ELETROMAGNETISM NOTES

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1. Spacetime and units

Spacetime is a four-dimensional real affine space whose underlying four-dimensional real vector space is endowed with a Minkowski-type inner product with signature $+---$ and a time-orientation. A point of spacetime is typically denoted by x. Using coordinates (defined by an origin and a positively time-oriented orthonormal basis), we write

$$x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4, \quad t = x^0, \quad \vec{x} = (x^1, x^2, x^3) \in \mathbb{R}^3, \quad x = (t, \vec{x}).$$

A vector in spacetime (i.e., an element of the vector space underlying spacetime) is typically denoted by $\dot{x}$ and, using coordinates:

$$\dot{x} = (\dot{x}^0, \dot{x}^1, \dot{x}^2, \dot{x}^3) \in \mathbb{R}^4, \quad \dot{t} = \dot{x}^0, \quad \dot{\vec{x}} = (\dot{x}^1, \dot{x}^2, \dot{x}^3), \quad \dot{x} = (\dot{t}, \dot{\vec{x}}).$$

Units are chosen such that

$$c = 1, \quad \varepsilon_0 = \frac{1}{4\pi}, \quad \mu_0 = 4\pi,$$

where c denotes the speed of light in vacuum, ε_0 denotes vacuum permittivity and μ_0 denotes vacuum permeability (Coulomb's constant then becomes equal to 1, i.e., Coulomb's force becomes the product of charges divided by the square of the distance). The spacetime Minkowski inner product $\langle \cdot, \cdot \rangle$ is given by:

$$\langle (\dot{t}_1, \dot{\vec{x}}_1), (\dot{t}_2, \dot{\vec{x}}_2) \rangle = \dot{t}_1 \dot{t}_2 - \langle \dot{\vec{x}}_1, \dot{\vec{x}}_2 \rangle, \quad \dot{t}_1, \dot{t}_2 \in \mathbb{R}, \ \dot{\vec{x}}_1, \dot{\vec{x}}_2 \in \mathbb{R}^3,$$

where the $\langle \cdot, \cdot \rangle$ in the righthand side of the equality denotes the standard Euclidean inner product (the Minkowski spacetime inner product and the Euclidean space inner product are both denoted by $\langle \cdot, \cdot \rangle$, but context should be sufficient to distinguish between them).

2. Density and flow of a physical quantity

Given some physical quantity $\mathfrak{a}$ which takes values in some real finite-dimensional vector space $\mathbb{E}$, its distribution over spacetime is described by a density function

$$\rho_{\mathfrak{a}} : \mathbb{R}^4 \to \mathbb{E}$$

and by a flow density

$$\vec{\jmath}_{\mathfrak{a}} : \mathbb{R}^4 \to \mathbb{R}^3 \otimes \mathbb{E}.$$

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The map $\vec{\jmath}_{\mathfrak{a}}$ should be thought of as a (time-dependent) $\mathbb{E}$-valued vector field over $\mathbb{R}^3$, i.e., we think of $\vec{\jmath}_{\mathfrak{a}}(x) \in \mathbb{R}^3 \otimes \mathbb{E}$ as having three components

$$\vec{\jmath}_{\mathfrak{a}}(x) = \big(\vec{\jmath}_{\mathfrak{a}}^{\,1}(x), \vec{\jmath}_{\mathfrak{a}}^{\,2}(x), \vec{\jmath}_{\mathfrak{a}}^{\,3}(x)\big)$$

with each $\vec{\jmath}_{\mathfrak{a}}^{\,i}(x)$ an element of $\mathbb{E}$. We can thus define the flow of the $\mathbb{E}$-valued vector field $\vec{\jmath}_{\mathfrak{a}}(t, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3 \otimes \mathbb{E}$ over an oriented surface of $\mathbb{R}^3$ and this is to be interpreted as the rate at time t of how much of the quantity $\mathfrak{a}$ flows through the given surface. The divergence

