mu}_{i+1}(t_i) = p$. This proves the claim.

Since $\tilde{\mu}_i(t_i) = \tilde{\mu}_{i+1}(t_i)$, for $i = 1, \ldots, r-1$, we can consider the curve $\tilde{\mu} : [a,b] \to \widetilde{X}$ such that $\tilde{\mu}|_{[t_{i-1}, t_i]} = \tilde{\mu}_i$, for $i = 1, \ldots, r$. The curve $\tilde{\mu}$ is a lifting of μ. Moreover, since $\pi|_{A_1}$ is injective, $\tilde{y}_0$ and $\tilde{\mu}(a)$ are in A_1 and $\pi(\tilde{y}_0) = \mu(a) = \pi\big(\tilde{\mu}(a)\big)$, it follows that $\tilde{\mu}(a) = \tilde{y}_0$. Therefore $L(\tilde{y}_0, \mu) = \tilde{\mu}$. The proof is completed by observing that $\tilde{\mu} \in \mathfrak{V}(P; A_1, \ldots, A_r)$. □

18.3. **Corollary.** *Let Y be a topological space and let $f : Y \times [a,b] \to X$ and $\tilde{f}_0 : Y \to \widetilde{X}$ be continuous maps such that for every $y \in Y$, the curve $\gamma_y : [a,b] \ni t \mapsto f(y,t) \in X$ has a unique lifting $\tilde{\gamma}_y : [a,b] \to \widetilde{X}$ such that $\tilde{\gamma}_y(a) = \tilde{f}_0(y)$. Then f has a unique lifting $\tilde{f} : Y \times [a,b] \to \widetilde{X}$ such that $\tilde{f}(y,a) = \tilde{f}_0(y)$, for all $y \in Y$.*

Proof. By Lemma 16.1, the map $F : Y \to \mathfrak{C}\big([a,b],X\big)$ defined by $F(y) = \gamma_y$, $y \in Y$, is continuous. By our hypotheses, the continuous map:

$$(\tilde{f}_0, F) : Y \longrightarrow \widetilde{X} \times \mathfrak{C}\big([a,b],\widetilde{X}\big)$$

takes values in $\mathcal{L}$. It is clear that there exists a unique map $\tilde{f} : Y \times [a,b] \to \widetilde{X}$ such that $\pi \circ \tilde{f} = f$ and $\tilde{f}(y,a) = \tilde{f}_0(y)$, for all $y \in Y$; such map is given by $\tilde{f}(y,t) = L\big(\tilde{f}_0(y), F(y)\big)(t)$, for all $y \in Y$, $t \in [a,b]$. It follows from Lemmas 18.2 and 16.2 that $\tilde{f}$ is indeed continuous. □

18.4. **Definition.** We say that π has the *unique lifting property for paths* if for any continuous map $\gamma : [a,b] \to X$ and any $\tilde{x}_0 \in \pi^{-1}\big(\gamma(a)\big)$ there exists a unique lifting $\tilde{\gamma} : [a,b] \to \widetilde{X}$ of γ with $\tilde{\gamma}(a) = \tilde{x}_0$.

18.5. **Definition.** By a *loop* in a topological space Y we mean a continuous map $\gamma : [a,b] \to Y$ with $\gamma(a) = \gamma(b)$. We say that the loop γ is *contractible* in Y if there exists a continuous map $H : [0,1] \times [a,b] \to Y$ such that:

- $H(0,t) = \gamma(t)$, for all $t \in [a,b]$;
- $H(s,a) = H(s,b)$, for all $s \in [0,1]$;
- the map $[a,b] \ni t \mapsto H(1,t) \in Y$ is constant.

We say that Y is *semi-locally simply-connected* if every point of Y has a neighborhood V such that any loop in V is contractible in Y.

18.6. **Lemma.** *Assume that π has the unique lifting property for paths. Let A be an arc-connected subset of $\widetilde{X}$ such that every loop in $\pi(A)$ is contractible in X. Then $\pi|_A$ is injective.*

Proof. Assume that $\tilde{x}_1, \tilde{x}_2 \in A$ and that $\pi(\tilde{x}_1) = \pi(\tilde{x}_2)$. Since A is arc-connected, there exists a continuous map $\tilde{\gamma} : [a,b] \to A$ with $\tilde{\gamma}(a) = \tilde{x}_1$ and $\tilde{\gamma}(b) = \tilde{x}_2$. Then $\gamma = \pi \circ \tilde{\gamma}$ is a loop in $\pi(A)$; therefore γ is contractible in X, i.e., there exists a continuous map $H : [0,1] \times [a,b] \to X$ as in Definition 18.5. Since π has the unique lifting property for paths, Corollary 18.3 gives us a lifting $\widetilde{H} : [0,1] \times [a,b] \to \widetilde{X}$ of H such that $\widetilde{H}(0,t) = \tilde{\gamma}(t)$, for all $t \in [a,b]$ (notice that $[a,b]$ plays the role of Y and $[0,1]$ plays the role of the interval $[a,b]$ in the statement of Corollary 18.3).

Since the map $[a,b] \ni t \mapsto H(1,t) \in X$ is constant, the unique lifting property for paths implies that its lifting $[a,b] \ni t \mapsto \widetilde{H}(1,t) \in \widetilde{X}$ is also constant. In particular, $\widetilde{H}(1,a) = \widetilde{H}(1,b)$; therefore, the paths:

$$[0,1] \ni s \longmapsto \widetilde{H}(1-s,a) \in \widetilde{X}, \quad [0,1] \ni s \longmapsto \widetilde{H}(1-s,b) \in \widetilde{X},$$

are liftings of the same path in X and they agree on $s = 0$. Again, by the unique lifting property for paths, it follows that $\widetilde{H}(1-s,a) = \widetilde{H}(1-s,b)$, for all $s \in [0,1]$. In particular:

$$\tilde{x}_1 = \tilde{\gamma}(a) = \widetilde{H}(0,a) = \widetilde{H}(0,b) = \tilde{\gamma}(b) = \tilde{x}_2.$$

This concludes the proof. □

18.7. **Corollary.** *Under the hypotheses of Lemma 18.6, if in addition the set A is open in $\widetilde{X}$ then $\pi(A)$ is open in X and $\pi|_A : A \to \pi(A)$ is a homeomorphism.*

Proof. Simply observe that, being a local homeomorphism, π is an open mapping; moreover, if A is open in $\widetilde{X}$ and the restriction of π to A is injective then $\pi|_A : A \to \pi(A)$ is a continuous, bijective open mapping. □

18.8. **Definition.** An open subset $U \subset X$ is called a *fundamental open subset* of X if $\pi^{-1}(U)$ equals a disjoint union $\bigcup_{i\in I} U_i$ of open subsets U_i of $\widetilde{X}$ such that $\pi|_{U_i} : U_i \to U$ is a homeomorphism for all $i \in I$. We say that π is a *covering map* if X can be covered by fundamental open subsets.

Obviously every covering map is a local homeomorphism.

18.9. **Corollary.** *Assume that π has the unique lifting property for paths and that $\widetilde{X}$ is locally arc-connected (recall Definition 6.1). Let U be an arc-connected open subset of X such that every loop in U is contractible in X. Then U is a fundamental open subset of X.*

Proof. Let $(U_i)_{i\in I}$ denote the arc-connected components of $\pi^{-1}(U)$. Since $\pi^{-1}(U)$ is open in $\widetilde{X}$, each U_i is open in $\widetilde{X}$, by Lemma 6.2. Obviously $\pi^{-1}(U) = \bigcup_{i\in I} U_i$ is a disjoint union. Let $i \in I$ be fixed and let us show that $\pi|_{U_i} : U_i \to U$ is a homeomorphism. Obviously $\pi(U_i) \subset U$. We claim that $\pi(U_i) = U$. Given $x \in U$, choose an arbitrary point $\tilde{x}_0 \in U_i$ and let $\gamma : [a,b] \to U$ be a continuous map with $\gamma(a) = \pi(\tilde{x}_0)$ and $\gamma(b) = x$. By the unique lifting property for paths, we can find a lifting $\tilde{\gamma} : [a,b] \to \widetilde{X}$ of γ such that $\tilde{\gamma}(a) = \tilde{x}_0$. Since $\tilde{\gamma}$ is a continuous curve in $\pi^{-1}(U)$ starting at a point of U_i and since U_i is an arc-connected component of $\pi^{-1}(U)$, it follows that $\tilde{\gamma}$ takes values in U_i. In particular $\tilde{\gamma}(b) \in U_i$ and $\pi\big(\tilde{\gamma}(b)\big) = \gamma(b) = x$. Finally, Corollary 18.7 implies that $\pi|_{U_i} : U_i \to U$ is a homeomorphism. □

18.10. **Corollary.** *Assume that π has the unique lifting property for paths and that X is locally arc-connected and semi-locally simply-connected. Then π is a covering map.*

Proof. Observe that, since π is a local homeomorphism and X is locally arc-connected, it follows that also $\widetilde{X}$ is locally arc-connected. The conclusion follows from Corollary 18.9 (recall also Corollary 6.3). □

Assume now that $\widetilde{X}$ is Hausdorff, so that Lemma 18.1 guarantees the uniqueness of the liftings of curves (with prescribed initial conditions). Now let $\gamma : [a,b] \to X$ be a continuous curve and let $\tilde{x}_0 \in \pi^{-1}\big(\gamma(a)\big)$ be such that γ *does not* admit a lifting $\tilde{\gamma} : [a,b] \to \widetilde{X}$ with $\tilde{\gamma}(a) = \tilde{x}_0$. Consider the set:

$$\big\{t \in \,]a,b] : \gamma|_{[a,t]} \text{ admits a lifting } \tilde{\gamma} : [a,t] \to \widetilde{X} \text{ with } \tilde{\gamma}(a) = \tilde{x}_0\big\}. \tag{18.2}$$

The set (18.2) is not empty; namely, if A is an open neighborhood of $\tilde{x}_0$ such that $\pi(A)$ is open in X and $\pi|_A : A \to \pi(A)$ is a homeomorphism then there exists $\varepsilon > 0$ with $\gamma\big([a, a + \varepsilon]\big) \subset \pi(A)$ and therefore $\tilde{\gamma} = (\pi|_A)^{-1} \circ \gamma|_{[a,a+\varepsilon]}$ is a lifting of $\gamma|_{[a,a+\varepsilon]}$ with $\tilde{\gamma}(a) = \tilde{x}_0$.

Obviously if t is in (18.2) and t' is in $]a, t]$ then also t' is in (18.2). Therefore (18.2) is an interval. Let $t_0 \in \,]a, b]$ be the supremum of (18.2). Then $]a, t_0[$ is contained in (18.2). For each $t \in \,]a, t_0[$, let $\tilde{\gamma}_t : [a, t] \to \widetilde{X}$ be a lifting of $\gamma|_{[a,t]}$ with $\tilde{\gamma}_t(a) = \tilde{x}_0$. Given $t, t' \in \,]a, t_0[$, with $t' < t$ then $\tilde{\gamma}_{t'}$ and $\tilde{\gamma}_t|_{[a,t']}$ are both liftings of the same curve having the same initial condition; therefore $\tilde{\gamma}_{t'} = \tilde{\gamma}_t|_{[a,t']}$. We can thus define a curve $\tilde{\gamma} : [a, t_0[\,\to \widetilde{X}$ by setting:

$$\tilde{\gamma}|_{[a,t]} = \tilde{\gamma}_t,$$

for all $t \in \,]a, t_0[$. The curve $\tilde{\gamma}$ is continuous, since its restriction to $]a, t]$ is continuous for all $t \in \,]a, t_0[$. Moreover, $\tilde{\gamma}$ is a lifting of $\gamma|_{[a,t_0[}$ satisfying the initial condition $\tilde{\gamma}(a) = \tilde{x}_0$. We call $\tilde{\gamma}$ the *maximal partial lifting* of γ starting at $\tilde{x}_0$.

We have the following:

18.11. **Lemma.** *Assume that $\widetilde{X}$ is Hausdorff. Let $\gamma : [a, b] \to X$ be a continuous curve and let $\tilde{x}_0 \in \pi^{-1}\big(\gamma(a)\big)$ be such that γ does not admit a lifting starting at $\tilde{x}_0$. Let $\tilde{\gamma} : [a, t_0[\,\to \widetilde{X}$ be the maximal partial lifting of γ starting at $\tilde{x}_0$, where $t_0 \in \,]a, b]$. Then $\gamma|_{[a,t_0]}$ does not admit a lifting starting at $\tilde{x}_0$ (i.e., t_0 is not in* (18.2)*).*

Proof. If $t_0 = b$ then $\gamma|_{[a,t_0]} = \gamma$ and, by our hypotheses, γ does not admit a lifting starting at $\tilde{x}_0$. Assume that $t_0 < b$ and assume by contradiction that $\gamma|_{[a,t_0]}$ admits a lifting $\tilde{\gamma} : [a, t_0] \to \widetilde{X}$ with $\tilde{\gamma}(a) = \tilde{x}_0$. Let A be an open neighborhood of $\tilde{\gamma}(t_0)$ in $\widetilde{X}$ such that $\pi(A)$ is open in X and $\pi|_A : A \to \pi(A)$ is a homeomorphism. Then $\gamma\big([t_0, t_0+\varepsilon]\big)$ is contained in $\pi(A)$ for some $\varepsilon > 0$. Consider the curve $\tilde{\mu} : [t_0, t_0 + \varepsilon] \to A$ defined by $\tilde{\mu} = (\pi|_A)^{-1} \circ \gamma|_{[t_0,t_0+\varepsilon]}$. Then $\tilde{\mu}$ is a lifting of $\gamma|_{[t_0,t_0+\varepsilon]}$ starting at $\tilde{\gamma}(t_0)$. Therefore the concatenation of $\tilde{\gamma}$ with $\tilde{\mu}$ is a lifting of $\gamma|_{[a,t_0+\varepsilon]}$ starting at $\tilde{x}_0$. This contradicts the maximality of $\tilde{\gamma}$ and concludes the proof. □

Recall that a point p in a topological space Y is called a *limit value* of a map $f : [a, b[\,\to Y$ at the point b if for any neighborhood V of p and any $\varepsilon > 0$ there exists $t \in \,]b - \varepsilon, b[$ with $f(t) \in V$. We have the following:

18.12. **Lemma.** *Assume that $\widetilde{X}$ and X are Hausdorff. Let $\gamma : [a, b] \to X$ be a continuous curve and let $\tilde{x}_0 \in \pi^{-1}\big(\gamma(a)\big)$ be such that γ does not admit a lifting starting at $\tilde{x}_0$. Let $\tilde{\gamma} : [a, t_0[\,\to \widetilde{X}$ be the maximal partial lifting of γ starting at $\tilde{x}_0$, where $t_0 \in \,]a, b]$. Then the map $\tilde{\gamma}$ has no limit values at the point t_0.*

Proof. Assume by contradiction that $p \in \widetilde{X}$ is a limit value of $\tilde{\gamma}$ at the point $\tilde{x}_0$. We claim that $\pi(p) = \gamma(t_0)$. Otherwise, we could find disjoint open sets $U_1, U_2 \subset X$ with $\pi(p) \in U_1$ and $\gamma(t_0) \in U_2$; then $\gamma\big(\,]t_0 - \varepsilon, t_0]\,\big) \subset U_2$ for some $\varepsilon > 0$ and there exists $t \in \,]t_0 - \varepsilon, t_0[$ with $\tilde{\gamma}(t) \in \pi^{-1}(U_1)$. This implies $\gamma(t) = \pi\big(\tilde{\gamma}(t)\big) \in U_1$, contradicting $U_1 \cap U_2 = \emptyset$. The claim is proved.

Let now A be an open neighborhood of p in $\widetilde{X}$ such that $\pi(A)$ is open in X and $\pi|_A : A \to \pi(A)$ is a homeomorphism. Since $\gamma(t_0) = \pi(p)$ is in $\pi(A)$, we can find $\varepsilon > 0$ with $\gamma\big(\,]t_0-\varepsilon, t_0]\,\big) \subset \pi(A)$. Now there exists $t \in\,]t_0-\varepsilon, t_0[$ with $\tilde{\gamma}(t) \in A$. define $\tilde{\mu} : [t, t_0] \to A$ by setting:

$$\tilde{\mu} = (\pi|_A)^{-1} \circ \gamma|_{[t,t_0]}.$$

Then $\tilde{\mu}$ is a lifting of $\gamma|_{[t,t_0]}$ starting at $\tilde{\gamma}(t)$; the concatenation of $\tilde{\gamma}|_{[a,t]}$ with $\tilde{\mu}$ is therefore a lifting of $\gamma|_{[a,t_0]}$ starting at $\tilde{x}_0$. This contradicts Lemma 18.11. □

18.13. **Corollary.** *Under the assumptions of Lemma 18.12, we have:*

(a) *if $(t_n)_{n\ge1}$ is a sequence in $[a, t_0[$ converging to t_0 then $\big(\tilde{\gamma}(t_n)\big)_{n\ge1}$ has no converging subsequence in $\widetilde{X}$;*

(b) *if K is a compact subset of $\widetilde{X}$ then there exists $\varepsilon > 0$ such that $\tilde{\gamma}\big(\,]t_0-\varepsilon, t_0[\,\big)$ is disjoint from K.*

Proof. If $\big(\tilde{\gamma}(t_n)\big)_{n\ge1}$ had a converging subsequence to a point $p \in \widetilde{X}$ then p would be a limit value of $\tilde{\gamma}$ at the point t_0. Thus (a) is proven. Let us prove (b). For each point $p \in K$, since p is not a limit value of $\tilde{\gamma}$ at the point t_0, we can find an open neighborhood U_p of p in $\widetilde{X}$ and a positive number $\varepsilon_p > 0$ such that $\tilde{\gamma}\big(\,]t_0-\varepsilon_p, t_0[\,\big)$ is disjoint from U_p. The open cover $\bigcup_{p\in K} U_p$ of K has a finite subcover $\bigcup_{i=1}^r U_{p_i}$. Let $\varepsilon = \min_{i=1}^r \varepsilon_{p_i} > 0$. Then $\tilde{\gamma}\big(\,]t_0-\varepsilon, t_0[\,\big)$ is disjoint from K. □

18.14. **Definition.** We will say that a continuous curve $\gamma : [a, b] \to X$ *admits liftings with arbitrary initial conditions* if for every $\tilde{x}_0 \in \pi^{-1}\big(\gamma(a)\big)$ there exists a continuous lifting $\tilde{\gamma} : [a, b] \to \widetilde{X}$ of γ with $\tilde{\gamma}(a) = \tilde{x}_0$.

18.15. **Lemma.** *Assume that $\widetilde{X}$ is Hausdorff and that the following property holds; for any point $p \in X$ there exists a neighborhood U of p in X, a point $p_0 \in X$ and a continuous map $H : [0, 1] \times U \to X$ such that:*

- *$H(0, x) = p_0$ and $H(1, x) = x$, for all $x \in U$;*
- *for any $x \in U$, the curves:*

$$[0, 1] \ni t \longmapsto H(t, x) \in X, \quad [0, 1] \ni t \longmapsto H(1-t, x) \in X, \tag{18.3}$$

admit liftings with arbitrary initial conditions.

Then π has the unique lifting property for paths.

Proof. Let $\gamma : [a, b] \to X$ be a continuous curve and let $\tilde{x}_0 \in \pi^{-1}\big(\gamma(a)\big)$ be fixed. Assume by contradiction that γ does not admit a lifting starting at $\tilde{x}_0$. Let $\tilde{\gamma} : [a, t_0[\, \to \widetilde{X}$ be the maximal partial lifting of γ starting at $\tilde{x}_0$, where $t_0 \in\,]a, b]$. Set $p = \gamma(t_0)$ and let U, p_0 and H be as in the statement of the lemma. Let $\varepsilon > 0$ be such that $\gamma\big([t_0-\varepsilon, t_0]\big) \subset U$. Let $\tilde{\mu} : [0, 1] \to \widetilde{X}$ be a lifting of the curve $[0, 1] \ni t \mapsto H\big(1-t, \gamma(t_0-\varepsilon)\big) \in X$ such that $\tilde{\mu}(0) = \tilde{\gamma}(t_0-\varepsilon)$. Then $\tilde{p}_0 = \tilde{\mu}(1)$ is a point in $\widetilde{X}$ such that $\pi(\tilde{p}_0) = p_0$. Since for every $x \in U$ the curve $[0, 1] \ni t \mapsto H(t, x) \in X$ admits a lifting starting at $\tilde{p}_0$, Corollary 18.3 gives us a lifting $\widetilde{H}:[0,1]\times U\to\widetilde{X}$ of H such that $\widetilde{H}(0,x)=\tilde{p}_0$, for all $x\in U$. The curves $[0,1]\ni t\mapsto\tilde{\mu}(1-t)\in\widetilde{X}$ and $[0,1]\ni t\mapsto\widetilde{H}\big(t,\gamma(t_0-\varepsilon)\big)\in\widetilde{X}$ are liftings of the same curve in X and they both start at the point $\tilde{p}_0$; therefore they are equal. In particular:

$$\tilde{\mu}(0)=\tilde{\gamma}(t_0-\varepsilon)=\widetilde{H}\big(1,\gamma(t_0-\varepsilon)\big).$$

Therefore $[t_0-\varepsilon,t_0]\ni t\mapsto\widetilde{H}\big(1,\gamma(t)\big)\in\widetilde{X}$ is a lifting of $\gamma|_{[t_0-\varepsilon,t_0]}$ starting at $\tilde{\gamma}(t_0-\varepsilon)$; setting $\tilde{\gamma}(t_0)=\widetilde{H}\big(1,\gamma(t_0)\big)$ we thus obtain a lifting of $\gamma|_{[a,t_0]}$ starting at $\tilde{x}_0$. This contradicts Lemma 18.11. □

18.16. **Definition.** If X is a manifold of class C^k ($1\le k\le\infty$ or $k=\omega$) then[9] a curve $\gamma:[a,b]\to X$ is called an *embedding of class* C^k if the following conditions hold:

- γ extends to a curve of class C^k defined in an open interval containing the interval $[a,b]$;
- $\gamma'(t)\ne 0$ for all $t\in[a,b]$;
- γ is injective.

18.17. **Corollary.** *Assume that the space X is a manifold of class C^k ($1\le k\le\infty$ or $k=\omega$) and that $\widetilde{X}$ is Hausdorff. Assume also that every embedding $\gamma:[a,b]\to X$ of class C^k admits liftings with arbitrary initial conditions. Then π has the unique lifting property for paths. In particular, by Corollary 18.10, π is a covering map.*

Proof. Let $p\in X$ be fixed and let $\varphi:U\to\widetilde{U}\subset\mathbb{R}^n$ be a local chart of class C^k on X with $p\in U$ and $\widetilde{U}$ a convex open subset of $\mathbb{R}^n$. Set $p_0=p$ and define $H:[0,1]\times U\to X$ by setting:

$$H(t,x)=\varphi^{-1}\big((1-t)\varphi(p)+t\varphi(x)\big),$$

for all $t\in[0,1]$, $x\in U$. For any $x\in U$, $x\ne p$, the curves (18.3) are embeddings of class C^k and therefore they admit liftings with arbitrary initial conditions. For $x=p$ the curves (18.3) are constant and therefore they obviously admit liftings with arbitrary initial conditions. The conclusion follows from Lemma 18.15. □

18.18. **Corollary.** *Assume that X is a Riemannian manifold and that $\widetilde{X}$ is Hausdorff. Assume also that every minimizing geodesic $\gamma:[a,b]\to X$ admits liftings with arbitrary initial conditions. Then π has the unique lifting property for paths. In particular, by Corollary 18.10, π is a covering map.*

Proof. Let $p\in X$ be fixed and let $r>0$ be such that the exponential map $\exp_p$ carries the open ball $\mathrm{B}(0;r)$ on T_pX diffeomorphically onto an open subset U of X. Set $p_0=p$ and define $H:[0,1]\times U\to X$ by setting:

$$H(t,x)=\exp_p\Big[t\Big(\big(\exp_p|_{\mathrm{B}(0;r)}\big)^{-1}(x)\Big)\Big],$$

[9]Recall that "class C^ω" means real-analytic.

for all $t \in [0,1]$, $x \in U$. Then for every $x \in U$, the curves (18.3) are minimizing geodesics and therefore they admit liftings with arbitrary initial conditions. The conclusion follows from Lemma 18.15. $\square$

In the next lemma we show that uniqueness of liftings works for covering maps even if the space $\widetilde{X}$ is not Hausdorff (compare with Lemma 18.1).

18.19. **Lemma.** *Assume that π is a covering map. Let Y be a connected topological space and let $\tilde{f}_1 : Y \to \widetilde{X}$, $\tilde{f}_2 : Y \to \widetilde{X}$ be continuous maps with $\pi \circ \tilde{f}_1 = \pi \circ \tilde{f}_2$. If $\tilde{f}_1$ and $\tilde{f}_2$ agree on some point of Y then $\tilde{f}_1 = \tilde{f}_2$.*

Proof. We proceed as in the proof of Lemma 18.1. We consider the set (18.1); since π is locally injective, (18.1) is open. Again, (18.1) is nonempty, by our hypotheses. We complete the proof by showing that (18.1) is closed (without using that $\widetilde{X}$ is Hausdorff). Let $y \in Y$ be a point not in (18.1), i.e., $\tilde{f}_1(y) \neq \tilde{f}_2(y)$. We have $\pi\big(\tilde{f}_1(y)\big) = \pi\big(\tilde{f}_2(y)\big)$; let $U \subset X$ be a fundamental open set containing $\pi\big(\tilde{f}_1(y)\big)$. Then $\pi^{-1}(U) = \bigcup_{i\in I} U_i$, where $(U_i)_{i\in I}$ is a family of disjoint open subsets of $\widetilde{X}$ and π maps U_i homeomorphically onto U, for all $i \in I$. We have $\tilde{f}_1(y) \in U_i$ and $\tilde{f}_2(y) \in U_j$, for some $i, j \in I$. Since $\pi|_{U_i}$ is injective, it must be $i \neq j$. Set $V = \tilde{f}_1^{-1}(U_i) \cap \tilde{f}_2^{-1}(U_j)$. Then V is an open neighborhood of y in Y. Moreover, $\tilde{f}_1(V) \subset U_i$, $\tilde{f}_2(V) \subset U_j$ and $U_i \cap U_j = \emptyset$; therefore V is disjoint from (18.1). This completes the proof. $\square$

18.20. **Lemma.** *If π is a covering map then π has the unique lifting property for paths.*

Proof. Let $\gamma : [a,b] \to X$ be a continuous map and let $\tilde{x}_0 \in \pi^{-1}\big(\gamma(a)\big)$ be fixed. We will show that γ has a lifting $\tilde{\gamma} : [a,b] \to \widetilde{X}$ with $\tilde{\gamma}(a) = \tilde{x}_0$; by Lemma 18.19, such lifting is unique.

Let us start with the case where the image of γ is contained in a fundamental open subset U of X. Write $\pi^{-1}(U) = \bigcup_{i\in I} U_i$, where $(U_i)_{i\in I}$ is a family of disjoint open subsets of $\widetilde{X}$ and π maps U_i homeomorphically onto U for all $i \in I$. Since $\tilde{x}_0 \in \pi^{-1}(U)$, we have $\tilde{x}_0 \in U_i$, for some $i \in I$. Then $\tilde{\gamma} = (\pi|_{U_i})^{-1} \circ \gamma$ is a lifting of γ with $\tilde{\gamma}(a) = \tilde{x}_0$.

Let us now go to the general case. Since the fundamental open subsets of X form an open cover of X, its inverse images by γ form an open cover of the compact metric space $[a,b]$; let $\delta > 0$ be a Lebesgue number for this open cover, i.e., every subset of $[a,b]$ having diameter less than δ is contained in the inverse image by γ of some fundamental open subset of X. Let $P = \{t_0, \ldots, t_r\}$ be a partition of $[a,b]$ with $t_i - t_{i-1} < \delta$, $i = 1, \ldots, r$. Then $\gamma\big([t_{i-1}, t_i]\big)$ is contained in a fundamental open subset of X; by the first part of the proof, the curve $\gamma|_{[t_{i-1},t_i]}$ admits liftings with arbitrary initial conditions, for all $i = 1, \ldots, r$. We construct a lifting $\tilde{\gamma}_i$ of $\gamma|_{[t_{i-1},t_i]}$ by induction on i as follows. Let $\tilde{\gamma}_1$ be a lifting of $\gamma|_{[t_0,t_1]}$ with $\tilde{\gamma}_1(a) = \tilde{x}_0$. Assuming that $\tilde{\gamma}_i$ is constructed for some $i < r$, we consider the lifting $\tilde{\gamma}_{i+1}$ of $\gamma|_{[t_i,t_{i+1}]}$ with $\tilde{\gamma}_{i+1}(t_i) = \tilde{\gamma}_i(t_i)$.

Since the continuous curves $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_r$ satisfy $\tilde{\gamma}_i(t_i) = \tilde{\gamma}_{i+1}(t_i)$, for all $i = 1, \ldots, r-1$, we can define a continuous curve $\tilde{\gamma} : [a,b] \to \widetilde{X}$ by setting $\tilde{\gamma}|_{[t_{i-1},t_i]} = \tilde{\gamma}_i$, for $i = 1, \ldots, r$. Then $\tilde{\gamma}$ is a lifting of γ and $\tilde{\gamma}(a) = \tilde{x}_0$. This concludes the proof. □

18.21. **Corollary.** *Assume that π is a covering map and that $\widetilde{X}$ is locally arc-connected. If U is an arc-connected open subset of X such that every loop in U is contractible in X (in particular, if U is simply-connected) then U is a fundamental open subset of X.*

Proof. Follows from Lemma 18.20 and Corollary 18.9. □

18.22. **Lemma.** *If π is a covering map then the image of π is closed in X.*

Proof. Let $x \in X$ be a point outside the image of π. Let U be a fundamental open subset of X containing x. Then $\pi^{-1}(U) = \bigcup_{i \in I} U_i$, where $(U_i)_{i \in I}$ is a family of disjoint open subsets of $\widetilde{X}$ and π maps U_i homeomorphically onto U for all $i \in I$. We claim that $I = \emptyset$; namely, otherwise there would exist some $i \in I$ and $U = \pi(U_i)$ would be contained in the image of π. Since $I = \emptyset$, it follows that $\pi^{-1}(U) = \emptyset$, i.e., U is disjoint from the image of π. □

18.23. **Corollary.** *If π is a covering map, $\widetilde{X}$ is nonempty and X is connected then π is surjective.*

Proof. The image of π is nonempty (because $\widetilde{X}$ is nonempty), open in X (because π is a local homeomorphism) and closed in X (by Lemma 18.22). □

Recall that a topological space X is said to be *simply-connected* if every loop in X is contractible in X.

18.24. **Lemma.** *Assume that π is a covering map, $\widetilde{X}$ is nonempty and arc-connected and X is connected and simply-connected. Then π is a homeomorphism.*

Proof. By Corollary 18.23, π is surjective and by Lemma 18.20, π has the unique lifting property for paths. It follows from Lemma 18.6 (with $A = \widetilde{X}$) that π is injective. Hence π is a homeomorphism. □

19. More on Covering Maps

In what follows $\widetilde{X}$ and X are topological spaces and $\pi : \widetilde{X} \to X$ is a local homeomorphism, i.e., given $\tilde{x} \in \widetilde{X}$, there exists an open subset A of $\widetilde{X}$ containing $\tilde{x}$ such that $\pi(A)$ is open in X and $\pi|_A : A \to \pi(A)$ is a homeomorphism.

19.1. **Definition.** A *section* of π is a continuous map $s : U \to \widetilde{X}$ defined on an open subset U of X such that $\pi \circ s$ equals the identity map of U.

19.2. **Lemma.** *If $s : U \to \widetilde{X}$, $s' : U' \to \widetilde{X}$ are sections of π such that $s(x) = s'(x)$ for some $x \in U \cap U'$ then there exists an open neighborhood V of x contained in $U \cap U'$ such that $s|_V = s'|_V$.*

Proof. Let A be an open neighborhood of $s(x) = s'(x)$ in $\widetilde{X}$ such that $\pi(A)$ is open and $\pi|_A : A \to \pi(A)$ is a homeomorphism. Set $V = s^{-1}(A) \cap s'^{-1}(A)$. Then V is open in X, $x \in V$ and $V \subset U \cap U'$. Moreover, for $y \in V$ we have $\pi\big(s(y)\big) = \pi\big(s'(y)\big) = y$ and $s(y), s'(y) \in A$; since $\pi|_A$ is injective, we get $s(y) = s'(y)$. □

19.3. **Corollary.** *Assume that $\widetilde{X}$ is Hausdorff. Let $s : U \to \widetilde{X}$, $s' : U \to \widetilde{X}$ be sections of π with U connected. If $s(x) = s'(x)$ for some $x \in U$ then $s = s'$.*

Proof. The set $E = \big\{y \in U : s(y) = s'(y)\big\}$ is nonempty and it is closed in U, since $\widetilde{X}$ is Hausdorff. By Lemma 19.2, E is open in U. Thus $E = U$. □

19.4. **Lemma.** *If $s : U \to \widetilde{X}$ is a section of π then $s(U)$ is open in $\widetilde{X}$ and $s : U \to s(U)$ is a homeomorphism.*

Proof. The map $s : U \to s(U)$ is continuous, bijective and its inverse, which is equal to $\pi|_{s(U)} : s(U) \to U$, is also continuous; thus $s : U \to s(U)$ is a homeomorphism. To complete the proof we show that $s(U)$ is open in $\widetilde{X}$. Given $x \in U$, we will find a neighborhood of $s(x)$ in $\widetilde{X}$ contained in $s(U)$. Let $A \subset \widetilde{X}$ be an open subset such that $s(x) \in A$, $\pi(A)$ is open in X and $\pi|_A : A \to \pi(A)$ is a homeomorphism. Then $s' = (\pi|_A)^{-1} : \pi(A) \to \widetilde{X}$ is a section of π and $s'(x) = s(x)$. By Lemma 19.2, there exists an open subset V of X with $x \in V$, $V \subset U \cap \pi(A)$ and $s|_V = s'|_V$. Since s' is a homeomorphism onto an open subset of $\widetilde{X}$, it follows that $s'(V)$ is open in $\widetilde{X}$; moreover, $s(x) \in s'(V) = s(V) \subset s(U)$. Hence $s'(V)$ is a neighborhood of $s(x)$ contained in $s(U)$. □

19.5. **Lemma.** *Assume that $\widetilde{X}$ is Hausdorff. Let U be a connected open subset of X satisfying the following property:*

(∗) *for every $x \in U$ and every $\tilde{x} \in \widetilde{X}$ with $\pi(\tilde{x}) = x$ there exists a section $s : U \to \widetilde{X}$ of π with $s(x) = \tilde{x}$.*

Then U is a fundamental open subset of X (recall Definition 18.8).

Proof. Let $\mathcal{S}$ be the set of all sections of π defined in U. We claim that:

$$\pi^{-1}(U) = \bigcup_{s \in \mathcal{S}} s(U).$$

Indeed, if $s \in \mathcal{S}$ then obviously $s(U) \subset \pi^{-1}(U)$; moreover, given $\tilde{x} \in \pi^{-1}(U)$ then $x = \pi(\tilde{x}) \in U$ and by property (∗) there exists $s \in \mathcal{S}$ with $s(x) = \tilde{x}$. Thus $\tilde{x} \in s(U)$. This proves the claim. Now observe that, by Lemma 19.4, $s(U)$ is open in $\widetilde{X}$ for all $s \in \mathcal{S}$; moreover, $\pi|_{s(U)} : s(U) \to U$ is a homeomorphism, being the inverse of $s : U \to s(U)$. To complete the proof, we show that the union $\bigcup_{s\in\mathcal{S}} s(U)$ is disjoint. Pick $s, s' \in \mathcal{S}$ with $s(U) \cap s'(U) \neq \emptyset$. Then there exists $x, y \in U$ with $s(x) = s'(y)$. Observe that:

$$x = \pi\big(s(x)\big) = \pi\big(s'(y)\big) = y,$$

and thus $s(x) = s'(x)$. Since U is connected and $\widetilde{X}$ is Hausdorff, Corollary 19.3 implies that $s = s'$. □

19.6. *Remark.* The converse of Lemma 19.5 holds. In fact, if U is a fundamental open subset of X then U has property $(*)$. Namely, write $\pi^{-1}(U) = \bigcup_{i\in I} U_i$, where $(U_i)_{i\in I}$ is a family of pairwise disjoint open subsets of $\widetilde{X}$ such that π maps U_i homeomorphically onto U for all $i \in I$. Given $x \in U$ and $\tilde{x} \in \pi^{-1}(x)$ then $\tilde{x} \in U_i$ for some $i \in I$. Let $s = (\pi|_{U_i})^{-1} : U \to \widetilde{X}$. Then s is a section of π and $s(x) = \tilde{x}$.

19.7. **Corollary.** *Assume that $\widetilde{X}$ is Hausdorff and that X is locally connected. If X can be covered by open sets satisfying condition $(*)$ above then π is a covering map.*

Proof. Given $x \in X$, there exists an open subset U of X containing x and satisfying condition $(*)$. Since X is locally connected, U contains an open connected neighborhood U' of x. Obviously U' also satisfies condition $(*)$. Thus U' is a fundamental open subset of X, by Lemma 19.5. □

19.8. **Lemma.** *If $U \subset X$ is a fundamental open subset then every open subset V of U is also fundamental.*

Proof. Write $\pi^{-1}(U) = \bigcup_{i\in I} U_i$, where $(U_i)_{i\in I}$ is a family of disjoint open subset of $\widetilde{X}$ and π maps U_i homeomorphically onto U, for every $i \in I$. Observe that $\pi^{-1}(V) = \bigcup_{i\in I} \big(\pi^{-1}(V) \cap U_i\big)$; moreover, $\big(\pi^{-1}(V) \cap U_i\big)_{i\in I}$ is a family of disjoint open subsets of $\widetilde{X}$ and π maps $\pi^{-1}(V) \cap U_i$ homeomorphically onto V, for every $i \in I$. □

19.9. **Lemma.** *Let Y be a subset of X. The map:*

$$\pi' = \pi|_{\pi^{-1}(Y)} : \pi^{-1}(Y) \longrightarrow Y$$

is a local homeomorphism; moreover, if $U \subset X$ is a fundamental open subset for π the $U \cap Y$ is a fundamental open subset (of Y) for π'.