$$\operatorname{div} \vec{\jmath}_{\mathfrak{a}}(x) = \sum_{i=1}^{3} \frac{\partial \vec{\jmath}_{\mathfrak{a}}^{\,i}}{\partial x^i}(x) \in \mathbb{E}$$

of $\vec{\jmath}_{\mathfrak{a}}$ with respect to the $\vec{x}$ variable is an $\mathbb{E}$-valued function over $\mathbb{R}^4$ and the divergence theorem holds. The expression

$$\frac{\partial \rho_{\mathfrak{a}}}{\partial t} + \operatorname{div} \vec{\jmath}_{\mathfrak{a}} \tag{2.1}$$

is the density of how much of the quantity $\mathfrak{a}$ is gained by the system. When the quantity $\mathfrak{a}$ is not exchanged with external systems, conservation of $\mathfrak{a}$ means that (2.1) is equal to zero. Otherwise, (2.1) should be equal to how much of $\mathfrak{a}$ is gained by interaction with external systems.

When a basis of $\mathbb{E}$ is given, we write $\vec{\jmath}_{\mathfrak{a}}^{\,ij}$ for the j-th coordinate of $\vec{\jmath}_{\mathfrak{a}}^{\,i}$ in the given basis of $\mathbb{E}$, so that:

$$[\operatorname{div} \vec{\jmath}_{\mathfrak{a}}(x)]^j = \sum_{i=1}^{3} \frac{\partial \vec{\jmath}_{\mathfrak{a}}^{\,ij}}{\partial x^i}(x).$$

One should pay attention to this convention for the ordering of i and j specially when $\mathbb{E}$ happens to be $\mathbb{R}^3$, when this can get confusing.

3. Maxwell's equations and Lorentz force

Let

$$E : \mathbb{R}^4 \longrightarrow \mathbb{R}^3 \quad \text{and} \quad B : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

denote, respectively, electric and magnetic fields and let

$$\rho : \mathbb{R}^4 \to \mathbb{R} \quad \text{and} \quad \vec{\jmath} : \mathbb{R}^4 \to \mathbb{R}^3$$

denote, respectively, charge density and current density. Maxwell's equations are:

$$\operatorname{div} B = 0, \quad \operatorname{rot} E + \frac{\partial B}{\partial t} = 0,$$
$$\operatorname{div} E = 4\pi\rho, \quad \operatorname{rot} B = 4\pi\vec{\jmath} + \frac{\partial E}{\partial t},$$

where div and rot denote, respectively, divergence and curl with respect to the $\vec{x}$ variable. Conservation of charge is expressed by the equation:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{\jmath} = 0.$$

Let $u : \mathbb{R} \to \mathbb{R}^3$ denote the (smooth) trajectory of a particle, so that

$$\big\{\big(t, u(t)\big) : t \in \mathbb{R}\big\} \subset \mathbb{R}^4$$

denotes its worldline. It is assumed that $\|u'(t)\| < 1$ for all t. If q denotes the charge of the particle, then the Lorentz force acting on it by the fields at time t is given by:

$$q\big[E\big(t, u(t)\big) + u'(t) \wedge B\big(t, u(t)\big)\big],$$

where $\wedge$ denotes the vector product. The dynamics for the motion of the particle is given by:

$$p'(t) = q\big[E\big(t, u(t)\big) + u'(t) \wedge B\big(t, u(t)\big)\big] + \text{other forces},$$

where the momentum $p : \mathbb{R} \to \mathbb{R}^3$ is defined by:

$$p(t) = \frac{mu'(t)}{(1 - \|u'(t)\|^2)^{\frac{1}{2}}}$$

and m denotes the mass of the particle.

3.1. **Potentials.** The scalar potential $\phi : \mathbb{R}^4 \to \mathbb{R}$ and the vector potential $\vec{A} : \mathbb{R}^4 \to \mathbb{R}^3$ are chosen such that the equalities

$$E = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}, \quad B = \operatorname{rot} \vec{A} \tag{3.1}$$

are satisfied, where ∇ denotes the gradient with respect to the $\vec{x}$ variable. The Lorenz gauge condition on the potentials is:

$$\frac{\partial \phi}{\partial t} + \operatorname{div} \vec{A} = 0.$$

Denote by $\triangle = \sum_{i=1}^3 \frac{\partial^2}{(\partial x^i)^2}$ the Laplace operator in $\mathbb{R}^3$ and by $\square = \frac{\partial^2}{\partial t^2} - \triangle$ the wave operator in $\mathbb{R}^4$. In terms of potentials ϕ and $\vec{A}$ satisfying the Lorenz gauge, Maxwell's equations become:

$$\square\phi = 4\pi\rho, \quad \square\vec{A} = 4\pi\vec{\jmath}. \tag{3.2}$$

4. Energy and momentum of the electromagnetic field

The energy density $\rho_{\mathfrak{e}} : \mathbb{R}^4 \to \mathbb{R}$ and the energy flow density $\vec{\jmath}_{\mathfrak{e}} : \mathbb{R}^4 \to \mathbb{R}^3$ of the electromagnetic field are given by:

$$\rho_{\mathfrak{e}} = \frac{1}{8\pi}(\|E\|^2 + \|B\|^2), \quad \vec{\jmath}_{\mathfrak{e}} = \frac{1}{4\pi}(E \wedge B).$$

The momentum density $\rho_{\mathfrak{p}} : \mathbb{R}^4 \to \mathbb{R}^3$ and the momentum flow density $\vec{\jmath}_{\mathfrak{p}} : \mathbb{R}^4 \to \mathbb{R}^3 \otimes \mathbb{R}^3$ of the electromagnetic field are given by:

$$\rho_{\mathfrak{p}} = \frac{1}{4\pi}(E \wedge B), \quad \vec{\jmath}_{\mathfrak{p}} = \frac{1}{8\pi}(\|E\|^2 + \|B\|^2)\mathrm{I} - \frac{1}{4\pi}(E \otimes E + B \otimes B),$$

where $\mathrm{I} \in \mathbb{R}^3 \otimes \mathbb{R}^3$ denotes the tensor represented in the canonical basis by the identity matrix (if we identify $\mathbb{R}^3 \otimes \mathbb{R}^3$ with the space of bilinear forms on the dual space $\mathbb{R}^{3^*}$, then I is identified with the Euclidean inner product in $\mathbb{R}^{3*}$). The conservation laws of energy and momentum are expressed by the equalities:

$$\frac{\partial \rho_{\mathfrak{e}}}{\partial t} + \operatorname{div} \vec{\jmath}_{\mathfrak{e}} = -\langle E, \vec{\jmath}\rangle, \quad \frac{\partial \rho_{\mathfrak{p}}}{\partial t} + \operatorname{div} \vec{\jmath}_{\mathfrak{p}} = -(\rho E + \vec{\jmath} \wedge B).$$

Note that $\langle E, \vec{\jmath}\rangle$ is the time-derivative of the amount of energy that the fields give to the charge distribution (power density) and $\rho E + \vec{\jmath} \wedge B$ is the time-derivative of the amount of momentum that the fields give to the charge distribution (force density).

5. Inner product on multilinear maps and Hodge star

Given real finite-dimensional vector spaces V_i, $i = 1, 2, \ldots, n$ endowed with indefinite inner products $\langle\cdot,\cdot\rangle$ (i.e., $\langle\cdot,\cdot\rangle$ is a nondegenerate symmetric bilinear form on V_i), we obtain an indefinite inner product on the tensor product $\bigotimes_{i=1}^n V_i$ by setting:

$$\langle v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_n\rangle = \prod_{i=1}^n \langle v_i, w_i\rangle, \quad v_i, w_i \in V_i.$$