Proof. Since π is a local homeomorphism, given $\tilde{x} \in \pi^{-1}(Y)$ we can find an open subset A of $\widetilde{X}$ with $\pi(A)$ open in X and $\pi|_A : A \to \pi(A)$ a homeomorphism. Now $A \cap \pi^{-1}(Y)$ is an open subset of $\pi^{-1}(Y)$ containing $\tilde{x}$ and $\pi\big(A \cap \pi^{-1}(Y)\big) = \pi(A) \cap Y$ is open in Y; moreover, π maps $A \cap \pi^{-1}(Y)$ homeomorphically onto $\pi(A) \cap Y$. Thus π' is a local homeomorphism. Now let us prove that $U \cap Y$ is fundamental for π'. Write $\pi^{-1}(U) = \bigcup_{i\in I} U_i$, where $(U_i)_{i\in I}$ is a family of disjoint open subsets of $\widetilde{X}$ and π maps U_i homeomorphically onto U, for all $i \in I$. We have:

$$\pi'^{-1}(U \cap Y) = \pi^{-1}(U) \cap \pi^{-1}(Y) = \bigcup_{i\in I} \big(U_i \cap \pi^{-1}(Y)\big),$$

and $\big(U_i \cap \pi^{-1}(Y)\big)_{i\in I}$ is a family of disjoint open subsets of $\pi^{-1}(Y)$. Moreover, π' maps $U_i \cap \pi^{-1}(Y)$ homeomorphically onto $U \cap Y$, for all $i \in I$. □

19.10. **Corollary.** *If π is a covering map and Y is a subset of X then $\pi|_{\pi^{-1}(Y)} : \pi^{-1}(Y) \to Y$ is also a covering map.* □

19.11. **Lemma.** *If π is a covering map, X is locally arc-connected and $\widetilde{Y}$ is an arc-connected component of $\widetilde{X}$ then $\pi|_{\widetilde{Y}} : \widetilde{Y} \to X$ is also a covering map.*

Proof. Let U be a fundamental arc-connected open subset of X (relatively to π). We will show that U is also fundamental relatively to $\pi|_{\widetilde{Y}}$. Write $\pi^{-1}(U) = \bigcup_{i\in I} U_i$, where $(U_i)_{i\in I}$ is a family of disjoint open subsets of $\widetilde{X}$ and π maps U_i homeomorphically onto U, for every $i \in I$. Since U_i is homeomorphic to U, we have that U_i is arc-connected for every $i \in I$; since $\widetilde{Y}$ is an arc-connected component of $\widetilde{X}$, we have either $U_i \subset \widetilde{Y}$ or $U_i \cap \widetilde{Y} = \emptyset$, for all $i \in I$. Set:

$$I' = \big\{ i \in I : U_i \subset \widetilde{Y} \big\}.$$

Then $(\pi|_{\widetilde{Y}})^{-1}(U) = \pi^{-1}(U) \cap \widetilde{Y} = \bigcup_{i\in I'} U_i$. This proves that U is fundamental for $\pi|_{\widetilde{Y}}$. Since π is a covering map and X is locally arc-connected, Lemma 19.8 implies that the fundamental arc-connected open subsets of X form a covering of X. This concludes the proof. □

19.12. **Corollary.** *Assume that π is a covering map. Let Y be a connected, locally arc-connected and simply-connected subset of X and let $\widetilde{Y}$ be an arc-connected component of $\pi^{-1}(Y)$. Then $\pi|_{\widetilde{Y}} : \widetilde{Y} \to Y$ is a homeomorphism.*

Proof. By Corollary 19.10, $\pi|_{\pi^{-1}(Y)} : \pi^{-1}(Y) \to Y$ is a covering map. Since Y is locally arc-connected and $\widetilde{Y}$ is an arc-connected component of $\pi^{-1}(Y)$, Lemma 19.11 implies that $\pi|_{\widetilde{Y}} : \widetilde{Y} \to Y$ is also a covering map. The conclusion follows from Lemma 18.24. □

19.13. **Corollary.** *Assume that π is a covering map and that X is simply-connected and locally arc-connected. Assume also that the image of π intersects every connected component of X. Then π admits a* global section*, i.e., a sections $s : X \to \widetilde{X}$ whose domain is X.*

Proof. Write $X = \bigcup_{i\in I} X_i$, where each X_i is a connected component of X. Since X is locally arc-connected (and, in particular, locally connected), each X_i is open in X; thus each X_i is also locally arc-connected. The fact that X is simply-connected implies that each X_i is also simply-connected. Let $\widetilde{X}_i$ be an arc-connected component of $\pi^{-1}(X_i)$; observe that, since the image of π intersects X_i, the set $\pi^{-1}(X_i)$ is nonempty and thus such an arc-connected component does exist. It follows from Corollary 19.12 that π maps $\widetilde{X}_i$ homeomorphically onto X_i. Let $s_i : X_i \to \widetilde{X}_i$ be the inverse of the homeomorphism $\pi|_{\widetilde{X}_i} : \widetilde{X}_i \to X_i$. Then each s_i is a section of π. The desired global section $s : X \to \widetilde{X}$ is obtained by setting $s|_{X_i} = s_i$, for every $i \in I$. □

20. Sheaves and Pre-Sheaves

Let X be a topological space. A *pre-sheaf* on X is a map $\mathfrak{P}$ that assigns to each open subset $U \subset X$ a set $\mathfrak{P}(U)$ and to each pair of open subsets $U, V \subset X$ with $V \subset U$ a map $\mathfrak{P}_{U,V} : \mathfrak{P}(U) \to \mathfrak{P}(V)$ such that the following properties hold:

- for every open subset $U \subset X$ the map $\mathfrak{P}_{U,U}$ is the identity map of the set $\mathfrak{P}(U)$;
- given open sets, $U, V, W \subset X$ with $W \subset V \subset U$ then:

$$\mathfrak{P}_{V,W} \circ \mathfrak{P}_{U,V} = \mathfrak{P}_{U,W}.$$

20.1. *Remark.* A pre-sheaf on X is simply a contravariant functor from the category of open subsets of X to the category of sets and maps. The morphisms in the category of open subsets of X are defined as follows; if $U, V \subset X$ are open then the set of morphisms from V to U has a single element if $V \subset U$ and it is empty otherwise.

A *sheaf* over a topological space X is a pair $(\mathcal{S}, \pi)$, where $\mathcal{S}$ is a topological space and $\pi : \mathcal{S} \to X$ is a local homeomorphism (see the beginning of Section 19).

20.2. **Example.** If $(\mathcal{S}, \pi)$ is a sheaf over the topological space X then the following pre-sheaf $\mathfrak{P}$ is naturally associated to $(\mathcal{S}, \pi)$; for every open subset $U \subset X$ let $\mathfrak{P}(U)$ be the set of sections of π whose domain is U (recall Definition 19.1). Given open subsets $U, V \subset X$ with $V \subset U$ then the map $\mathfrak{P}_{U,V}$ is defined by:

$$\mathfrak{P}_{U,V}(s) = s|_V,$$

for all $s \in \mathfrak{P}(U)$.

Let $\mathfrak{P}$ be a pre-sheaf over a topological space X. Given a point $x \in X$, consider the disjoint union of all sets $\mathfrak{P}(U)$, where U is an open neighborhood of x in X. We define an equivalence relation $\sim$ on such disjoint union as follows; given $f_1 \in \mathfrak{P}(U_1)$, $f_2 \in \mathfrak{P}(U_2)$, where U_1, U_2 are open neighborhoods of x in X then $f_1 \sim f_2$ if and only if there exists an open neighborhood V of x contained in $U_1 \cap U_2$ such that $\mathfrak{P}_{U_1,V}(f_1) = \mathfrak{P}_{U_2,V}(f_2)$. If U is an open neighborhood of x in X and $f \in \mathfrak{P}(U)$ then the equivalence class of f corresponding to the equivalence relation $\sim$ will be denote by $[f]_x$ and will be called the *germ* of f at the point x. We set:

$$\mathcal{S}_x = \big\{[f]_x : f \in \mathfrak{P}(U), \text{ for some open neighborhood } U \text{ of } x \text{ in } X\big\}.$$

20.3. *Remark.* The set $\mathcal{S}_x$ is simply the direct limit of the net $U \mapsto \mathfrak{P}(U)$, where U runs over the set of open neighborhoods of x ordered by reverse inclusion.

Let $\mathcal{S}$ denote the disjoint union of all $\mathcal{S}_x$, with $x \in X$. Let $\pi : \mathcal{S} \to X$ denote the map that carries $\mathcal{S}_x$ to the point x. Our goal now is to define a topology on $\mathcal{S}$. Given an open subset $U \subset X$ and an element $f \in \mathfrak{P}(U)$ we set:

$$\mathcal{V}(f) = \big\{[f]_x : x \in U\big\} \subset \mathcal{S}.$$

Observe that if V is an open subset of U then:

$$\mathcal{V}\big(\mathfrak{P}_{U,V}(f)\big) = \big\{[f]_x : x \in V\big\};$$

namely, we have $\big[\mathfrak{P}_{U,V}(f)\big]_x = [f]_x$, for all $x \in V$.

We claim that the set:

$$\big\{\mathcal{V}(f) : f \in \mathfrak{P}(U),\ U \text{ an open subset of } X\big\} \tag{20.1}$$

is a basis for a topology on $\mathcal{S}$. First, it is obvious that (20.1) is a covering of $\mathcal{S}$. Second, we have to prove the following property; given open subsets $U_1, U_2 \subset X$, $f_1 \in \mathfrak{P}(U_1)$, $f_2 \in \mathfrak{P}(U_2)$ and $\mathfrak{g} \in \mathcal{V}(f_1) \cap \mathcal{V}(f_2)$, there exists an element of (20.1) containing $\mathfrak{g}$ and contained in $\mathcal{V}(f_1) \cap \mathcal{V}(f_2)$. Let us find such element of (20.1). Since $\mathfrak{g} \in \mathcal{V}(f_1) \cap \mathcal{V}(f_2)$ we have $\mathfrak{g} = [f_1]_x = [f_2]_x$, for some $x \in U_1 \cap U_2$. Since $[f_1]_x = [f_2]_x$, there must exist an open neighborhood V of x contained in $U_1 \cap U_2$ such that $\mathfrak{P}_{U_1,V}(f_1) = \mathfrak{P}_{U_2,V}(f_2)$. Now it is easy to see that $\mathcal{V}\big(\mathfrak{P}_{U_1,V}(f_1)\big)$ is an element of (20.1) containing $\mathfrak{g}$ and contained in $\mathcal{V}(f_1) \cap \mathcal{V}(f_2)$.

In what follows we consider the set $\mathcal{S}$ endowed with the topology having (20.1) as a basis. Our goal is to show that $(\mathcal{S}, \pi)$ is a sheaf over X. We start with the following:

20.4. **Lemma.** *Let $U \subset X$ be an open subset. Given $x \in U$ and $f \in \mathfrak{P}(U)$ then the set:*

$$\big\{\mathcal{V}\big(\mathfrak{P}_{U,V}(f)\big) : V \text{ an open neighborhood of } x \text{ contained in } U\big\} \tag{20.2}$$

is a fundamental system of open neighborhoods of $[f]_x$ in $\mathcal{S}$ (i.e., every neighborhood of $[f]_x$ in $\mathcal{S}$ contains an element of (20.2)).

Proof. Let $\mathcal{W}$ be a neighborhood of $[f]_x$ in $\mathcal{S}$; since (20.1) is a basis of open subsets for $\mathcal{S}$, we can find an open subset $U_1 \subset X$ and $f_1 \in \mathfrak{P}(U_1)$ with $[f]_x \in \mathcal{V}(f_1) \subset \mathcal{W}$. Since $[f]_x \in \mathcal{V}(f_1)$, it must be $x \in U_1$ and $[f]_x = [f_1]_x$; thus there exists an open neighborhood V of x contained in $U \cap U_1$ such that $\mathfrak{P}_{U,V}(f) = \mathfrak{P}_{U_1,V}(f_1)$. Then $\mathcal{V}\big(\mathfrak{P}_{U_1,V}(f_1)\big)$ belongs to (20.2) and is contained in $\mathcal{W}$. □

Given an open subset $U \subset X$ and an element $f \in \mathfrak{P}(U)$ we define a map $\hat{f} : U \to \mathcal{S}$ by setting:

$$\hat{f}(x) = [f]_x,$$

for all $x \in U$.

20.5. **Lemma.** *If $U \subset X$ is an open subset and $f \in \mathfrak{P}(U)$ then the map $\hat{f}$ maps U homeomorphically onto $\mathcal{V}(f)$.*

Proof. It is clear hat $\hat{f} : U \to \mathcal{V}(f)$ is a bijection. Moreover, if V is open in U (and hence in X), we have $\hat{f}(V) = \mathcal{V}\big(\mathfrak{P}_{U,V}(f)\big)$; thus $\hat{f}$ is an open mapping. To complete the proof, we show that $\hat{f}$ is continuous. Let $x \in U$ be fixed and let $\mathcal{V}\big(\mathfrak{P}_{U,V}(f)\big)$ be an element of the fundamental system of neighborhoods (20.2) of $\hat{f}(x) = [f]_x$; by V we denote an open neighborhood of x contained in U. Then $\hat{f}(V) = \mathcal{V}\big(\mathfrak{P}_{U,V}(f)\big)$; this proves the continuity of $\hat{f}$ and completes the proof of the lemma. □

20.6. **Corollary.** *The map $\pi : \mathcal{S} \to X$ is a local homeomorphism. Thus $(\mathcal{S}, \pi)$ is a sheaf over X.*

Proof. If $U \subset X$ is an open subset and $f \in \mathfrak{P}(U)$ then π maps the open set $\mathcal{V}(f)$ homeomorphically onto the open subset U of X; namely, the map $\pi|_{\mathcal{V}(f)} : \mathcal{V}(f) \to U$ is the inverse of the map $\hat{f} : U \to \mathcal{V}(f)$. The conclusion follows by observing that the sets $\mathcal{V}(f)$ cover $\mathcal{S}$. □

We call $(\mathcal{S}, \pi)$ the *sheaf of germs* associated to the pre-sheaf $\mathfrak{P}$. Observe that if U is an open subset of X and $f \in \mathfrak{P}(U)$ then $\hat{f}$ is a section of the sheaf of germs defined in U.

20.7. **Definition.** We say that the pre-sheaf $\mathfrak{P}$ has the *localization property* if, given a family $(U_i)_{i\in I}$ of open subsets of X and setting $U = \bigcup_{i\in I} U_i$ then the map:

$$\mathfrak{P}(U) \ni f \longmapsto \big(\mathfrak{P}_{U,U_i}(f)\big)_{i\in I} \in \prod_{i\in I} \mathfrak{P}(U_i) \tag{20.3}$$

is injective and its image consists of the families $(f_i)_{i\in I}$ in $\prod_{i\in I} \mathfrak{P}(U_i)$ such that $\mathfrak{P}_{U_i,U_i\cap U_j}(f_i) = \mathfrak{P}_{U_j,U_i\cap U_j}(f_j)$, for all $i, j \in I$.

20.8. *Remark.* Observe that if $\mathfrak{P}$ has the localization property then the set $\mathfrak{P}(\emptyset)$ has exactly one element. Namely, consider the empty family $(U_i)_{i\in I}$, i.e., I is the empty set. Then $U = \bigcup_{i\in I} U_i$ is the empty set and the image of the map (20.3) has exactly one element (the empty family $(f_i)_{i\in I}$). Thus $\mathfrak{P}(\emptyset)$ has exactly one element as well.

20.9. **Definition.** Given pre-sheafs $\mathfrak{P}$ and $\mathfrak{P}'$ over a topological space X then an *isomorphism* from $\mathfrak{P}$ to $\mathfrak{P}'$ is a map λ that associates to each open subset $U \subset X$ a bijection $\lambda_U : \mathfrak{P}(U) \to \mathfrak{P}'(U)$ such that, given open subsets $U, V \subset X$ with $V \subset U$ then the diagram:

$$\begin{array}{ccc} \mathfrak{P}(U) & \xrightarrow{\lambda_U} & \mathfrak{P}'(U) \\ {\scriptstyle \mathfrak{P}_{U,V}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{P}'_{U,V}} \\ \mathfrak{P}(V) & \xrightarrow[\lambda_V]{} & \mathfrak{P}'(V) \end{array} \tag{20.4}$$

commutes.

20.10. **Lemma.** *If the pre-sheaf $\mathfrak{P}$ has the localization property then, for every open subset $U \subset X$, the map $f \mapsto \hat{f}$ gives a bijection between the set $\mathfrak{P}(U)$ and the set of sections of the sheaf of germs defined in U. More precisely, such bijections give an isomorphism between the pre-sheaf $\mathfrak{P}$ and the pre-sheaf naturally associated to the sheaf of germs $(\mathcal{S}, \pi)$ (recall Example 20.2).*

Proof. We start by observing that, once we prove that the maps $f \mapsto \hat{f}$ are bijections, it will follow easily that they give an isomorphism of pre-sheaves (i.e., diagram (20.4) commutes). Namely, given open subsets $U, V \subset X$ with $V \subset U$ and given $f \in \mathfrak{P}(U)$, the commutativity of diagram (20.4) is equivalent to $\hat{g} = \hat{f}|_V$, where $g = \mathfrak{P}_{U,V}(f)$.

Let $U \subset X$ be an open subset. Let us prove that the map $\mathfrak{P}(U) \ni f \mapsto \hat{f}$ is injective. Let $f_1, f_2 \in \mathfrak{P}(U)$ be fixed and assume that $\widehat{f_1} = \widehat{f_2}$. For every $x \in U$ we have $[f_1]_x = [f_2]_x$ and thus there exists an open neighborhood U_x of x contained in U such that $\mathfrak{P}_{U,U_x}(f_1) = \mathfrak{P}_{U,U_x}(f_2)$. Now $U = \bigcup_{x \in U} U_x$ and thus the localization property implies that $f_1 = f_2$. This proves the injectivity of $f \mapsto \hat{f}$.

Now let $s : U \to \mathcal{S}$ be a section of π and let us find $f \in \mathfrak{P}(U)$ with $s = \hat{f}$. For every $x \in U$, $s(x)$ is an element of $\mathcal{S}_x$; thus there exists an open neighborhood U_x of x and an element $f_x \in \mathfrak{P}(U_x)$ such that $s(x) = [f_x]_x$. Since s and $\widehat{f_x}$ are both sections of the local homeomorphism π and since $s(x) = \widehat{f_x}(x)$, there exists an open neighborhood V_x of x contained in $U_x \cap U$ such that $s|_{V_x} = \widehat{f_x}|_{V_x}$ (recall Lemma 19.2). Set $g_x = \mathfrak{P}_{U_x,V_x}(f_x)$, for all $x \in U$; we claim that there exists $f \in \mathfrak{P}(U)$ with $\mathfrak{P}_{U,V_x}(f) = g_x$, for all $x \in U$. Since $\bigcup_{x \in U} V_x$ is an open cover of U, by the localization property, in order to prove the claim it suffices to show that for every $x, y \in U$ we have:

$$\mathfrak{P}_{V_x, V_x \cap V_y}(g_x) = \mathfrak{P}_{V_y, V_x \cap V_y}(g_y).$$

Let $x, y \in U$ be fixed and set $h_1 = \mathfrak{P}_{V_x, V_x \cap V_y}(g_x)$, $h_2 = \mathfrak{P}_{V_y, V_x \cap V_y}(g_y)$. We have:

$$\widehat{h_1} = \widehat{g_x}|_{V_x \cap V_y} = \widehat{f_x}|_{V_x \cap V_y} = s|_{V_x \cap V_y} = \widehat{f_y}|_{V_x \cap V_y} = \widehat{g_y}|_{V_x \cap V_y} = \widehat{h_2}.$$

By the first part of the proof, we get $h_1 = h_2$. This proves the claim, i.e., there exists $f \in \mathfrak{P}(U)$ with $\mathfrak{P}_{U,V_x}(f) = g_x$, for all $x \in U$. This implies $[f]_x = [g_x]_x = [f_x]_x = s(x)$, for all $x \in U$. Hence $\hat{f} = s$. □

20.11. *Remark.* It is easily seen that the pre-sheaf naturally associated to a sheaf (recall Example 20.2) always satisfy the localization property. Thus the localization property is indeed an essential hypothesis in Lemma 20.10.

20.12. **Definition.** We say that the pre-sheaf $\mathfrak{P}$ has the *uniqueness property* if for every connected open subset $U \subset X$ and every nonempty open subset $V \subset U$ the map $\mathfrak{P}_{U,V}$ is injective.