We identify the tensor product $\bigotimes_{i=1}^n V_i^*$ of the dual spaces with the space of multilinear maps $V_1 \times \cdots \times V_n \to \mathbb{R}$ by setting:

$$(\alpha_1 \otimes \cdots \otimes \alpha_n)(v_1, \ldots, v_n) = \prod_{i=1}^n \alpha_i(v_i), \quad \alpha_i \in V_i^*,\ v_i \in V_i. \tag{5.1}$$

An indefinite inner product on a real finite-dimensional vector space V yields an isomorphism $v \mapsto \langle v, \cdot\rangle$ between V and V^* and this isomorphism induces an indefinite inner product on V^*. Thus, the indefinite inner products on the spaces V_i induce also an indefinite inner product on $\bigotimes_{i=1}^n V_i^*$ and hence on the space of multilinear maps $V_1 \times \cdots \times V_n \to \mathbb{R}$.

Given a real finite-dimensional vector space V, we identify the exterior product $\bigwedge_k V^*$ with the space of k-linear alternate forms on V by setting:

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \ldots, v_k) = \det\big(\alpha_i(v_j)\big)_{k\times k}, \quad \alpha_i \in V^*,\ v_i \in V.$$

Thus $\bigwedge_k V^*$ is identified with a subspace of $\bigotimes_k V^*$ and we have:

$$\alpha_1 \wedge \cdots \wedge \alpha_k = \sum_{\sigma \in S^k} \operatorname{sgn}(\sigma)(\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(k)}), \quad \alpha_i \in V^*,$$

where S^k denotes the group of permutations of $\{1, \ldots, k\}$ and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ. An indefinite inner product on V thus induces an indefinite inner product on $\bigwedge_k V^*$ and we have:

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k\rangle = k! \det\big(\langle \alpha_i, \beta_j\rangle\big)_{k\times k}, \quad \alpha_i, \beta_i \in V^*.$$

5.1. **Hodge star.** Given an indefinite inner product $\langle\cdot,\cdot\rangle$ in a real oriented vector space V of dimension $n < +\infty$, there exists a unique n-form on V, denoted $\mathrm{vol} \in \bigwedge_n V^*$, such that

$$\mathrm{vol}(e_1, \dots, e_n) = 1,$$

for any positive orthonormal basis $(e_i)_{i=1}^n$ of V; *orthonormal* means that $\langle e_i, e_j\rangle = 0$, for $i \neq j$, and $|\langle e_i, e_i\rangle| = 1$, for all i. We have:

$$\mathrm{vol} = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n,$$

where $(\mathrm{d}x^i)_{i=1}^n$ is dual to some positive orthonormal basis of V. For any $\omega \in \bigwedge_k V^*$, we define $\star\omega \in \bigwedge_{n-k} V^*$ by requiring that:

$$\lambda \wedge (\star\omega) = \frac{1}{k!}\langle\lambda, \omega\rangle\,\mathrm{vol},$$

for all $\lambda \in \bigwedge_k V^*$. If $(\mathrm{d}x^i)_{i=1}^n$ is dual to a positive orthonormal basis $(e_i)_{i=1}^n$ of V, then

$$\star(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}) = \pm\mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_{n-k}},$$

where $\{i_1, \dots, i_k, j_1, \dots, j_{n-k}\} = \{1, 2, \dots, n\}$ and the sign is determined by the equality

$$\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \wedge \big(\star(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k})\big) = \langle e_{i_1}, e_{i_1}\rangle \cdots \langle e_{i_k}, e_{i_k}\rangle\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

If $\omega \in \bigwedge_k V^*$, then:

$$\star\star\omega = (-1)^{k(n-k)}(-1)^{\mathrm{ind}}\omega,$$

where ind denotes the index of the indefinite inner product (i.e., the number of minus signs in its signature).

5.2. **Codifferential.** If ω is a smooth differential k-form on a semi-Riemannian oriented[1] manifold M, then the codifferential $\delta\omega$ is the smooth differential $(k-1)$-form on M such that:

$$\frac{1}{(k-1)!}\int_M \langle\lambda, \delta\omega\rangle\,\mathrm{vol} = \frac{1}{k!}\int_M \langle \mathrm{d}\lambda, \omega\rangle\,\mathrm{vol}, \tag{5.2}$$

for any smooth differential $(k-1)$-form λ on M having compact support. By $\mathrm{d}\lambda$ we have denoted the exterior differential of λ. If $k = 0$ we set $\delta\omega = 0$. We have the following explicit formula for $\delta\omega$ in terms of the Hodge star and the exterior differential:

$$\delta\omega = (-1)^{\mathrm{ind}}(-1)^{n(k-1)+1}\star\mathrm{d}\star\omega,$$

where ω is a differential k-form, n denotes the dimension of M and ind denotes the index of the semi-Riemannian metric.

[1] Orientation is not needed, but without an orientation one has to use the volume density of M in (5.2) rather than the volume form. Though the Hodge star $\star$ depends on a choice of orientation, the codifferential δ does not, since it has two Hodge stars.

6. Fundamental solution for the wave operator

Let $C \subset \mathbb{R}^4$ denote the lightcone:

$$C = \big\{x \in \mathbb{R}^4 : \langle x, x\rangle = 0\big\} = \big\{(t, \vec{x}) \in \mathbb{R}^4 : |t| = \|x\|\big\}$$

and C^+ denote the future lightcone:

$$C^+ = \big\{(t, \vec{x}) \in C : t \geq 0\big\}.$$

On what follows we will define a Borel measure μ_C on C. Consider the mapping

$$\mathbb{R} \times S^2 \ni (t, u) \longmapsto (t, tu) \in C, \tag{6.1}$$

where $S^2 = \big\{u \in \mathbb{R}^3 : \|u\| = 1\big\}$ denotes the Euclidean unit sphere of $\mathbb{R}^3$. Let $\mathfrak{m}_{\mathbb{R}\times S^2}$ denote the Borel measure in $\mathbb{R} \times S^2$ defined as the product of the Lebesgue measure of $\mathbb{R}$ by the standard area measure of S^2. We define μ_C to be the push-forward under (6.1) of the measure on $\mathbb{R} \times S^2$ given by:

$$\int |t| \, \mathrm{d}\mathfrak{m}_{\mathbb{R}\times S^2}(t, u), \tag{6.2}$$

i.e., μ_C calculated in a Borel subset of C is equal to the integral (6.2) over the inverse image of such Borel subset by the mapping (6.1). We denote by μ_{C^+} the restriction of μ_C to C^+. We can think of μ_{C^+} as a Borel measure over $\mathbb{R}^4$ (whose value at a Borel subset of $\mathbb{R}^4$ is the value of μ_{C^+} at the intersection of such Borel subset with C^+) and, since such measure is finite over compact subsets, it defines a (Schwartz) distribution over $\mathbb{R}^4$. More explicitly, given a test function $\varphi : \mathbb{R}^4 \to \mathbb{R}$ (smooth, with compact support), we set:

$$\langle\!\langle \mu_{C^+}, \varphi \rangle\!\rangle = \int_{C^+} \varphi \, \mathrm{d}\mu_{C^+},$$

where the bracket $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes evaluation of a distribution at a test function. If $\Box$ denotes the wave operator in $\mathbb{R}^4$ (see Subsection 3.1), then:

$$\Box \mu_{C^+} = 4\pi \delta_0,$$

where δ_0 denotes the Dirac delta distribution $\langle \delta_0, \varphi\rangle = \varphi(0)$. In other words, $\frac{1}{4\pi}\mu_{C^+}$ is a fundamental solution for the wave operator $\Box$ in $\mathbb{R}^4$.