20.13. **Lemma.** *If the pre-sheaf $\mathfrak{P}$ has the uniqueness property and if X is locally connected and Hausdorff then the space $\mathcal{S}$ is Hausdorff.*

Proof. Let $U_1, U_2 \subset X$ be open sets, $f_1 \in \mathfrak{P}(U_1)$, $f_2 \in \mathfrak{P}(U_2)$, $x \in U_1$, $y \in U_2$ be fixed with $[f_1]_x \neq [f_2]_y$. We have to find disjoint open neighborhoods of $[f_1]_x$ and $[f_2]_y$ in $\mathcal{S}$. If $x \neq y$, we can find disjoint open subsets $V_1, V_2 \subset X$ with $x \in V_1$ and $y \in V_2$. Then $\pi^{-1}(V_1)$ and $\pi^{-1}(V_2)$ are disjoint open neighborhoods of $[f_1]_x$ and $[f_2]_y$, respectively. Assume now that $x = y$. Let U be a connected open neighborhood of x contained in $U_1 \cap U_2$. Then $\mathcal{V}\big(\mathfrak{P}_{U_1,U}(f_1)\big)$ is an open neighborhood of $[f_1]_x$ and $\mathcal{V}\big(\mathfrak{P}_{U_2,U}(f_2)\big)$ is an open neighborhood of $[f_2]_x$. We claim that $\mathcal{V}\big(\mathfrak{P}_{U_1,U}(f_1)\big)$ and $\mathcal{V}\big(\mathfrak{P}_{U_2,U}(f_2)\big)$ are disjoint. Otherwise, there would exist $z \in U$ with $[f_1]_z = [f_2]_z$ and thus there would exist an open neighborhood V of z contained in U such that $\mathfrak{P}_{U_1,V}(f_1) = \mathfrak{P}_{U_2,V}(f_2)$. This implies:

$$(\mathfrak{P}_{U,V} \circ \mathfrak{P}_{U_1,U})(f_1) = (\mathfrak{P}_{U,V} \circ \mathfrak{P}_{U_2,U})(f_2);$$

by the uniqueness property, $\mathfrak{P}_{U,V}$ is injective and so $\mathfrak{P}_{U_1,U}(f_1) = \mathfrak{P}_{U_2,U}(f_2)$. In particular, $[f_1]_x = [f_2]_x$, contradicting our hypothesis. □

20.14. **Definition.** We say that an open subset $U \subset X$ has the *extension property with respect to the pre-sheaf* $\mathfrak{P}$ if for every connected nonempty open subset V of U the map $\mathfrak{P}_{U,V}$ is surjective. We say that the pre-sheaf $\mathfrak{P}$ has the *extension property* if X can be covered by open sets having the extension property with respect to $\mathfrak{P}$.

20.15. **Lemma.** *Assume that X is locally connected. If U is an open subset of X having the extension property with respect to the pre-sheaf $\mathfrak{P}$ then U has the property $(*)$ in the statement of Lemma 19.5 with respect to the local homeomorphism $\pi : \mathcal{S} \to X$.*

Proof. Let $x \in U$ and $\tilde{x} \in \mathcal{S}$ be fixed, with $\pi(\tilde{x}) = x$. We have to find a section $s : U \to \mathcal{S}$ of π with $s(x) = \tilde{x}$. Since $\tilde{x} \in \mathcal{S}_x$, there exists an open neighborhood W of x and $f \in \mathfrak{P}(W)$ with $\tilde{x} = [f]_x$. Let V be a connected open neighborhood of x contained in $U \cap W$. Since U has the extension property with respect to $\mathfrak{P}$, we can find $g \in \mathfrak{P}(U)$ with $\mathfrak{P}_{U,V}(g) = \mathfrak{P}_{W,V}(f)$. Hence $s = \hat{g}$ is a section of π defined in U and $s(x) = [g]_x = [f]_x = \tilde{x}$. □

20.16. **Corollary.** *Assume that X is Hausdorff and locally connected and that the pre-sheaf $\mathfrak{P}$ has the uniqueness property. If U is a connected open subset of X having the extension property with respect to the pre-sheaf $\mathfrak{P}$ then U is a fundamental open subset of X with respect to the map π.*

Proof. By Lemma 20.15, U has the property $(*)$ and by Lemma 20.13 the space $\mathcal{S}$ is Hausdorff. The conclusion follows from Lemma 19.5. □

20.17. **Corollary.** *Assume that X is Hausdorff and locally connected and that the pre-sheaf $\mathfrak{P}$ has the uniqueness property and the extension property. Then the map $\pi : \mathcal{S} \to X$ is a covering map.*

Proof. By Lemma 20.13, $\mathcal{S}$ is Hausdorff. The conclusion follows from Corollary 19.7. □

The following is a converse of Lemma 20.15.

20.18. **Lemma.** *Assume that the pre-sheaf $\mathfrak{P}$ has the localization property and the uniqueness property. If an open subset $U \subset X$ has the property $(*)$ in the statement of Lemma 19.5 with respect to the local homeomorphism $\pi : \mathcal{S} \to X$ then U has the extension property with respect to the pre-sheaf $\mathfrak{P}$.*

Proof. Let V be a connected nonempty open subset of U. Let $f \in \mathfrak{P}(V)$ be fixed. We have to find an element $g \in \mathfrak{P}(U)$ with $\mathfrak{P}_{U,V}(g) = f$. Choose an arbitrary point $x \in V$. The germ $[f]_x$ is an element of $\mathcal{S}$ with $\pi\big([f]_x\big) = x$. Since $x \in U$ and U has the property $(*)$, it follows that there exists a section $s : U \to \mathcal{S}$ of π with $s(x) = [f]_x$. Since $\mathfrak{P}$ has the localization property, Lemma 20.10 gives us an element $g \in \mathfrak{P}(U)$ with $s = \hat{g}$. Then $[g]_x = s(x) = [f]_x$ and therefore there exists an open neighborhood W of x contained in V such that $\mathfrak{P}_{U,W}(g) = \mathfrak{P}_{V,W}(f)$; thus:

$$\mathfrak{P}_{V,W}\big(\mathfrak{P}_{U,V}(g)\big) = \mathfrak{P}_{V,W}(f).$$

Since $\mathfrak{P}$ has the uniqueness property and W is a nonempty open subset of the connected open set V, we have $\mathfrak{P}_{U,V}(g) = f$. This concludes the proof. □

Finally, we prove our main results.

20.19. **Lemma.** *Assume that X is Hausdorff, locally arc-connected and that the pre-sheaf $\mathfrak{P}$ has the localization property, the uniqueness property and the extension property. If U is an arc-connected open subset of X such that every loop in U is contractible in X (in particular, if U is simply-connected) then U has the extension property.*

Proof. By Corollary 20.17, the map $\pi : \mathcal{S} \to X$ is a covering map. Observe that, since X is locally arc-connected and $\pi : \mathcal{S} \to X$ is a local homeomorphism then $\mathcal{S}$ is also locally arc-connected; thus, by Corollary 18.21, U is a fundamental open subset of X. By Remark 19.6, U has property $(*)$ and hence Lemma 20.18 implies that U has the extension property. □

20.20. **Corollary.** *Assume that X is Hausdorff, locally arc-connected, arc-connected, simply-connected and that the pre-sheaf $\mathfrak{P}$ has the localization property, the uniqueness property and the extension property. Then for every connected nonempty open subset $V \subset X$ and every $f \in \mathfrak{P}(V)$ there exists $g \in \mathfrak{P}(X)$ with $\mathfrak{P}_{X,V}(g) = f$.*

Proof. It follows from Lemma 20.19 that X itself is an open subset of X having the extension property. Thus, since V is open, connected and nonempty, it follows that the map $\mathfrak{P}_{X,V} : \mathfrak{P}(X) \to \mathfrak{P}(V)$ is surjective. □

20.21. **Lemma.** *Assume that X is Hausdorff, locally arc-connected and simply-connected and that the pre-sheaf $\mathfrak{P}$ has the localization property, the uniqueness property and the extension property. Assume also that every* *connected component of X contains a nonempty open set U such that $\mathfrak{P}(U)$ is nonempty. Then the set $\mathfrak{P}(X)$ is nonempty.*

Proof. By Corollary 20.17, the map $\pi : \mathcal{S} \to X$ is a covering map. Since every connected component of X contains a nonempty set U such that $\mathfrak{P}(U)$ is nonempty, it follows that the image of π intersects every connected component of X. It follows from Corollary 19.13 that π admits a global section $s : X \to \mathcal{S}$. By Lemma 20.10, there exists $f \in \mathfrak{P}(X)$ with $s = \hat{f}$. Hence $\mathfrak{P}(X)$ is nonempty. □

20.22. **Example.** Let X be a Hausdorff, simply-connected smooth manifold and let θ be a smooth closed 1-form on X. Let us prove that θ is exact. For every open subset $U \subset X$ let $\mathfrak{P}(U)$ be the set of smooth maps $f : U \to \mathbb{R}$ with $\mathrm{d}f = \theta|_U$. If $U, V \subset X$ are open subsets with $V \subset U$, define:

$$\mathfrak{P}_{U,V}(f) = f|_V,$$

for all $f \in \mathfrak{P}(U)$. It is immediate that $\mathfrak{P}$ is a pre-sheaf over X satisfying the localization property. If U is a connected open subset of X and if $f_1, f_2 \in \mathfrak{P}(U)$ are equal at one point of U then $f_1 = f_2$; this implies that $\mathfrak{P}$ satisfies the uniqueness property. Assuming the well-known fact that every smooth closed 1-form on an open ball in Euclidean space is exact, we conclude that for every open subset U of X that is diffeomorphic to an open ball in Euclidean space the set $\mathfrak{P}(U)$ is nonempty; in particular, every connected component of X contains a nonempty open subset U such that $\mathfrak{P}(U)$ is nonempty. Finally, let us prove that $\mathfrak{P}$ has the extension property. To this aim, we prove that if U is an open subset of X that is diffeomorphic to an open ball in Euclidean space then U has the extension property with respect to $\mathfrak{P}$. Namely, let V be a connected nonempty open subset of U and let $f \in \mathfrak{P}(V)$ be fixed. Since U is diffeomorphic to an open ball in Euclidean space, there exists a smooth map $f_1 : U \to \mathbb{R}$ with $\mathrm{d}f_1 = \theta|_U$. Since V is connected, $f_1|_V - f$ is constant and equal to some $c \in \mathbb{R}$. Hence $f_1 - c \in \mathfrak{P}(U)$ and $(f_1 - c)|_V = f$. This concludes the proof of the extension property. Now Lemma 20.21 implies that $\mathfrak{P}(X)$ is nonempty, i.e., there exists a smooth map $f : X \to \mathbb{R}$ with $\mathrm{d}f = \theta$. Hence θ is exact.

21. Two-Coloring of Abelian Groups

In what follows we identify abelian groups with $\mathbb{Z}$-modules.

21.1. **Definition.** Let G be an abelian group. A subset $S \subset G$ is said to be *admissible* if given $x_1, \ldots, x_k \in S$ and integers $n_1, \ldots, n_k \in \mathbb{Z}$ with $\sum_{i=1}^k n_i x_i = 0$ then $\sum_{i=1}^k n_i$ is even.

In the definition above we do not require the elements $x_1, \ldots, x_k \in S$ to be distinct; nevertheless, it is easy to see that $S \subset G$ is admissible if and only if given *distinct* elements $x_1, \ldots, x_k \in S$ and integers $n_1, \ldots, n_k \in \mathbb{Z}$ with $\sum_{i=1}^k n_i x_i = 0$ then $\sum_{i=1}^k n_i$ is even.

The following statements concerning admissible subsets are trivial:

- the empty set is admissible;
- a subset of an admissible subset is admissible;
- a subset $S \subset G$ is admissible if and only if every finite subset of S is admissible;
- if H is a subgroup of G and S is a subset of H then S is admissible in H if and only if S is admissible in G;
- if $S \subset G$ is admissible then $0 \notin S$.

21.2. **Definition.** Let G be an abelian group and $S \subset G$ be a subset. A *two-coloring* of (G, S) is a map $\alpha : G \to \{0, 1\}$ such that, for all $x, y \in G$ with $x - y \in S$ we have $\alpha(x) \neq \alpha(y)$.

In what follows, if S is a subset of an abelian group G we denote by $\langle S \rangle$ the subgroup of G spanned by S, i.e.:

$$\langle S \rangle = \left\{ \sum_{i=1}^{k} n_i x_i : x_1, \ldots, x_k \in S,\ n_1, \ldots, n_k \in \mathbb{Z} \right\}.$$

We set $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$.

21.3. **Lemma.** *Let G be an abelian group and $S \subset G$ be a subset. The following statements are equivalent:*

(a) *S is admissible;*
(b) *there exists a homomorphism $f : \langle S \rangle \to \mathbb{Z}_2$ that carries S to $\bar{1}$;*
(c) *there exists a two-coloring of (G, S).*

Proof.

(a)⇒(b). Set:

$$f\Big(\sum_{i=1}^{k} n_i x_i\Big) = \Big(\sum_{i=1}^{k} n_i\Big) + 2\mathbb{Z} \in \mathbb{Z}_2,$$

for all $x_1, \ldots, x_k \in S$, $n_1, \ldots, n_k \in \mathbb{Z}$. If $\sum_{i=1}^{k} n_i x_i = \sum_{i=1}^{l} m_i y_i$ with $x_1, \ldots, x_k, y_1, \ldots, y_l \in S$ and $n_1, \ldots, n_k, m_1, \ldots, m_l \in \mathbb{Z}$ then:

$$\sum_{i=1}^{k} n_i x_i - \sum_{i=1}^{l} m_i y_i = 0,$$

and, since S is admissible, $\sum_{i=1}^{k} n_i - \sum_{i=1}^{l} m_i$ is even. Thus f is well-defined. It is easy to see that f is a homomorphism that carries S to $\bar{1}$.

(b)⇒(c). Denote by $q : G \to G/\langle S \rangle$ the quotient map and let $\mathfrak{s} : G/\langle S \rangle \to G$ be a right inverse for q, i.e., $\mathfrak{s}$ choses an element for each class on $G/\langle S \rangle$. Observe that for any $x \in G$ we have $x - \mathfrak{s}\big(q(x)\big) \in \langle S \rangle$; set:

$$\alpha(x) = f\Big(x - \mathfrak{s}\big(q(x)\big)\Big),$$

for all $x \in G$. We claim that $\alpha : G \to \mathbb{Z}_2 \cong \{0, 1\}$ is a two-coloring of (G, S). Let $x, y \in G$ be fixed with $x - y \in S$. Then $q(x) = q(y)$; set

$z = \mathfrak{s}\big(q(x)\big) = \mathfrak{s}\big(q(y)\big)$. We have:

$$f(x-y) = f\big((x-z)-(y-z)\big) = f(x-z) - f(y-z) = \alpha(x) - \alpha(y).$$

Since $x - y \in S$, we have $f(x-y) = \bar{1}$ and thus $\alpha(x) \neq \alpha(y)$.

(c)⇒(a). Given $x \in G$ and $y \in S$ we have:

$$\alpha(x) \neq \alpha(x+y) \quad \text{and} \quad \alpha(x) \neq \alpha(x-y);$$

it follows easily by induction on $\sum_{i=1}^k |n_i|$ that $\alpha(x) = \alpha\big(x + \sum_{i=1}^k n_i y_i\big)$ if and only if $\sum_{i=1}^k n_i$ is even, for all $x \in G$, $y_1, \ldots, y_k \in S$, $n_1, \ldots, n_k \in \mathbb{Z}$. In particular, setting $x = 0$, we get that S is admissible. □

21.4. **Lemma.** *Let G be an abelian group and $S \subset G$ be a nonempty subset. Then S is admissible if and only if there exists a subgroup $H \subset G$ and an element $x \in G$, $x \notin H$, such that $2x \in H$ and S is contained in the coset $x + H$.*

Proof. Assume that S is admissible. By Lemma 21.3, there exists a homomorphism $f : \langle S \rangle \to \mathbb{Z}_2$ that carries S to $\bar{1}$. Set $H = \mathrm{Ker}(f)$ and choose $x \in S$. Then $f(x) = \bar{1}$ and $f(2x) = \bar{0}$, i.e., $x \notin H$ but $2x \in H$. Moreover, given $y \in S$ we have $f(y - x) = \bar{0}$ and thus $y - x \in H$; hence $y = x + (y - x) \in x + H$. Assume now that H is a subgroup of G, $x \in G$, $x \notin H$ and $2x \in H$. We show that $x + H$ (and thus any subset of $x + H$) is admissible. Let $y_1, \ldots, y_k \in H$ and $n_1, \ldots, n_k \in \mathbb{Z}$ be fixed with $\sum_{i=1}^k n_i(x + y_i) = 0$. Then:

$$\Big(\sum_{i=1}^k n_i\Big) x = -\sum_{i=1}^k n_i y_i \in H;$$

since $2x \in H$, if $\sum_{i=1}^k n_i$ where odd, it would follow that $x \in H$, contradicting our hypothesis. □

22. Inductive Limits of Locally Convex Spaces

Let $\mathbb{K}$ denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.

23. Non separable metric spaces

23.1. **Lemma.** *Let (M, d) be a metric space. Then M is non separable if and only if there exists an $\varepsilon > 0$ and a uncountable subset A of M such that $d(x, y) > \varepsilon$, for all $x, y \in A$ with $x \neq y$.*

Proof. Clearly the existence of A and ε an in the statement imply that M is non separable, since the open balls of radius $\frac{\varepsilon}{2}$ centered at the points of A constitute an uncountable family of non empty pairwise disjoint open subsets (and every dense subset must intersect all such open subsets and therefore must be uncountable). Conversely, assume that M is not separable. We use transfinite recursion to construct a family $(x_\alpha)_{\alpha \in \aleph_1}$ of points of M indexed in the first uncountable ordinal $\aleph_1$ as follows: given $\alpha \in \aleph_1$, if the points

$x_\beta \in M$ are defined for $\beta < \alpha$ then $\{x_\beta : \beta < \alpha\}$ is a countable subset of M and therefore cannot be dense. We can thus choose a point $x_\alpha \in M$ not in the closure of $\{x_\beta : \beta < \alpha\}$. Once the family $(x_\alpha)_{\alpha \in \aleph_1}$ is constructed, let ε_α denote the (positive) distance between x_α and the set $\{x_\beta : \beta < \alpha\}$, for each $\alpha \in \aleph_1$. Given distinct ordinals $\alpha, \beta \in \aleph_1$, we have:

$$d(x_\alpha, x_\beta) \geq \varepsilon_{\max\{\alpha,\beta\}}.$$

For each $n \geq 1$, set:

$$I_n = \big\{\alpha \in \aleph_1 : \varepsilon_\alpha > \tfrac{1}{n}\big\}.$$

Since $\aleph_1 = \bigcup_{n \geq 1} I_n$, there must exist $n \geq 1$ such that I_n is uncountable. The proof is concluded by setting:

$$A = \big\{x_\alpha : \alpha \in I_n\big\}$$

and $\varepsilon = \frac{1}{n}$. Notice that the map $\alpha \mapsto x_\alpha$ is injective, so that A is uncountable like I_n. Moreover, $d(x_\alpha, x_\beta) > \frac{1}{n}$, for all $\alpha, \beta \in I_n$ with $\alpha \neq \beta$. □

24. Fibered implicit function theorem

24.1. **Theorem.** *Let E, F, M, N be differentiable manifolds, $p : E \to M$, $q : F \to N$ be smooth submersions and $\phi : E \to F$, $f : M \to N$ be smooth maps such that the diagram:*

$$\begin{array}{ccc} E & \xrightarrow{\phi} & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow[f]{} & N \end{array}$$

commutes. Let $e_0 \in E$ be such that the differential $\mathrm{d}\phi(e_0)$ maps $\mathrm{Ker}\big(\mathrm{d}p(e_0)\big)$ isomorphically onto $\mathrm{Ker}\big[\mathrm{d}q\big(\phi(e_0)\big)\big]$. Given a smooth map $g : M \to F$ such that the diagram:

$$\begin{array}{ccc} & & F \\ & {\scriptstyle g}\nearrow & \downarrow{\scriptstyle q} \\ M & \xrightarrow[f]{} & N \end{array}$$

commutes and $\phi(e_0) = g\big(p(e_0)\big)$, then there exists an open neighborhood U of e_0 in E such that the set:

$$\big\{e \in U : \phi(e) = g\big(p(e)\big)\big\}$$

equals the image of a smooth map $s : p(U) \to U$ such that $p \circ s$ is the identity map of $p(U)$.

Proof. Using the local form of submersions, the general case is easily reduced to the case in which $E = M \times E_0$, $F = N \times F_0$, with E_0, F_0 open subsets of Euclidean spaces, and p, q are the first projection maps of such cartesian products. In that case, the map ϕ is of the form:

$$\phi(m, y) = \big(f(m), \psi(m, y)\big), \quad (m, y) \in M \times E_0,$$

with $\psi : M \times E_0 \to F_0$ a smooth map. Set $(m_0, y_0) = e_0$. The assumption that $\mathrm{d}\phi(e_0)$ maps $\mathrm{Ker}\big(\mathrm{d}p(e_0)\big)$ isomorphically onto $\mathrm{Ker}\big[\mathrm{d}q\big(\phi(e_0)\big)\big]$ means that $\frac{\partial\psi}{\partial y}(m_0, y_0) : T_{y_0}E_0 \to T_{\psi(m_0,y_0)}F_0$ is an isomorphism. The map g is of the form:

$$g(m) = \big(f(m), h(m)\big), \quad m \in M,$$

with $h : M \to F_0$ a smooth map. Apply the standard implicit function theorem to the equation:

$$\psi(m, y) - h(m) = 0$$

to find an open neighborhood V of m_0 in M, an open neighborhood W of y_0 in E_0 and a smooth map $\sigma : V \to W$ such that the set:

$$\big\{(m, y) \in V \times W : \psi(m, y) = h(m)\big\}$$

equals the graph of σ. The proof is concluded by setting $U = V \times W$ and by defining $s : V \to U \subset M \times E_0$ by $s(m) = \big(m, \sigma(m)\big)$, $m \in V$. □

24.2. *Remark.* If $p : E \to M$, $q : F \to N$ are smooth submersions and $\phi : E \to F$ is a smooth map then, for $e_0 \in E$, the assumption that $\mathrm{d}\phi(e_0)$ maps $\mathrm{Ker}\big(\mathrm{d}p(e_0)\big)$ isomorphically onto $\mathrm{Ker}\big[\mathrm{d}q\big(\phi(e_0)\big)\big]$ holds if ϕ restricts to a smooth diffeomorphism from $p^{-1}\big(p(e_0)\big)$ to $q^{-1}\big[q\big(\phi(e_0)\big)\big]$.

24.1. **Application to groupoids.** Consider a small category $\mathcal{G}$ with set of objects M and set of morphisms G; denote by $s : G \to M$, $t : G \to M$, respectively, the source and target maps. The composition of morphisms operation is a map $G \star G \to G$ defined in the set:

$$G \star G = \big\{(g, h) \in G \times G : s(g) = t(h)\big\}.$$

If every morphism of $\mathcal{G}$ is an isomorphism we say that $\mathcal{G}$ is a *groupoid.* Denote by $1 : M \to G$ the map $x \mapsto 1_x$ that associates to each object $x \in M$ the identity morphism 1_x of x. Assume that both M and G are endowed with the structure of a differentiable manifold and that the maps t and s are smooth submersions. In this case, the map $s \times t : G \times G \to M \times M$ is a smooth submersion as well and in particular it is transverse to the diagonal of $M \times M$; it follows that $G \star G$ is an embedded submanifold of $G \times G$. If both the multiplication map $G \star G \to G$ and the map $1 : M \to G$ are smooth, we say that $\mathcal{G}$ is a *Lie groupoid.* Given $(g, h) \in G \star G$, the tangent space $T_{(g,h)}(G \star G)$ is equal to the inverse image by $\mathrm{d}s_g \times \mathrm{d}t_h$ of the diagonal of $T_xM \times T_xM$, where $x = s(g) = t(h)$; in other words:

$$T_{(g,h)}(G \star G) = \big\{(v, w) \in T_gG \times T_hG : \mathrm{d}s_g(v) = \mathrm{d}t_h(w)\big\}, \quad (g, h) \in G \star G. \tag{24.1}$$

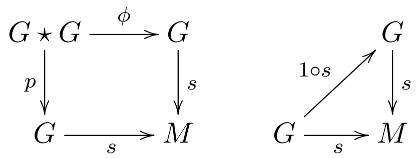
It follows from (24.1) and from the fact that s and t are smooth submersions that the projection maps:

$$G \star G \ni (g, h) \longmapsto g \in G, \quad G \star G \ni (g, h) \longmapsto h \in G$$

are smooth submersions as well.

24.3. **Lemma.** *The inversion map $G \ni g \mapsto g^{-1} \in G$ of a Lie groupoid is smooth.*

Proof. Apply Theorem 24.1 to the following set up:



where ϕ is the multiplication map, p is the projection $(g,h) \mapsto h$ and s is the source map. The validity of the assumption about the map ϕ appearing in the statement of Theorem 24.1 is checked by keeping in mind Remark 24.2: notice that, for a fixed $h \in G$, the restriction of ϕ to $p^{-1}(h) = s^{-1}\big(t(h)\big) \times \{h\}$ is a diffeomorphism onto $s^{-1}\big(s(h)\big)$. Namely, the inverse of such restriction is given by $k \mapsto (kh^{-1}, h)$ and it is therefore smooth. □

25. Tubular neighborhood trick improved

Given topological spaces X, Y, we say that a map $f : X \to Y$ is a *quasi-local homeomorphism* if every $x \in X$ has an open neighborhood U in X such that $f|_U : U \to f(U)$ is a homeomorphism (it is not assumed that $f(U)$ be open in Y, so f might not be a local homeomorphism). Obviously, a quasi-local homeomorphism is continuous and locally injective. Moreover, if X is locally compact and Y is Hausdorff then any continuous locally injective map $f : X \to Y$ is a quasi-local homeomorphism. Namely, if $x \in X$ and U_0 is an open neighborhood of x in which f is injective and if U is an open neighborhood of x contained in a compact subset K of U_0 then $f|_K : K \to f(K)$ is a homeomorphism and thus $f|_U : U \to f(U)$ is a homeomorphism.

25.1. **Lemma** (tubular neighborhood trick improved)**.** *Let X, Y be topological spaces, with Y hereditarily paracompact and Hausdorff. Let $f : X \to Y$ be a quasi-local homeomorphism; if $S \subset X$ is a subset such that $f|_S : S \to f(S)$ is a homeomorphism then there exists an open subset $Z \subset X$ containing S such that $f|_Z$ is injective.*

We need a preparatory lemma.

25.2. **Lemma.** *Let X, Y be topological spaces, $f : X \to Y$ be a continuous map and $S \subset X$ be a subset such that $f|_S : S \to f(S)$ is an open map. Given $x \in S$ and an open neighborhood U of x in X then we can find an open neighborhood U' of x contained in U and an open subset V of Y such that $f(U' \cap S) = V \cap f(S)$ and $f(U') \subset V$.*

Proof. The set $U \cap S$ is open in S and thus $f(U \cap S)$ is open in $f(S)$; let $V \subset Y$ be an open set with $f(U \cap S) = V \cap f(S)$. Then $U' = U \cap f^{-1}(V)$

is an open neighborhood of x contained in U. Obviously $f(U') \subset V$ and $f(U' \cap S) \subset V \cap f(S)$; moreover:

$$V \cap f(S) = f(U \cap S) = f(U' \cap S).$$

The last equality above follows by observing that $U \cap S \subset f^{-1}(V)$ and hence $U \cap S = U' \cap S$. □

Proof of Lemma 25.1. For each $x \in S$, let U'_x be an open neighborhood of x in X such that $f|_{U'_x} : U'_x \to f(U'_x)$ is a homeomorphism. By Lemma 25.2, we can replace U'_x with a smaller open neighborhood of x (so that the map $f|_{U'_x} : U'_x \to f(U'_x)$ remains a homeomorphism) and obtain an open subset V'_x of Y such that $f(U'_x) \subset V'_x$ and:

$$f(U'_x \cap S) = V'_x \cap f(S). \tag{25.1}$$

The set:

$$Y_0 = \bigcup_{x \in S} V'_x$$

is open in Y and it contains $f(S)$. Moreover, Y_0 is Hausdorff and paracompact; therefore, by Lemma 8.10, Y_0 is also T4. Let $Y_0 = \bigcup_{i \in I} V_i$ be a locally finite open refinement of the open cover $Y_0 = \bigcup_{x \in S} V'_x$ of Y_0 (the family $(V_i)_{i \in I}$ is locally finite *in* Y_0). For each $i \in I$, choose $x \in S$ with $V_i \subset V'_x$ and set:

$$U_i = f^{-1}(V_i) \cap U'_x.$$

Then $U_i \subset U'_x$ is open in X, $f|_{U_i} : U_i \to f(U_i)$ is a homeomorphism and from (25.1) we get:

$$f(U_i \cap S) = V_i \cap f(S), \tag{25.2}$$

for all $i \in I$. By Lemma 7.1, there exists a shrinking $Y_0 = \bigcup_{i \in I} W_i$ of the open cover $Y_0 = \bigcup_{i \in I} V_i$ of Y_0, i.e., $\overline{W_i} \subset V_i$ for all $i \in I$ (the closure on W_i will always be taken with respect to the space Y_0). For each $i \in I$ set:

$$Z_i = f^{-1}(W_i) \cap U_i.$$

Then $Z_i \subset U_i$ is open in X, $f|_{Z_i} : Z_i \to f(Z_i)$ is a homeomorphism and from (25.2) we get:

$$f(Z_i \cap S) = W_i \cap f(S), \tag{25.3}$$

for all $i \in I$. We claim that:

$$S \subset \bigcup_{i \in I} Z_i. \tag{25.4}$$

Namely, given $x \in S$, there exists $i \in I$ with $f(x) \in W_i$. Then $f(x) \in W_i \cap f(S)$ and therefore, by (25.3), we can find $y \in Z_i \cap S$ with $f(x) = f(y)$. Since $f|_S$ is injective, we obtain $x = y \in Z_i$, proving the claim.

Now for $x \in S$, we set:

$$I_x = \big\{ i \in I : f(x) \in \overline{W_i} \big\};$$

since the cover $Y_0 = \bigcup_{i\in I} \overline{W_i}$ is locally finite, the set I_x is finite and nonempty. Observe that for $i \in I_x$ we have, using (25.2):

$$f(x) \in \overline{W_i} \cap f(S) \subset V_i \cap f(S) = f(U_i \cap S)$$

and thus the injectivity of $f|_S$ implies $x \in U_i$. We have just shown that:

$$x \in \bigcap_{i\in I_x} U_i, \tag{25.5}$$

for all $x \in S$.

Now, given $x \in S$, $i \in I_x$, the intersection $\bigcap_{j\in I_x} U_j$ is open in U_i and thus $f\big(\bigcap_{j\in I_x} U_j\big)$ is open in $f(U_i)$; we can therefore find an open subset $H_{x,i}$ of Y_0 with:

$$f\Big(\bigcap_{j\in I_x} U_j\Big) = H_{x,i} \cap f(U_i). \tag{25.6}$$

Then:

$$f\Big(\bigcap_{i\in I_x} U_i\Big) \subset \bigcap_{i\in I_x} H_{x,i}. \tag{25.7}$$

Our next goal is to find for each $x \in S$ an open neighborhood G_x of $f(x)$ in Y_0 with the following properties:

(i) for each $i \in I$, G_x intersects W_i if and only if $i \in I_x$;

(ii) $G_x \subset \bigcap_{i\in I_x} H_{x,i}$.

The desired set G_x can be defined by:

$$G_x = \Big(\bigcap_{i\in I_x} H_{x,i}\Big) \cap \Big(Y_0 \setminus \bigcup_{i\in I\setminus I_x} \overline{W_i}\Big).$$

It follows from (25.5) and (25.7) that $f(x) \in G_x$. The fact that G_x is open follows using Lemma 8.2. Property (ii) is obvious. For property (i), observe that $i \in I_x$ implies $f(x) \in G_x \cap \overline{W_i}$ and thus $G_x \cap W_i \neq \emptyset$; moreover, for $i \in I \setminus I_x$ we obviously have $G_x \cap W_i = \emptyset$.

Now set $G = \bigcup_{x\in S} G_x$ and finally:

$$Z = f^{-1}(G) \cap \bigcup_{i\in I} Z_i.$$

Obviously Z is open in X and $S \subset Z$, by (25.4). We complete the proof by showing that $f|_Z$ is injective. Let $x, y \in Z$ be chosen with $f(x) = f(y)$. We can find $i, j \in I$ with $x \in Z_i$ and $y \in Z_j$. Moreover, $f(x) = f(y) \in G_z$ for some $z \in S$. We have $f(x) \in G_z \cap W_i$ and $f(y) \in G_z \cap W_j$, so that $i, j \in I_z$, by property (i). Property (ii) implies $G_z \subset H_{z,i}$. Now $x \in Z_i \subset U_i$, and $f(x) \in H_{z,i} \cap f(U_i)$, so (25.6) implies:

$$f(x) \in f\Big(\bigcap_{k\in I_z} U_k\Big).$$

We can thus find $p \in \bigcap_{k\in I_z} U_k \subset U_i \cap U_j$ with $f(p) = f(x) = f(y)$. Since f is injective in U_i and in U_j, we conclude that $x = p = y$. □

26. Perturbation of operators

The following lemma is well-known.

26.1. **Lemma.** *Let X be a normed space and Y be a nondense subspace of X. Then, for every $\varepsilon > 0$, there exists $x \in X$ with $\|x\| = 1$ and $d(x, Y) > 1 - \varepsilon$.*

Proof. Choose $z \in X$ not in the closure of Y, so that $d(z, Y) > 0$. Let $(y_n)_{n\geq 1}$ be a sequence in Y such that $\lim_{n\to+\infty} \|z - y_n\| = d(z, Y)$. Setting $x_n = \frac{z-y_n}{\|z-y_n\|}$ then:

$$d(x_n, Y) = \frac{1}{\|z - y_n\|}\, d(z - y_n, Y) = \frac{1}{\|z - y_n\|}\, d(z, Y),$$

so that $\lim_{n\to+\infty} d(x_n, Y) = 1$. □

The following immediate corollary is rarely mentioned.

26.2. **Corollary.** *Let X be a normed space and Y be a subspace of X. If:*

$$\sup\big\{d(x, Y) : x \in X,\ \|x\| = 1\big\} < 1$$

then Y is dense in X. □

Given Banach spaces X, Y and a bounded operator $T : X \to Y$, we set:

$$\rho(T) = \inf\big\{\|T(x)\| : x \in X,\ \|x\| = 1\big\}.$$

Clearly, $\rho(T)$ is the largest $c \geq 0$ with $\|T(x)\| \geq c\|x\|$, for all $x \in X$. Moreover, $\rho(T) > 0$ if and only if T is a homeomorphism onto its range, if and only if T is injective with closed range.

26.3. **Lemma.** *Let X, Y be Banach spaces and $T : X \to Y$, $S : X \to Y$ be bounded operators. Then:*

$$|\rho(T) - \rho(S)| \leq \|T - S\|.$$

Proof. Given $x \in X$ with $\|x\| = 1$ we have:

$$\|S(x)\| \geq \|T(x)\| - \|T(x) - S(x)\| \geq \rho(T) - \|T - S\|,$$

yielding $\rho(S) \geq \rho(T) - \|T - S\|$. Thus $\rho(T) - \rho(S) \leq \|T - S\|$. □

We obtain now the following interesting proof of a well-known result.

26.4. **Proposition.** *Let X be a Banach space and $H : X \to X$ be a bounded operator with $\|H\| < 1$. If Id denotes the identity operator of X then $\mathrm{Id} + H$ is an isomorphism of X.*

Proof. Lemma 26.3 yields:

$$|\rho(\mathrm{Id} + H) - \rho(\mathrm{Id})| \leq \|H\| < 1,$$

and, since $\rho(\mathrm{Id}) = 1$, we have $\rho(\mathrm{Id} + H) > 0$. This proves that $\mathrm{Id} + H$ is injective with closed range. Now we use Corollary 26.2 to establish that the range of $\mathrm{Id} + H$ is dense in X. Namely, simply note that, for $x \in X$ with $\|x\| = 1$ we have:

$$d\big(x, (\mathrm{Id} + H)[X]\big) \leq \big\|x - \big(x + H(x)\big)\big\| = \|H(x)\| \leq \|H\| < 1. \qquad \square$$

26.5. **Lemma.** *Let X be a Banach space. If Y is a closed subspace of X and V is a finite-dimensional subspace of X then $Y+V$ is closed in X.*

Proof. Consider the quotient map $q : X \to X/Y$. Since $q[V]$ is a finite-dimensional subspace of the Banach space X/Y, it is closed. Hence:

$$Y + V = q^{-1}\big[q[V]\big]$$

is closed in X. □

26.6. **Lemma.** *Let X be a finite-dimensional normed space. Given $\varepsilon > 0$, there exists a finite subset F of the open unit ball of X such that $d(x, F) < \varepsilon$, for all $x \in X$ with $\|x\| \le 1$.*

Proof. Follows by observing that that the open unit ball of X is totally bounded and dense in the unit closed ball of X. □

26.7. **Corollary.** *Let X be a Banach space and Y be a finite-codimensional closed subspace of X. Then, for any $\varepsilon > 0$, there exists a finite subset F of the open unit ball of X such that $d(x, F + Y) < \varepsilon$, for all $x \in X$ with $\|x\| \le 1$.*

Proof. The lemma yields a finite subset F_1 of the open unit ball of X/Y such that $d(z, F_1) < \varepsilon$, for all $z \in X/Y$ with $\|z\| \le 1$. Let F be a finite subset of the open unit ball of X with $q[F] = F_1$, where $q : X \to X/Y$ denotes the quotient map. The conclusion follows. □

26.8. **Lemma.** *Let X, Y be Banach spaces and $T : X \to Y$ be a bounded operator with $\rho(T) = c > 0$. Assume that the range of T has finite codimension in Y. Then, for any bounded operator $S : X \to Y$ with $\|S - T\| < \frac{c}{2}$, we have that S is injective with closed range and that the range of S has finite codimension in Y.*

Proof. From Lemma 26.3 we obtain:

$$\rho(S) \ge \rho(T) - \|S - T\| > \frac{c}{2} > 0,$$

so that S is injective with closed range. Let us prove that the range of S has finite codimension. Let $\varepsilon > 0$ be fixed (to be specified later). Since the range of T is closed with finite codimension, Corollary 26.7 yields a finite subset F of the open unit ball of Y such that $d(y, F + T[X]) < \varepsilon$, for all $y \in Y$ with $\|y\| \le 1$. Let V be the linear span of F. By Lemma 26.5, $S[X] + V$ is closed in Y. Let us prove that (for an adequate choice of ε), $S[X]+V$ is also dense in Y. For this purpose we use Corollary 26.2. Let $y \in Y$ with $\|y\| = 1$ be fixed and let us estimate $d(y, S[X] + V)$. From $d(y, F + T[X]) < \varepsilon$ we obtain $x \in X$ and $z \in F$ with $\big\|y - \big(z + T(x)\big)\big\| < \varepsilon$. Note that $\|z\| < 1$ and therefore:

$$\|T(x)\| \le \big\|y - \big(z + T(x)\big)\big\| + \|y\| + \|z\| < 2 + \varepsilon.$$

Using $\rho(T) = c$ this yields:

$$\|x\| < \frac{2+\varepsilon}{c}$$

and therefore:

$$\|T(x) - S(x)\| \le \frac{2+\varepsilon}{c}\,\|T - S\|.$$

Moreover:

$$d(y, S[X] + V) \le \big\|y - \big(z + S(x)\big)\big\| \le \big\|y - \big(z + T(x)\big)\big\| + \|T(x) - S(x)\| < \varepsilon + \frac{2+\varepsilon}{c}\,\|T - S\|.$$

Since $\|T - S\| < \frac{c}{2}$, we could have chosen $\varepsilon > 0$ (depending only on S, not on y) with:

$$\varepsilon + \frac{2+\varepsilon}{c}\,\|T - S\| < 1.$$

This concludes the proof. □

26.9. **Corollary.** *Let X, Y be Banach spaces and $T : X \to Y$ be a bounded operator with $\rho(T) = c > 0$. Assume that the range of T has infinite codimension in Y. Then, for any bounded operator $S : X \to Y$ with $\|S-T\| < \frac{c}{3}$, we have that S is injective with closed range and that the range of S has infinite codimension in Y.*

Proof. Set $\rho(S) = c'$. From Lemma 26.3 we obtain:

$$c' \ge \rho(T) - \|T - S\| > \frac{2}{3}\,c > 0,$$

and therefore S is injective with closed range. Now $\|T - S\| < \frac{c}{3} < \frac{c'}{2}$. Assuming by contradiction that the range of S has finite codimension, the lemma yields that the range of T also has finite codimension, contradicting our assumptions. □

27. Ordered sets

Recall that a topological space X is called *Lindelöf* if every open cover of X admits a countable subcover and that X is called *hereditarily Lindelöf* if every subspace of X is Lindelöf. (For example, second countable spaces are hereditarily Lindelöf.) Observe that if X is hereditarily Lindelöf then X satisfies the *countable chain condition*, i.e., every family of nonempty disjoint open subsets of X is countable. Namely, if $(U_i)_{i\in I}$ is a family of nonempty disjoint open subsets of X then $Y = \bigcup_{i\in I} U_i$ is Lindelöf and therefore there exists a countable subset I' of I such that $Y = \bigcup_{i\in I'} U_i$. But then $I = I'$.