We give an alternative description of μ_{C^+}. Consider the mapping:

$$\mathbb{R}^3 \ni \vec{x} \longmapsto (\|\vec{x}\|, \vec{x}) \in C^+. \tag{6.3}$$

If $\mathfrak{m}_{\mathbb{R}^3}$ denotes the Lebesgue measure of $\mathbb{R}^3$, then μ_{C^+} is equal to the push-forward under (6.3) of the measure on $\mathbb{R}^3$ given by:

$$\int \frac{1}{\|\vec{x}\|} \, \mathrm{d}m(\vec{x}).$$

7. Lorentz boosts of electric and magnetic fields

Given a vector $v \in \mathbb{R}^3$ with $\|v\| < 1$, we define a linear transformation $L_v : \mathbb{R}^4 \to \mathbb{R}^4$ by setting:

$$L_v(\dot{t}, \dot{\vec{x}}) = \frac{1}{(1 - \|v\|^2)^{\frac{1}{2}}} \left(\dot{t} + \langle \dot{\vec{x}}, v\rangle, v\dot{t} + \operatorname{proj}_v \dot{\vec{x}}\right) + (0, \operatorname{proj}_{v^\perp} \dot{\vec{x}}),$$

for all $(\dot{t}, \dot{\vec{x}}) \in \mathbb{R}^4$, where

$$\operatorname{proj}_v \dot{\vec{x}} = \frac{\langle \dot{\vec{x}}, v\rangle}{\langle v, v\rangle} v$$

denotes the orthogonal projection of $\dot{\vec{x}}$ onto the one-dimensional subspace spanned by v and $\operatorname{proj}_{v^\perp} \dot{\vec{x}} = \dot{\vec{x}} - \operatorname{proj}_v \dot{\vec{x}}$ denotes the orthogonal projection of $\dot{\vec{x}}$ onto the orthogonal complement of v. The map L_v is a Lorentz transformation, i.e., a linear isometry of the Minkowski inner product. Moreover, it preserves time and space orientation and it leaves the two-dimensional subspace spanned by $(1, 0)$ and $(0, v)$ invariant. The map L_v gives a "boost of velocity v" to particles, i.e., it sends the worldline $\mathbb{R} \times \{0\}$ of a particle at rest to the worldline $\big\{(t, vt) : t \in \mathbb{R}\big\}$ of a particle with velocity v. The map that sends the coordinates $(\dot{t}, \dot{\vec{x}})$ to the coordinates used by an observer that, with respect to the coordinates $(\dot{t}, \dot{\vec{x}})$, moves with velocity v is $L_{-v} = L_v^{-1}$.

The push-forward of the electric and magnetic fields E, B by the Lorentz transformation L_v (which gives a "boost of velocity v" to such fields) are the fields E' and B' given by:

$$E'(x') = \operatorname{proj}_v E(x) + \frac{1}{(1 - \|v\|^2)^{\frac{1}{2}}} \big(\operatorname{proj}_{v^\perp} E(x) + B(x) \wedge v\big),$$

$$B'(x') = \operatorname{proj}_v B(x) + \frac{1}{(1 - \|v\|^2)^{\frac{1}{2}}} \big(\operatorname{proj}_{v^\perp} B(x) + v \wedge E(x)\big),$$

for all $x' \in \mathbb{R}^4$, where $x = L_v^{-1}(x') = L_{-v}(x')$.

8. Explicitly Lorentz invariant formulation

Given an electric field E and a magnetic field B, we define a differential 2-form $F : \mathbb{R}^4 \to \bigwedge_2 \mathbb{R}^{4*}$ over spacetime by setting:

$$F_x\big((\dot{t}_1, \dot{\vec{x}}_1), (\dot{t}_2, \dot{\vec{x}}_2)\big) = \dot{t}_1 \langle E(x), \dot{\vec{x}}_2\rangle - \dot{t}_2 \langle E(x), \dot{\vec{x}}_1\rangle - \det\big(B(x), \dot{\vec{x}}_1, \dot{\vec{x}}_2\big),$$
$$x \in \mathbb{R}^4,\ \dot{t}_1, \dot{t}_2 \in \mathbb{R},\ \dot{\vec{x}}_1, \dot{\vec{x}}_2 \in \mathbb{R}^3.$$