27.1. **Lemma.** *Let X be a linearly ordered set endowed with the order topology. Assume that X is hereditarily Lindelöf. Let S be an uncountable subset of X and $\phi : S \to X$ be a map such that $\phi(x) > x$ and $]x, \phi(x)[\ne \emptyset$, for all $x \in S$. Then there exists $x \in S$ such that $S \cap\,]x, \phi(x)[$ is uncountable.*

Proof. Set $Y = \bigcup_{x\in S} \,]x,\phi(x)[$. Note that the intervals $]x,\phi(x)[$, with x varying in $S\setminus Y$, are pairwise disjoint. Namely, if $x_1,x_2\in S\setminus Y$, $x_1\neq x_2$ (so that, for instance, $x_1<x_2$) and $z\in\,]x_1,\phi(x_1)[\,\cap\,]x_2,\phi(x_2)[$ then:

$$x_1<x_2<z<\phi(x_1),$$

yielding $x_2\in Y$. Since X satisfies the countable chain condition, it follows that $S\setminus Y$ is countable and therefore $S\cap Y$ is uncountable. Now, since Y is Lindelöf, there exists a countable subset S' of S such that:

$$Y=\bigcup_{x\in S'}\,]x,\phi(x)[\,.$$

If $S\cap\,]x,\phi(x)[$ were countable for all $x\in S'$ then $S\cap Y$ would also be countable. Hence, there exists $x\in S'$ such that $S\cap]x,\phi(x)[$ is uncountable. □

27.2. **Corollary.** *Under the assumptions of Lemma 27.1, there exists a sequence $(x_n)_{n\ge1}$ in S such that $x_n<x_{n+1}<\phi(x_n)$, for all $n\ge1$.*

Proof. By Lemma 27.1, there exists $x_1\in S$ such that $S_1=S\cap\,]x_1,\phi(x_1)[$ is uncountable. Assuming that we are given $x_n\in S$ such that:

$$S_n=S\cap\,]x_n,\phi(x_n)[$$

is uncountable, apply Lemma 27.1 to $\phi|_{S_n}$ obtaining $x_{n+1}\in S_n$ such that $S_n\cap\,]x_{n+1},\phi(x_{n+1})[$ is uncountable. Then $x_{n+1}\in S$, $x_n<x_{n+1}<\phi(x_n)$ and $S\cap\,]x_{n+1},\phi(x_{n+1})[$ is uncountable. □

27.3. **Lemma.** *Let X be a linearly ordered set endowed with the order topology. Assume that X satisfies the countable chain condition. Let $\varepsilon>0$ and $f:X\to\mathbb{R}$ be a map. Denote by S the set of those $x\in X$ such that there exists $y\in X$ with $y>x$, $]x,y[\,\neq\emptyset$ and $|f(z)-f(x)|\ge\varepsilon$, for all $z\in\,]x,y[$. Then S is countable.*

Proof. For each $n\in\mathbb{Z}$, set $S_n=S\cap f^{-1}\big(\big[n\frac{\varepsilon}{2},(n+1)\frac{\varepsilon}{2}\big]\big)$. For each $x\in S$, choose $\phi(x)\in X$ with $\phi(x)>x$, $]x,\phi(x)[\,\neq\emptyset$ and $|f(z)-f(x)|\ge\varepsilon$, for all $z\in\,]x,\phi(x)[$. We claim that the intervals $]x,\phi(x)[$, with x varying in S_n, are pairwise disjoint. Namely, if $x_1,x_2\in S_n$, $x_1\neq x_2$ (so that, for instance, $x_1<x_2$) and $z\in\,]x_1,\phi(x_1)[\,\cap\,]x_2,\phi(x_2)[$ then:

$$x_1<x_2<z<\phi(x_1)\Longrightarrow x_2\in\,]x_1,\phi(x_1)[\,\Longrightarrow|f(x_2)-f(x_1)|\ge\varepsilon.$$

But $f(x_1),f(x_2)\in\big[n\frac{\varepsilon}{2},(n+1)\frac{\varepsilon}{2}\big]$, so that $|f(x_2)-f(x_1)|\le\frac{\varepsilon}{2}$. Since X satisfies the countable chain condition, it follows that S_n is countable and hence $S=\bigcup_{n\in\mathbb{Z}}S_n$ is countable. □

27.4. **Corollary.** *Let X be a linearly ordered set endowed with the order topology. Assume that X satisfies the countable chain condition and let $f:X\to\mathbb{R}$ be a map. Denote by S the set of those $x\in X$ such that there exists $y\in X$ with $y>x$, $]x,y[\,\neq\emptyset$ and:*

$$\inf\big\{|f(z)-f(x)|:z\in\,]x,y[\,\big\}>0.$$

Then S is countable.

Proof. For each $\varepsilon > 0$, the set S^ε of those $x \in X$ such that there exists $y \in X$ with $y > x$, $]x, y[\neq \emptyset$ and $|f(z) - f(x)| \geq \varepsilon$ for all $z \in]x, y[$ is countable, by Lemma 27.3. Then $S = \bigcup_{k=1}^{\infty} S^{\frac{1}{k}}$ is countable. □

28. Weak-star topology and Fréchet-Urisohn spaces

A topological space $\mathfrak{X}$ is called a *Fréchet–Urysohn space* if given a subset A of $\mathfrak{X}$ and a point $x \in \mathfrak{X}$ in the closure of A, there exists a sequence $(x_n)_{n\geq 1}$ in A converging to x. Obviously, every first-countable (in particular, every metrizable) topological space is Frechét–Urysohn.

28.1. **Lemma.** *Let $\mathfrak{X}$, $\mathfrak{Y}$ be topological spaces and $q : \mathfrak{X} \to \mathfrak{Y}$ be a continuous, surjective closed map. If $\mathfrak{X}$ is Frechét–Urysohn then so is $\mathfrak{Y}$.*

Proof. Let A be a subset of $\mathfrak{Y}$ and $y \in \mathfrak{Y}$ be in the closure of A. Let B denote the closure of $q^{-1}[A]$. Then $q[B]$ is closed and contains A, so that $y = q(x)$ for some $x \in B$. Since $\mathfrak{X}$ is Fréchet–Urisohn, x is the limit of a sequence $(x_n)_{n\geq 1}$ in $q^{-1}[A]$. Hence $\big(q(x_n)\big)_{n\geq 1}$ is a sequence in A converging to y. □

Given a Banach space X, we denote by $B_X = \big\{x \in X : \|x\| \leq 1\big\}$ its closed unit ball.

28.2. **Lemma.** *Let $(X, \|\cdot\|)$ be a Banach space such that the dual ball B_{X^*} is Fréchet–Urysohn in its weak-star topology. Let V be a norm-closed finite-codimensional subspace of X^*. If V is weak-star dense in X^* then V is weak-star sequentially dense in X^*, i.e., every point of X^* is the weak-star limit of a sequence in V.*

Proof. Define a semi-norm p on X by setting:

$$p(x) = \sup\big\{|\alpha(x)| : \alpha \in V,\ \|\alpha\| \leq 1\big\}.$$

Clearly:

$$p(x) \leq \|x\|, \tag{28.1}$$

for all $x \in X$. The fact that V is weak-star dense in X^* implies that V separates the points of X; thus, if $x \in X$ is nonzero, there exists $\alpha \in V$ with $\|\alpha\| \leq 1$ and $\alpha(x) \neq 0$. So p is in fact a norm in X. From (28.1) it follows that every p-bounded linear functional on X is also $\|\cdot\|$-bounded, i.e., the dual space of (X, p) is a vector[10] subspace W of X^*. If $\alpha \in V$ and $\|\alpha\| \leq 1$ then, obviously:

$$|\alpha(x)| \leq p(x),$$

for all $x \in X$, so that $\alpha \in W$. Then $V \subset W$. Since V is norm-closed in X^* and has finite codimension, it follows that W is also norm-closed in X^* (being the inverse image by the quotient map $X^* \to X^*/V$ of W/V). Denote by

[10]Of course, the topology of $(X, p)^*$ defined by the norm associated to p might not coincide with the induced topology from X^*.

$(\widehat{X}, \hat{p})$ the completion of the normed space (X, p) and by $i : (X, p) \to (\widehat{X}, \hat{p})$ the corresponding inclusion map. From (28.1), it follows that the operator $T : (X, \|\cdot\|) \to (\widehat{X}, \hat{p})$ defined by $T(x) = i(x)$ is bounded. Since every bounded linear functional on (X, p) has a (unique) bounded linear extension to $(\widehat{X}, \hat{p})$, it follows that the range of the adjoint T^* is precisely W. The fact that T^* has closed range implies that T has closed range. But, since the range of T is dense, it follows that T is surjective: this means that $\widehat{X} = X$, i.e., that the norm p was complete to begin with. It also means that the norms p and $\|\cdot\|$ are equivalent, so that there exists a constant $c > 0$ such that:

$$p(x) \geq c\|x\|,$$

for all $x \in X$. We claim that the weak-star closure in X^* of the unit ball $B_V = V \cap B_{X^*}$ of V contains the ball cB_{X^*} of radius c. Namely, assume that $\alpha \in X^*$ is not in the weak-star closure of B_V and let us show that $\|\alpha\| > c$. Since B_V is convex, applying the Hahn–Banach separation theorem to the (locally convex topological vector space) X^*, endowed with the weak-star topology, we obtain a weak-star continuous linear functional $\gamma : X^* \to \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) whose real part $\Re\gamma$ separates α from B_V, i.e.:

$$|\gamma(\alpha)| \geq \Re\gamma(\alpha) > \sup_{\beta \in B_V} \Re\gamma(\beta) = \sup_{\beta \in B_V} |\gamma(\beta)|.$$

The weak*-continuity of γ means that γ is given by evaluation at a vector x in X. Then:

$$|\alpha(x)| > \sup_{\beta \in B_V} |\beta(x)| = p(x) \geq c\|x\|,$$

which yields $\|\alpha\| > c$. Since B_{X^*} is Fréchet–Urysohn in its weak-star topology, it follows that every point of cB_{X^*} is the weak-star limit of a sequence in B_V. The conclusion follows. □

28.3. **Lemma.** *Let X be a Banach space such that the dual ball B_{X^*} is Fréchet–Urysohn in its weak-star topology. Let Γ be a finite-dimensional subspace of the bidual X^{**} such that $\Gamma \cap X = \{0\}$ (where X is identified with a subspace of X^{**} in the usual way). Let $(\gamma_1, \ldots, \gamma_m)$ be a basis of Γ. Given scalars $c_1, \ldots, c_m \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), there exists a sequence $(\alpha_n)_{n\geq 1}$ in X^*, weak-star convergent to zero, such that $\gamma_i(\alpha_n) = c_i$, for all $i = 1, \ldots, m$ and all $n \geq 1$.*

Proof. Let $P : X^* \to \mathbb{K}^m$ be the bounded linear operator defined by:

$$P(\alpha) = \big(\gamma_1(\alpha), \ldots, \gamma_m(\alpha)\big), \quad \alpha \in X^*.$$

The fact that $\gamma_1, \ldots, \gamma_m$ are linearly independent implies that P is onto (because the annihilator of the range of P is the kernel of the adjoint of P, which maps the i-th vector of the canonical basis of $(\mathbb{K}^m)^*$ to γ_i). Let $V = \mathrm{Ker}(P) \subset X^*$ be the subspace annihilated by Γ. The annihilator V^{o} of V in X^{**} is equal to the range of the adjoint of P and therefore equal to Γ. Clearly, V is norm-closed in X^* and finite-codimensional. We claim that V is weak-star dense in X^*. Otherwise, we obtain a nonzero weak-star continuous linear functional $\gamma : X^* \to \mathbb{K}$ that annihilates V, i.e., $\gamma \in \Gamma \cap X$, contradicting the assumption that $\Gamma \cap X$ is zero. By Lemma 28.2, V is weak-star sequentially dense in X^*. Pick $\beta \in X^*$ with $P(\beta) = (c_1, \ldots, c_m)$ and let $(\beta_n)_{n\geq 1}$ be a sequence in V weak-star convergent to β. Setting $\alpha_n = \beta - \beta_n$, then $(\alpha_n)_{n\geq 1}$ is weak-star convergent to zero and:

$$P(\alpha_n) = P(\beta) = (c_1, \ldots, c_m),$$

for all n. This concludes the proof. □

28.4. **Lemma.** *Let X be a Banach space such that the dual ball B_{X^*} is Fréchet–Urysohn in its weak-star topology. Let Γ be a finite-dimensional subspace of the bidual X^{**} such that $\Gamma \cap X = \{0\}$ (where X is identified with a subspace of X^{**} in the usual way). Then, for every bounded operator $M : \Gamma \to \ell_\infty \equiv c_0^{**}$ there exists a bounded operator $T : X \to c_0$ such that M is the restriction of T^{**} to Γ.*

Proof. Let $(\gamma_1, \ldots, \gamma_m)$ be a basis of Γ and set $u^i = (u^i_n)_{n\geq 1} = M(\gamma_i) \in \ell_\infty$, for $i = 1, \ldots, m$. By Lemma 28.3, for each $i = 1, \ldots, m$, there exists a sequence $(\alpha^i_n)_{n\geq 1}$ in X^*, weak-star convergent to zero, such that $\gamma_i(\alpha^i_n) = 1$ and $\gamma_j(\alpha^i_n) = 0$, for $j \neq i$ and all $n \geq 1$. Set:

$$\alpha_n = \sum_{i=1}^{m} u^i_n \alpha^i_n, \quad n \geq 1.$$

Since the sequence $(u^i_n)_{n\geq 1}$ is bounded for all i, the sequence $(\alpha_n)_{n\geq 1}$ is weak-star convergent to zero. Therefore, we obtain a bounded operator $T : X \to c_0$ by setting:

$$T(x) = \big(\alpha_n(x)\big)_{n\geq 1}, \quad x \in X.$$

Also, it is easily checked that $T^{**}(\gamma_i) = \big(\gamma_i(\alpha_n)\big)_{n\geq 1}$. But $\gamma_i(\alpha_n) = u^i_n$ and therefore $T^{**}(\gamma_i) = M(\gamma_i)$, i.e., $T^{**}|_\Gamma = M$. □

29. Inverses of perturbations of identity

Consider a category such that, for each pair of objects X, Y, the set of morphisms $\mathrm{Hom}(X, Y)$ is endowed with an abelian group structure (denoted additively) such that the composition of morphisms (denoted multiplicatively) is distributive with respect to addition. Given an object X, we denote its identity morphism by I_X.

29.1. **Lemma.** *Let X, Y be objects and $T : X \to Y$, $L : Y \to X$ be morphisms. If $A : X \to X$ is a left inverse (resp., right inverse) for $\mathrm{I}_X + LT$ then:*

$$B = \mathrm{I}_Y - TAL : Y \longrightarrow Y$$

is a left inverse (resp., right inverse) for $\mathrm{I}_Y + TL$.

Proof. Assuming that A is a left inverse for $\mathrm{I}_X + LT$, we compute:

$$\begin{aligned} B(\mathrm{I}_Y + TL) = (\mathrm{I}_Y - TAL)(\mathrm{I}_Y + TL) &= \mathrm{I}_Y + TL - TA(\mathrm{I}_X + LT)L \\ &= \mathrm{I}_Y + TL - TL = \mathrm{I}_Y. \end{aligned}$$

Similarly, assuming that A is a right inverse for $\mathrm{I}_X + LT$, we compute:

$$\begin{aligned} (\mathrm{I}_Y + TL)B = (\mathrm{I}_Y + TL)(\mathrm{I}_Y - TAL) &= \mathrm{I}_Y + TL - T(\mathrm{I}_X + LT)AL \\ &= \mathrm{I}_Y + TL - TL = \mathrm{I}_Y. \quad \square \end{aligned}$$

30. Associative algebras without divisors of zero

Let K be a field. By an *algebra over* K we mean a vector space A over K endowed with a bilinear binary operation:

$$A \ni (a_1, a_2) \longmapsto a_1 a_2 \in A,$$

called the *multiplication* of the algebra. The algebra is called *associative* if its multiplication is associative. In what follows, *all algebras are assumed to be associative.* An algebra is called *commutative* if its multiplication is commutative. A *unit* for an algebra A is a (necessarily unique) element $\mathbf{1}$ of A that is a bilateral neutral element for the multiplication; we have $\mathbf{1} \neq 0$, unless $A = \{0\}$. If A has a unit $\mathbf{1}$, then an element $a \in A$ is called *invertible* if it admits a (necessarily unique) bilateral multiplicative inverse. We say that an algebra A has *no divisors of zero* if $a_1 a_2 = 0$ implies $a_1 = 0$ or $a_2 = 0$, for all $a_1, a_2 \in A$. We call an algebra A a *division algebra* if A has a unit $\mathbf{1}$ and if every nonzero element of A is invertible. A division algebra has no divisors of zero. A *subalgebra* of A is a vector subspace of A that is closed under multiplication and an *ideal* of A is a vector subspace I of A such that $a_1 a_2 \in I$ for all $a_1, a_2 \in A$, provided that either $a_1 \in I$ or $a_2 \in I$. The subalgebra of A *spanned* by a subset S of A is the smallest subalgebra of A containing S; it is equal to the vector subspace of A spanned by the set:

$$\big\{a_1 a_2 \cdots a_k : a_1, \ldots, a_k \in S,\ k \geq 1\big\}$$

of all (nonvacuous) finite products of elements of S. Notice that the subalgebra spanned by S is commutative if and only if $a_1 a_2 = a_2 a_1$, for all $a_1, a_2 \in S$. If A and B are algebras (over the same field K), then an *algebra homomorphism* from A to B is a linear map $\phi : A \to B$ such that:

$$\phi(a_1 a_2) = \phi(a_1)\phi(a_2), \tag{30.1}$$

for all $a_1, a_2 \in A$. Notice that a linear map $\phi : A \to B$ is an algebra homomorphism if and only if (30.1) holds for all a_1, a_2 belonging to a subset of A that spans A as a vector space. A bijective algebra homomorphism is called an *algebra isomorphism* and its inverse is automatically an algebra homomorphism as well. The kernel of an algebra homomorphism is an ideal in the domain of the homomorphism and the range of an algebra homomorphism is a subalgebra of its counterdomain.

A field K is itself a commutative division algebra over K and also a commutative division algebra over any subfield of K. In particular, the field of real numbers $\mathbb{R}$ is a commutative division algebra over $\mathbb{R}$ and the field $\mathbb{C}$ of complex numbers is a commutative division algebra over $\mathbb{R}$ and over $\mathbb{C}$. Set $\mathbb{H} = \mathbb{R}^4$ and denote the canonical basis of $\mathbb{H}$ by $\{1, i, j, k\}$. The space $\mathbb{H}$ becomes a division algebra over $\mathbb{R}$ by endowing $\mathbb{H}$ with the unique bilinear multiplication such that 1 is the neutral element and:

$$i^2 = -1, \quad j^2 = -1, \quad k^2 = -1,$$
$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

We call $\mathbb{H}$ the *algebra of quaternions*. If V is a vector space over K, then the space $\mathrm{Lin}(V)$ of all linear endomorphisms of V, endowed with the operation of composition, is an algebra over K whose unit is the identity map of V. The space $K[X]$ of polynomials with coefficients in K, endowed with the standard operations, is a commutative algebra over K with unit and no divisors of zero. It is well-known that every ideal of $K[X]$ is *principal*, i.e., it is of the form:

$$\langle p(X)\rangle = \big\{p(X)q(X) : q(X) \in K[X]\big\},$$

for some $p(X) \in K[X]$. If I is a nonzero ideal of $K[X]$, then the unique monic polynomial $p(X)$ in $K[X]$ with $I = \langle p(X)\rangle$ is called the *generator* of the ideal I. The generator of I is the unique monic element of I having the least degree among nonzero elements of I. Let A be an algebra over K with a unit $\mathbf{1}$. The map:

$$K \ni \lambda \longmapsto \lambda\mathbf{1} \in A \tag{30.2}$$

is an algebra homomorphism and its range (which is the subalgebra of A spanned by $\{\mathbf{1}\}$) is called the *subalgebra of scalars* of A. If $\mathbf{1} \neq 0$, then (30.2) is an algebra isomorphism from K to the subalgebra of scalars of A. If K is a subfield of a field K' and if $\phi : K' \to A$ is a homomorphism of algebras (over K) that extends (30.2), then we can extend the operation $K \times A \to A$ of multiplication by scalars of the vector space A to $K' \times A$, turning A into a vector space over K', by defining:

$$\lambda a = \phi(\lambda)a, \quad \lambda \in K',\ a \in A. \tag{30.3}$$

If every element in the range of ϕ commutes with every element of A, then the multiplication of A is bilinear over K', i.e., A becomes an algebra over K' endowed with the multiplication by scalars defined in (30.3).

For $p(X) \in K[X]$ and $a \in A$ we define:

$$p(a) = c_0\mathbf{1} + \sum_{i=1}^{n} c_i a^i \in A,$$

if $p(X) = c_0 + \sum_{i=1}^{n} c_i X^i$, $c_i \in K$, $i = 0, 1, \ldots, n$. The *evaluation map*:

$$K[X] \ni p(X) \longmapsto p(a) \in A \tag{30.4}$$

is then an algebra homomorphism whose range is a commutative subalgebra of A. The range of (30.4) is the subalgebra of A spanned by $\{\mathbf{1}, a\}$. If (30.4) is not injective, then the kernel of (30.4) is a nonzero ideal of $K[X]$ and its generator $m_a(X) \in K[X]$ is called the *minimal polynomial* of a. Notice that, since $K[X]$ is infinite dimensional, the map (30.4) is never injective if A is finite dimensional. If A has no divisors of zero and $A \neq \{0\}$, then clearly the minimal polynomial of an element of A (if it exists) is an irreducible polynomial.

30.1. **Lemma.** *Let A be a finite dimensional algebra with a unit $\mathbf{1}$ over a field K. If $a \in A$ is invertible, then $\mathbf{1}$ belongs to the subalgebra of A spanned by $\{a\}$.*

Proof. Let $m_a(X) = c_0 + \sum_{i=1}^n c_i X^i$ be the minimal polynomial of a. If $c_0 = 0$, then $m_a(X) = Xp(X)$ for some $p(X) \in K[X]$; then $ap(a) = 0$ and the invertibility of a implies $p(a) = 0$, contradicting the fact that $m_a(X)$ is the minimal polynomial of a. So $c_0 \neq 0$. It then follows that:

$$\mathbf{1} = -\frac{1}{c_0} \sum_{i=1}^n c_i a^i,$$

concluding the proof. □

30.2. **Lemma.** *If A is a finite dimensional algebra with no divisors of zero over a field K then A (has a unit and) is a division algebra.*

Proof. Assume $A \neq \{0\}$, otherwise the result is trivial. For each $a \in A$, consider the linear maps:

$$L_a : A \ni x \longmapsto ax \in A, \quad R_a : A \ni x \longmapsto xa \in A.$$

The fact that A has no divisors of zero implies that L_a and R_a are injective, for all nonzero $a \in A$. Thus, since A is finite dimensional, L_a and R_a are linear isomorphisms, for all nonzero $a \in A$. The sets:

$$\big\{L_a : a \in A\big\}, \quad \big\{R_a : a \in A\big\}$$

are subalgebras of $\mathrm{Lin}(A)$ containing an invertible element of $\mathrm{Lin}(A)$. It follows from Lemma 30.1 that they contain the unit of $\mathrm{Lin}(A)$, i.e., there exist $a, b \in A$ such that L_a and R_b are equal to the identity map of A. Then:

$$a = R_b(a) = ab = L_a(b) = b,$$

so that $\mathbf{1} = a$ is a unit for A. For nonzero $x \in A$, the surjectivity of L_x and R_x yield $y, z \in A$ with $xy = \mathbf{1}$ and $zx = \mathbf{1}$. Then:

$$y = \mathbf{1}y = (zx)y = z(xy) = z\mathbf{1} = z,$$

so that x is invertible. □

30.3. **Lemma.** *Let A be a finite dimensional algebra over an algebraically closed field K. If A has no divisors of zero, then A (has a unit and) is equal to its subalgebra of scalars. In particular, if $A \neq \{0\}$, then A is isomorphic (as an algebra) to K.*

Proof. By Lemma 30.2, A has a unit $\mathbf{1}$. Assume $A \neq \{0\}$, otherwise the result is trivial. Given $a \in A$, then $m_a(X) \in K[X]$ is a monic irreducible polynomial. Since K is algebraically closed, we have $m_a(X) = X - \lambda$, for some $\lambda \in K$. Then $a = \lambda\mathbf{1}$. □

30.4. **Lemma.** *Let A be a nonzero finite dimensional commutative algebra over $\mathbb{R}$. If A has no divisors of zero, then A is isomorphic (as an algebra) to either $\mathbb{R}$ or $\mathbb{C}$.*

Proof. By Lemma 30.2, A has a unit $\mathbf{1}$. If A is equal to its subalgebra of scalars, then A is isomorphic to $\mathbb{R}$. Otherwise, let $a \in A$ be an element that does not belong to the subalgebra of scalars of A. Then the minimal polynomial $m_a(X) \in \mathbb{R}[X]$ cannot have degree 1. Since $m_a(X)$ is irreducible, it must be a degree 2 polynomial with no real roots, i.e., it must be of the form $m_a(X) = (X + \lambda)^2 + \mu$, for some $\lambda, \mu \in \mathbb{R}$ with $\mu > 0$. Set:

$$b = \frac{1}{\sqrt{\mu}}\,(a + \lambda\mathbf{1}) \in A.$$

Then $b^2 = -\mathbf{1}$. This implies that $\mathbf{1}$ and b are linearly independent. Let $\phi : \mathbb{C} \to A$ be the linear map such that $\phi(1) = \mathbf{1}$ and $\phi(i) = b$. We have that $\phi : \mathbb{C} \to \phi[\mathbb{C}]$ is an algebra isomorphism. We can extend the operation $\mathbb{R} \times A \to A$ of multiplication by scalars of the vector space A to $\mathbb{C} \times A$, turning A into a complex vector space, by defining:

$$wx = \phi(w)x, \quad w \in \mathbb{C},\ x \in A. \tag{30.5}$$

The fact that A is commutative implies that A becomes an algebra over $\mathbb{C}$ when endowed with (30.5). Obviously, A remains finite dimensional over $\mathbb{C}$ (since it is finitely generated as a real vector space) and the property of having no divisors of zero does not depend on the scalar field. It then follows from Lemma 30.3 that $A = \phi[\mathbb{C}]$. □

30.5. **Lemma.** *Let A be a nonzero finite dimensional algebra over $\mathbb{R}$. If A has no divisors of zero, then A is isomorphic (as an algebra) to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

Proof. By Lemma 30.2, A has a unit $\mathbf{1}$. If A is equal to its subalgebra of scalars, then A is isomorphic to $\mathbb{R}$. Otherwise, we can pick an element of A not in the subalgebra of scalars; such element and the unit $\mathbf{1}$ span a commutative subalgebra of A which, by Lemma 30.4, must be isomorphic to $\mathbb{C}$. Then there exists an element $a \in A$ with $a^2 = -\mathbf{1}$. The subspace of A spanned by $\{\mathbf{1}, a\}$ is a subalgebra C of A isomorphic to $\mathbb{C}$. If $C = A$, we are done. Assume $C \neq A$. Consider the linear map $T : A \to A$ defined by:

$$T(x) = [a, x] \stackrel{\text{def}}{=} ax - xa, \quad x \in A.$$

The kernel of T consists of those elements of A that commute with a. Since C is commutative, we have $C \subset \operatorname{Ker}(T)$. Moreover, if $x \in \operatorname{Ker}(T)$, then the subalgebra of A spanned by $\{\mathbf{1}, a, x\}$ is commutative and hence isomorphic to $\mathbb{C}$, by Lemma 30.4. This shows that $x \in C$ and therefore $\operatorname{Ker}(T) = C$.

A straightforward computation using $a^2 = -\mathbf{1}$ shows that every element in the range of T anticommutes with a, i.e.,

$$ay + ya = 0,$$

for every $y \in \mathrm{Im}(T)$. Since only zero can both commute and anticommute with a, we have $\mathrm{Ker}(T) \cap \mathrm{Im}(T) = \{0\}$. By a dimension argument, we obtain:

$$A = \mathrm{Ker}(T) \oplus \mathrm{Im}(T) = C \oplus \mathrm{Im}(T).$$

Since $C \neq A$, there exists a nonzero element $y \in \mathrm{Im}(T)$. Using Lemma 30.4 we obtain that the subalgebra of A spanned by $\{\mathbf{1}, y\}$ is isomorphic to $\mathbb{C}$, and therefore there exist $\alpha, \beta \in \mathbb{R}$ such that $(\alpha\mathbf{1} + \beta y)^2 = -\mathbf{1}$. Then:

$$\alpha^2\mathbf{1} + 2\alpha\beta y + \beta^2 y^2 = -\mathbf{1};$$

since y anticommutes with a, it follows that y^2 commutes with a and the equality above implies then that $2\alpha\beta y$ commutes with a. But $2\alpha\beta y$ also anticommutes with a, so $\alpha\beta = 0$. Since $\beta = 0$ implies $\alpha^2 = -1$, the only possibility is $\alpha = 0$, i.e., $\beta^2 y^2 = -\mathbf{1}$. Set $b = \beta y$ and $c = ab$. We have that both b and c anticommute with a and that $b^2 = -\mathbf{1}$ and $c^2 = -\mathbf{1}$. Moreover, b anticommutes with c, $bc = a$ and $ca = b$. Hence the unique linear map $\phi : \mathbb{H} \to A$ such that:

$$\phi(1) = \mathbf{1}, \quad \phi(i) = a, \quad \phi(j) = b, \quad \phi(k) = c,$$

is an algebra homomorphism. The kernel of ϕ is an ideal of $\mathbb{H}$; being a division algebra, $\mathbb{H}$ has only the trivial ideals $\{0\}$ and $\mathbb{H}$, so ϕ must be injective and $\phi[\mathbb{H}]$ is a subalgebra of A isomorphic to $\mathbb{H}$. To conclude the proof, we have to show that $\phi[\mathbb{H}] = A$. Since $C \subset \phi[\mathbb{H}]$, it suffices to check that $\mathrm{Im}(T) \subset \phi[\mathbb{H}]$. Let $z \in \mathrm{Im}(T)$. Then z anticommutes with a, so bz commutes with a, and thus $bz = \gamma\mathbf{1} + \delta a$, for some $\gamma, \delta \in \mathbb{R}$. Using $b^2 = -\mathbf{1}$, we obtain:

$$z = -\gamma b + \delta c,$$

proving that $z \in \phi[\mathbb{H}]$. □

31. Disjoint refinement of a family of sets

In what follows, $|X|$ denotes the cardinality of a set X.

31.1. **Lemma.** *Let κ be a cardinal and let $(A_\alpha)_{\alpha\in\kappa}$ be a family of sets with $|A_\alpha| \geq \kappa$, for all $\alpha \in \kappa$. Then there exists an injective function f with domain κ such that $f(\alpha) \in A_\alpha$, for all $\alpha \in \kappa$.*

Proof. Define f by transfinite recursion choosing $f(\alpha)$ in $A_\alpha \setminus \big\{f(\beta) : \beta \in \alpha\big\}$, for all $\alpha \in \kappa$. □

31.2. **Lemma.** *Let κ be an infinite cardinal and let $(A_\alpha)_{\alpha\in\kappa}$ be a family of sets with $|A_\alpha| \geq \kappa$, for all $\alpha \in \kappa$. Then there exists an injective function f with domain $\kappa \times \kappa$ such that $f(\alpha, \beta) \in A_\alpha$, for all $\alpha, \beta \in \kappa$.*

Proof. Let us define by transfinite recursion a family $(f_\alpha)_{\alpha\in\kappa}$ such that:

(a) f_α is an injective function with domain $\alpha \times \alpha$, for all $\alpha \in \kappa$;
(b) $f_\alpha(\beta, \gamma) \in A_\beta$, for all $\beta, \gamma \in \alpha$ and all $\alpha \in \kappa$;
(c) f_α extends f_β, for $\beta \leq \alpha \in \kappa$.

Let $f_0 = \emptyset$ and, for $\alpha \in \kappa$ a limit ordinal, set $f_\alpha = \bigcup_{\beta \in \alpha} f_\beta$. Now, given $\alpha \in \kappa$, let us define $f_{\alpha+1}$ in terms of f_α. Set $B_\beta = A_\beta \setminus \mathrm{Im}(f_\alpha)$, for all $\beta \in \kappa$, so that $|B_\beta| \geq \kappa$. By Lemma 31.1, there exists an injective map g with domain κ and $g(\beta) \in B_\beta$, for all $\beta \in \kappa$. Let h be an injective map with domain α and with image contained in

$$A_\alpha \setminus \big(\mathrm{Im}(f_\alpha) \cup \{g(\beta) : \beta \leq \alpha\}\big)$$

and let $f_{\alpha+1}$ be the extension of f_α such that $f_{\alpha+1}(\alpha, \beta) = h(\beta)$, for all $\beta \in \alpha$, and such that $f_{\alpha+1}(\beta, \alpha) = g(\beta)$, for all $\beta \leq \alpha$. It is easily seen that the family $(f_\alpha)_{\alpha \in \kappa}$ satisfies conditions (a), (b) and (c) above. To conclude the proof, set $f = \bigcup_{\alpha \in \kappa} f_\alpha$. □

31.3. **Corollary** (disjoint refinement)**.** *Let κ be an infinite cardinal and let $(A_\alpha)_{\alpha \in \kappa}$ be a family of sets with $|A_\alpha| \geq \kappa$, for all $\alpha \in \kappa$. Then there exists a family $(B_\alpha)_{\alpha \in \kappa}$ of pairwise disjoint sets such that $B_\alpha \subset A_\alpha$ and $|B_\alpha| = \kappa$, for all $\alpha \in \kappa$.*

Proof. Take f as in Lemma 31.2 and set $B_\alpha = \{f(\alpha, \beta) : \beta \in \kappa\}$, for all $\alpha \in \kappa$. □

31.4. **Corollary.** *Let κ be an infinite cardinal and let $(A_\alpha)_{\alpha \in \kappa}$ be a family of sets with $|A_\alpha| \geq \kappa$, for all $\alpha \in \kappa$. Then there exists a family $(B_\alpha)_{\alpha \in \kappa}$ of pairwise disjoint sets such that:*

- $B_\alpha \subset \bigcup_{\beta \in \kappa} A_\beta$, *for all* $\alpha \in \kappa$;
- $|B_\alpha| = \kappa$, *for all* $\alpha \in \kappa$;
- $B_\alpha \cap A_\beta \neq \emptyset$, *for all* $\alpha, \beta \in \kappa$.

Proof. Take f as in Lemma 31.2 and set $B_\alpha = \{f(\beta, \alpha) : \beta \in \kappa\}$, for all $\alpha \in \kappa$. □

31.5. **Corollary.** *Denote by $\mathfrak{c}$ the cardinal of the continuum. There exists a family $(B_\alpha)_{\alpha \in \mathfrak{c}}$ of pairwise disjoint subsets of $[0, 1]$ such that $|B_\alpha| = \mathfrak{c}$ and the outer Lebesgue measure of B_α is equal to 1, for all $\alpha \in \mathfrak{c}$.*

Proof. The collection of all closed subsets of $[0, 1]$ has cardinality $\mathfrak{c}$ and therefore there exists a family $(F_\alpha)_{\alpha \in \mathfrak{c}}$ such that $\{F_\alpha : \alpha \in \mathfrak{c}\}$ is the collection of all closed subsets of $[0, 1]$ with positive Lebesgue measure. We have that $|F_\alpha| = \mathfrak{c}$, for all $\alpha \in \mathfrak{c}$, since every uncountable closed subset of $\mathbb{R}$ has cardinality $\mathfrak{c}$. Applying Corollary 31.4 to the family $(F_\alpha)_{\alpha \in \mathfrak{c}}$, we obtain a family $(B_\alpha)_{\alpha \in \mathfrak{c}}$ of pairwise disjoint subsets of $[0, 1]$ such that:

- $|B_\alpha| = \mathfrak{c}$, for all $\alpha \in \mathfrak{c}$;
- for all $\alpha \in \mathfrak{c}$, B_α intersects every closed subset of $[0, 1]$ with positive Lebesgue measure.

To conclude the proof, let us show that B_α has outer Lebesgue measure equal to 1, for all $\alpha \in \mathfrak{c}$. If the outer Lebesgue measure of B_α were less than 1, there would exist a Lebesgue measurable subset M of $\mathbb{R}$ containing B_α with measure less than 1. Then $[0,1] \setminus M$ would have positive Lebesgue measure and hence would contain a closed subset F with positive Lebesgue measure. But this implies $B_\alpha \cap F = \emptyset$ and yields a contradiction. □

32. Generalized Cayley–Hamilton theorem

In this section, rings are assumed to have a unity element and homomorphisms of rings are assumed to send the unity of the domain to the unity of the counterdomain. The zeroth power r^0 of an element r of a ring is defined as being equal to the unity of the ring. If R is a commutative ring, we denote by $M_n(R)$ the ring of $n \times n$ matrices with entries in R and by $R[X]$ the ring of polynomials in the indeterminate X with coefficients in R. We regard R as a subring of $R[X]$ and $M_n(R)$ as a subring of $M_n\big(R[X]\big)$. The ring $M_n(R)$ carries the structure of an R-module and it is an associative R-algebra. Since the ring $R[X]$ is commutative as well, we have that the ring $M_n\big(R[X]\big)$ carries the structure of an $R[X]$-module and it is an associative $R[X]$-algebra. The unity of the ring $M_n(R)$ is denoted by I. For $p(X) \in R[X]$ and $A \in M_n\big(R[X]\big)$, we write

$$p(A) = \sum_{k=0}^{m} r_k A^k,$$

where $p(X) = \sum_{k=0}^{m} r_k X^k$ and $r_0, r_1, \ldots, r_m \in R$.

A family $(g_i)_{i\in I}$ in an abelian group is said to be *almost null* if the set $\big\{i \in I : g_i \neq 0\big\}$ is finite. For an almost null family $(g_i)_{i\in I}$, the sum $\sum_{i\in I} g_i$ is defined in the obvious way.

32.1. **Lemma.** *If R is a commutative ring, then for every $A \in M_n\big(R[X]\big)$, there exists a unique almost null sequence $(A_k)_{k\geq 0}$ in $M_n(R)$ such that:*

$$A = \sum_{k=0}^{\infty} X^k A_k. \tag{32.1}$$

Proof. Equality (32.1) is equivalent to the statement that the (i,j)-entry of A_k equals the coefficient of X^k in the (i,j)-entry of A, for all $i,j = 1,\ldots,n$ and all $k \geq 0$. The conclusion follows by observing that the sequence $(A_k)_{k\geq 0}$ in $M_n(R)$ defined by the latter requirement is almost null. □

32.2. **Lemma.** *Let R and S be rings and $h : R \to S$ be a homomorphism of the underlying additive abelian groups. If $\mathcal{A} \subset R$ spans R as an additive abelian group and if*

$$h(ab) = h(a)h(b), \tag{32.2}$$

for all $a, b \in \mathcal{A}$, then (32.2) *holds for all $a, b \in R$.*

Proof. If $b \in \mathcal{A}$, then the set

$$\big\{a \in R : h(ab) = h(a)h(b)\big\}$$

is a subgroup of R containing $\mathcal{A}$ and thus it is equal to R. It follows that, for all $a \in R$, the set

$$\big\{b \in R : h(ab) = h(a)h(b)\big\} \tag{32.3}$$

is a subgroup of R containing $\mathcal{A}$. Hence (32.3) is equal to R for all $a \in R$ and the conclusion follows. □

32.3. **Lemma.** *Let R be a commutative ring, S be a ring and $h : M_n(R) \to S$ be a ring homomorphism. If $s \in S$ commutes with every element in the image of h, then there exists a unique extension $\tilde{h} : M_n\big(R[X]\big) \to S$ of h such that $\tilde{h}$ is a ring homomorphism and $\tilde{h}(X\mathrm{I}) = s$.*

Proof. By Lemma 32.1, every $A \in M_n\big(R[X]\big)$ can be written uniquely as

$$A = \sum_{k=0}^{\infty} X^k A_k = \sum_{k=0}^{\infty} (X\mathrm{I})^k A_k,$$

with $(A_k)_{k\ge 0}$ an almost null sequence in $M_n(R)$. It follows that $\tilde{h}(A)$ is necessarily given by:

$$\tilde{h}(A) = \sum_{k=0}^{\infty} s^k h(A_k). \tag{32.4}$$

Defining $\tilde{h}$ by (32.4), it is readily seen that $\tilde{h}$ is a homomorphism of the underlying additive abelian groups and it follows from Lemma 32.2 with

$$\mathcal{A} = \big\{X^k B : k \ge 0,\ B \in M_n(R)\big\}$$

that $\tilde{h}$ is a homomorphism of rings. □

32.4. **Corollary.** *Let R be a commutative ring and $\mathcal{M}$ be a left $M_n(R)$-module. If $f : \mathcal{M} \to \mathcal{M}$ is $M_n(R)$-linear, then the operation of multiplication by elements of $M_n(R)$ of $\mathcal{M}$ extends in a unique way to an operation of multiplication by elements of $M_n\big(R[X]\big)$ in such a way that $\mathcal{M}$ becomes a left $M_n\big(R[X]\big)$-module and $(X\mathrm{I})m = f(m)$, for all $m \in \mathcal{M}$.*

Proof. The left $M_n(R)$-module structure of $\mathcal{M}$ is the same thing as a structure of abelian group in $\mathcal{M}$ together with a homomorphism of rings

$$h : M_n(R) \longrightarrow \mathrm{End}(\mathcal{M}),$$

where $\mathrm{End}(\mathcal{M})$ is the ring of homomorphisms of the abelian group $\mathcal{M}$ to itself. The assumption that f is $M_n(R)$-linear means that $f \in \mathrm{End}(\mathcal{M})$ commutes with every element in the image of h. By Lemma 32.3, h extends to a unique homomorphism of rings

$$\tilde{h} : M_n\big(R[X]\big) \longrightarrow \mathrm{End}(\mathcal{M})$$

with $\tilde{h}(X\mathrm{I}) = f$. □

32.5. **Corollary.** *Let R be a commutative ring and consider the ring $M_n(R)$ as a left $M_n(R)$-module in the canonical way. Given $U \in M_n(R)$, then the operation of multiplication by elements of $M_n(R)$ extends in a unique way to an operation $\star$ of multiplication by elements of $M_n\big(R[X]\big)$ in such a way that $M_n(R)$ becomes a left $M_n\big(R[X]\big)$-module and $(X\mathrm{I}) \star A = AU$, for all $A \in M_n(R)$.*

Proof. Note that $f : M_n(R) \ni A \mapsto AU \in M_n(R)$ is (left) $M_n(R)$-linear and apply Corollary 32.4. □

32.6. **Lemma.** *Let R be a commutative ring and $U \in M_n(R)$. If $M_n(R)$ is regarded as a left $M_n\big(R[X]\big)$-module as in Corollary 32.5, then*

$$\big(p(X)\mathrm{I}\big) \star A = Ap(U),$$

for every $p(X) \in R[X]$ and every $A \in M_n(R)$.

Proof. If $p(X) = \sum_{k=0}^{m} r_k X^k$, with $r_k \in R$, $k = 0, 1, \ldots, m$, then:

$$\big(p(X)\mathrm{I}\big) \star A = \sum_{k=0}^{m} (r_k\mathrm{I})(X\mathrm{I})^k \star A = \sum_{k=0}^{m} r_k A U^k = A \sum_{k=0}^{m} r_k U^k = Ap(U). \qquad \square$$

32.7. **Lemma** (generalized Cayley–Hamilton)**.** *Let R be a commutative ring, $U \in M_n(R)$ and let $A_0, A_1, \ldots, A_m \in M_n(R)$ satisfy*

$$\sum_{k=0}^{m} A_k U^k = 0.$$

If we set $A = \sum_{k=0}^{m} X^k A_k \in M_n\big(R[X]\big)$ and $p(X) = \det(A) \in R[X]$, then $p(U) = 0$.

Proof. Let $M_n(R)$ be endowed with the struture of left $M_n\big(R[X]\big)$-module defined in Corollary 32.5. We have:

$$A \star \mathrm{I} = \sum_{k=0}^{m} (X\mathrm{I})^k \star (A_k \star \mathrm{I}) = \sum_{k=0}^{m} A_k U^k = 0.$$

Let $B \in M_n\big(R[X]\big)$ be the classical adjoint of A, so that $BA = p(X)\mathrm{I}$. Using Lemma 32.6, we compute:

$$0 = B \star (A \star \mathrm{I}) = (BA) \star \mathrm{I} = \big(p(X)\mathrm{I}\big) \star \mathrm{I} = p(U). \qquad \square$$

32.8. **Corollary** (Cayley–Hamilton)**.** *Let R be a commutative ring. Given $U \in M_n(R)$, if $p(X) = \det(X\mathrm{I} - U)$ is the characteristic polynomial of U, then $p(U) = 0$.*

Proof. Use Lemma 32.7 with $A_0 = -U$, $A_1 = \mathrm{I}$ and $m = 1$. □

33. Nice criteria for a map to be a homeomorphism

33.1. **Lemma.** *Let K and Y be topological spaces, X be a dense subset of K and $f : K \to Y$ be a continuous map. Consider the following conditions:*

(a) *$f|_X : X \to f[X]$ is a homeomorphism;*
(b) *for all $p \in K$ and all $x \in X$, if $f(p) = f(x)$, then $p = x$.*

If K is Hausdorff, then condition (a) implies condition (b). If Y is Hausdorff and K is compact, then condition (b) implies condition (a).

Proof. Assume that K is Hausdorff and that condition (a) holds. Pick $p \in K$ and $x \in X$ with $p \neq x$ and let us show that $f(p) \neq f(x)$. Let U and V be disjoint open subsets of K with $p \in U$ and $x \in V$. Since X is dense in K and U is open in K, it follow that $X \cap U$ is dense in U; in particular, p belongs to the closure of $X \cap U$. Thus, $f(p)$ belongs to the closure of $f[X \cap U]$. From (a), we obtain that $f[X \cap U]$ and $f[X \cap V]$ are disjoint open subsets in $f[X]$ and, since $f(x) \in f[X \cap V]$, we conclude that $f(x)$ is not in the closure of $f[X \cap U]$ in $f[X]$. Hence $f(x) \neq f(p)$.

Now assume that K is compact, Y is Hausdorff and that condition (b) holds. It follows immediately from (b) that $f|_X$ is injective and therefore $f|_X : X \to f[X]$ is a continuous bijection. Let us show that $f|_X : X \to f[X]$ is a closed map. Let F be a closed subset of K. It follows easily from (b) that

$$f[F \cap X] = f[F] \cap f[X].$$

Since $f[F]$ is compact and Y is Hausdorff, we obtain that $f[F]$ is closed in Y and hence that $f[F] \cap f[X]$ is closed in $f[X]$. □

34. Conditions in which FIP implies nonempty intersection

A nonempty collection of sets has the *finite intersection property* (FIP) if every nonempty finite subcollection has a nonempty intersection. It is a basic fact that a topological space X is compact if and only if every nonempty collection of closed subsets of X with FIP has nonempty intersection. It follows that for an arbitrary topological space X, if a collection of closed subsets of X has a compact member and has FIP, then the intersection of the collection is nonempty. We will now generalize this result.

34.1. **Lemma.** *Let $(X_i)_{i \in I}$ be a family of topological spaces and let $\mathcal{F}$ be a nonempty collection of closed subsets of the product space $X = \prod_{i \in I} X_i$ with FIP. Assume that for each $i \in I$, there exists $F \in \mathcal{F}$ such that $\pi_i[F]$ is contained in a compact subset of X_i, where $\pi_i : X \to X_i$ denotes the i-th projection. Then the collection $\mathcal{F}$ has a nonempty intersection.*

Proof. Since the collection $\mathcal{F}$ has FIP, it is contained in a ultrafilter $\mathcal{U}$ of subsets of X. Let $i \in I$ be given and consider the ultrafilter $(\pi_i)_* \mathcal{U}$ of subsets of X_i having

$$\big\{\pi_i[F] : F \in \mathcal{U}\big\}$$

as a filter basis. Our assumptions imply that $(\pi_i)_*\mathcal{U}$ has a compact member K_i and thus $(\pi_i)_*\mathcal{U} \cap \wp(K_i)$ is an ultrafilter of subsets of the compact space K_i. Then $(\pi_i)_*\mathcal{U} \cap \wp(K_i)$ converges to a point $x_i \in K_i$ and this implies that also $(\pi_i)_*\mathcal{U}$ converges to x_i. Hence $\mathcal{U}$ converges to the point $x = (x_i)_{i\in I}$ and x belongs to the intersection of the family $\mathcal{F}$. □

34.2. **Corollary.** *Let $(X_i)_{i\in I}$ be a family of topological spaces and let $\mathcal{F}$ be a nonempty collection of closed subsets of the product space $X = \prod_{i\in I} X_i$. Assume that for each $i \in I$, there exists $F \in \mathcal{F}$ such that $\pi_i[F]$ is contained in a compact subset of X_i, where $\pi_i : X \to X_i$ denotes the i-th projection. If the intersection of the collection $\mathcal{F}$ is contained in an open subset U of X, then the intersection of some nonempty finite subcollection of $\mathcal{F}$ is contained in U.*

Proof. Apply Lemma 34.1 to the collection $\mathcal{F} \cup \{X \setminus U\}$. □

34.3. **Corollary.** *Let $(X_i)_{i\in I}$ be a family of Hausdorff topological spaces and for each $i \in I$ let K_i be a compact subset of X_i. Let U be an open subset of the product space $X = \prod_{i\in I} X_i$ such that $\prod_{i\in I} K_i \subset U$. Then there exists a finite subset J of I such that U contains the set:*

$$K_J = \big\{(x_i)_{i\in I} \in X : x_i \in K_i, \text{ for all } i \in J\big\}.$$

Proof. Apply Corollary 34.2 to the collection:

$$\mathcal{F} = \big\{K_J : J \text{ a finite subset of } I\big\}.$$ □

35. Proper maps and regular measures

Let X and Y be topological spaces. A map $f : X \to Y$ is called *proper* if $f^{-1}[K]$ is compact, for every compact subset K of Y. A subset F of a topological space Y is called *sequentially closed* if every limit[11] in Y of every sequence in F is in F. We say that Y is a *sequential space* if every sequentially closed subset of Y is closed in Y. This happens, for instance, if Y is a first countable space.

35.1. **Lemma.** *Let X and Y be topological spaces with Y Hausdorff. Assume that Y is either locally compact or sequential. Then every continuous proper map $f : X \to Y$ is closed.*

Proof. Let F be a closed subset of X and let us show that $f[F]$ is closed in Y. If Y is sequential, it suffices to check that $f[F]$ is sequentially closed in Y. Let $(x_n)_{n\geq 1}$ be a sequence in F such that $\big(f(x_n)\big)_{n\geq 1}$ converges to a point $y \in Y$. Then $K = \{y\} \cup \big\{f(x_n) : n \geq 1\big\}$ is a compact subset of Y and hence $(x_n)_{n\geq 1}$ is a sequence in the compact subset $f^{-1}[K]$. Thus $(x_n)_{n\geq 1}$ has a cluster point $x \in K$ which is also in F, since F is closed. It

[11] Since we are not assuming that Y is Hausdorff, a sequence could have more than one limit.

follows that $f(x)$ is a cluster point of the sequence $\big(f(x_n)\big)_{n\geq 1}$ and since Y is Haussdorff this implies that $f(x) = y$.

Now assume that Y is locally compact. Given $y \in Y \setminus f[F]$, we show that y has a neighborhood disjoint from $f[F]$. Let V be a compact neighborhood of y. Then $F \cap f^{-1}[V]$ is a compact subset of X and thus $H = f\big[F \cap f^{-1}[V]\big]$ is a compact subset of Y. Since Y is Hausdorff, it follows that H is closed in Y. Moreover, since $y \notin f[F]$, we have $y \notin H$ and thus $V \setminus H$ is a neighborhood of y. It is easily seen that $V \setminus H$ is disjoint from $f[F]$. □

35.2. **Definition.** Let X be a locally compact Hausdorff topological space. By a *regular measure* on X we mean a nonnegative countably additive measure μ on the Borel σ-algebra of X satisfying the following conditions:

(i) $\mu(B) = \inf\big\{\mu(U) : U \supset B \text{ open in } X\big\}$, for every Borel subset B of the space X;
(ii) $\mu(U) = \sup\big\{\mu(K) : K \subset U \text{ compact}\big\}$, for every open subset U of the space X;
(iii) $\mu(K) < +\infty$, for every compact subset K of the space X.

Note that condition (i) is trivially satisfied when $\mu(B) = +\infty$.

If X and Y are topological spaces, μ is a nonnegative countably additive measure on the Borel σ-algebra of X and $f : X \to Y$ is a continuous map, we denote by $f_*\mu$ the nonnegative countably additive measure on the Borel σ-algebra of Y defined by

$$(f_*\mu)(B) = \mu\big(f^{-1}[B]\big), \tag{35.1}$$

for every Borel subset B of Y.

35.3. **Lemma.** *Let X and Y be locally compact Hausdorff topological spaces, $f : X \to Y$ be a continuous proper map and μ be a regular measure on X. Then $f_*\mu$ is a regular measure on Y.*

Proof. Let B be a Borel subset of Y with $(f_*\mu)(B) < +\infty$. Given $\varepsilon > 0$, there exists an open subset U of X containing $f^{-1}[B]$ such that:

$$\mu(U) < \mu\big(f^{-1}[B]\big) + \varepsilon.$$

It follows from Lemma 35.1 that $V = Y \setminus f[X \setminus U]$ is open in Y. Moreover, one readily checks that $B \subset V$ and that $f^{-1}[V] \subset U$, so that:

$$(f_*\mu)(V) \leq \mu(U) < (f_*\mu)(B) + \varepsilon.$$

Now let U be an open subset of Y and fix $M < (f_*\mu)(U)$. Since $f^{-1}[U]$ is open in X and $M < \mu\big(f^{-1}[U]\big)$, there must exist a compact subset K of $f^{-1}[U]$ with $\mu(K) > M$. Then $f[K]$ is a compact subset of U and:

$$(f_*\mu)\big(f[K]\big) = \mu\big(f^{-1}[f[K]]\big) \geq \mu(K) > M.$$

Finally, note that for every compact subset K of Y we have that $(f_*\mu)(K)$ is finite, since $f^{-1}[K]$ is a compact subset of X. □

35.4. **Lemma.** *Let X be a locally compact Hausdorff topological space and let μ and ν be nonnegative countably additive measures on the Borel σ-algebra of X. Assume that $\nu(B) \le \mu(B)$ and that*

$$\nu(B) < +\infty \Longrightarrow \mu(B) < +\infty,$$

for every Borel subset B of X. If μ is regular, then also ν is regular.

Proof. Let B be a Borel subset of X with $\nu(B) < +\infty$ and let $\varepsilon > 0$ be given. Since $\mu(B) < +\infty$, there exists an open subset U of X containing B with $\mu(U) < \mu(B) + \varepsilon$. But since $\mu(B)$ is finite this implies:

$$\nu(U) - \nu(B) = \nu(U \setminus B) \le \mu(U \setminus B) = \mu(U) - \mu(B) < \varepsilon,$$

which proves that $\nu(U) < \nu(B) + \varepsilon$. Now let U be an open subset of X and let us prove that:

$$\nu(U) = \sup\big\{\nu(K) : K \subset U \text{ compact}\big\}. \tag{35.2}$$

Assume first that $\nu(U) < +\infty$ and let $\varepsilon > 0$ be given. Since $\mu(U) < +\infty$, there exists a compact subset K of U such that $\mu(K) > \mu(U) - \varepsilon$. Then:

$$\nu(U) - \nu(K) = \nu(U \setminus K) \le \mu(U \setminus K) = \mu(U) - \mu(K) < \varepsilon,$$

which proves that $\nu(K) > \nu(U) - \varepsilon$ and establishes (35.2). Now assume that $\nu(U) = +\infty$. Then $\mu(U) = +\infty$ and, for each positive integer n, there exists a compact subset K_n of U with $\mu(K_n) > n$. Replacing K_n with $\bigcup_{i=1}^n K_i$, we may assume that the sequence of compact sets $(K_n)_{n\ge1}$ is increasing. Then $\mu\big(\bigcup_{n=1}^\infty K_n\big) = +\infty$, so that

$$\sup_{n\ge1} \nu(K_n) = \lim_{n\to+\infty} \nu(K_n) = \nu\Big(\bigcup_{n=1}^\infty K_n\Big) = +\infty,$$

which again establishes (35.2). Finally, it is obvious that

$$\nu(K) \le \mu(K) < +\infty,$$

for every compact subset K of X. □

Recall that if μ is a signed countably additive measure on a σ-algebra $\mathcal{A}$, then there exists a decomposition $\mu = \mu_1 - \mu_2$ with μ_1 and μ_2 nonnegative countably additive measures on $\mathcal{A}$ and either μ_1 or μ_2 finite; moreover, there is a unique such decomposition $\mu = \mu_+ - \mu_-$ which is *minimal* in the sense that $\mu_+ \le \mu_1$ and $\mu_- \le \mu_2$ for every other decomposition $\mu = \mu_1 - \mu_2$. The decomposition $\mu = \mu_+ - \mu_-$ is called the *Jordan decomposition* of μ.

35.5. **Definition.** Let X be a locally compact Hausdorff topological space. A signed countably additive measure μ on the Borel σ-algebra of X is called *regular* if both μ_+ and μ_- are regular.

35.6. **Lemma.** *Let X be a locally compact Hausdorff topological space and μ be a signed countably additive measure on the Borel σ-algebra of X. Assume that $\mu = \mu_1 - \mu_2$, where μ_1 and μ_2 are nonnegative countably additive* *measures on the Borel σ-algebra of X and either μ_1 or μ_2 finite. If both μ_1 and μ_2 are regular, then so is μ.*

Proof. We have $\mu_+ \le \mu_1$, $\mu_- \le \mu_2$ and

$$\mu_+(B) < +\infty \Longrightarrow \mu(B) < +\infty \Longrightarrow \mu_1(B) < +\infty,$$
$$\mu_-(B) < +\infty \Longrightarrow \mu(B) > -\infty \Longrightarrow \mu_2(B) < +\infty,$$

for every Borel subset B of X. The conclusion follows from Lemma 35.4. □

Clearly definition (35.1) for $f_*\mu$ also makes sense when μ is a signed measure. Though obviously $f_*\mu = f_*(\mu_+) - f_*(\mu_-)$, it is not true in general that this is the Jordan decomposition for $f_*\mu$. Nevertheless, we have the following result.

35.7. **Corollary.** *Let X and Y be locally compact Hausdorff topological spaces, $f : X \to Y$ be a continuous proper map and μ be a signed regular measure on X. Then $f_*\mu$ is a signed*[12] *regular measure on Y.*

Proof. Simply note that $f_*\mu = f_*(\mu_+) - f_*(\mu_-)$ and apply the results of Lemmas 35.3 and 35.6. □

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[12]We are assuming the convention according to which the set of signed measures includes the set of nonnegative measures. It may happen that $\mu(B) < 0$ for some B but that $f_*\mu$ is nonnegative. This is the case, for instance, if $X = \{x_1, x_2\}$ has two points, Y has a unique point, $\mu\big(\{x_1\}\big) > 0$, $\mu\big(\{x_2\}\big) < 0$ and $\mu(X) > 0$. Note also that in this example $(f_*\mu)_- = 0$, while $f_*(\mu_-)$ is not zero.