From the charge density ρ and current density $\vec{\jmath}$, we obtain a vector field $j : \mathbb{R}^4 \to \mathbb{R}^4$ over spacetime by setting:

$$j = (\rho, \vec{\jmath}).$$

The flow[2] of the vector field j over a spacelike 3-surface (which can be understood as a region of space at some instant of time, with respect to some general coordinate system) gives the amount of charge in that 3-surface and the flow of j over a timelike surface (which can be understood as a moving 2-surface in space, with respect to some general coordinate system) gives the amount of charge flowing through that surface. In terms of F and j, Maxwell's equations become:

$$\mathrm{d}F = 0, \quad \delta F = \star\mathrm{d}\star F = -4\pi\langle j, \cdot\rangle,$$

where $\mathrm{d}F$ denotes the exterior differential of F.

8.1. **Lorentz force.** Let $\gamma : \mathbb{R} \to \mathbb{R}^4$ denote a parametrization by arc-length of the worldline of a particle with charge q (parametrization by arc-length means that $\langle\gamma'(s), \gamma'(s)\rangle = 1$, for all s). We assume also that the parametrization of γ is positively time-oriented, i.e., that $\gamma'(s)$ points to the future, for all $s \in \mathbb{R}$. The Lorentz force law is expressed as:

$$m\langle\gamma''(s), \cdot\rangle = qF\big(\cdot, \gamma'(s)\big),$$

where m denotes the mass of the particle. This equation defines the dynamics of the particle, in case there are no other forces acting on it.

8.2. **Potentials.** If ϕ denotes the scalar potential and $\vec{A}$ denotes the vector potential, we define a vector field $A : \mathbb{R}^4 \to \mathbb{R}^4$ over spacetime by setting:

$$A = (\phi, \vec{A}).$$

We have:

$$\mathrm{d}\langle A, \cdot\rangle = F,$$

where $\mathrm{d}\langle A, \cdot\rangle$ denotes the exterior differential of the 1-form $\langle A, \cdot\rangle$. The Lorenz gauge condition is given by:

$$\delta\langle A, \cdot\rangle = \star\mathrm{d}\star\langle A, \cdot\rangle = 0.$$

8.3. **Energy and momentum.** The energy density, energy flow density, momentum density and momentum flow density of the electromagnetic field defined in Section 4 can be assembled into a energy-momentum tensor T, which is a tensor field $T : \mathbb{R}^4 \to \mathbb{R}^4 \otimes \mathbb{R}^4$ over spacetime. We identify $\mathbb{R}^4 \otimes \mathbb{R}^4$ with the space of bilinear forms in $\mathbb{R}^{4^*}$ by setting:

$$(\dot{x} \otimes \dot{y})(\alpha, \beta) = \alpha(\dot{x})\beta(\dot{y}),$$

for all $\alpha, \beta \in \mathbb{R}^{4^*}$, $\dot{x}, \dot{y} \in \mathbb{R}^4$. For a linear functional $\alpha \in \mathbb{R}^{4^*}$, we write:

$$\alpha = (\alpha_0, \vec{\alpha}),$$

[2]The flow of j over a 3-surface is the integral of the 3-form $\mathrm{vol}(j, \cdot, \cdot, \cdot)$ over that surface, where vol denotes the volume form of spacetime (canonically associated to the Minkowski inner product). If the 3-surface is not lightlike, this is equal to the integral over the 3-surface (with respect to the measure defined by the metric given by the restriction of the Minkowski inner product) of the Minkowski inner product of j with the unit normal vector.

with $\alpha_0 = \alpha(1,0) \in \mathbb{R}$ and $\vec{\alpha} \in \mathbb{R}^{3*}$ defined by $\vec{\alpha}(\dot{\vec{x}}) = \alpha(0, \dot{\vec{x}})$, for all $\dot{\vec{x}} \in \mathbb{R}^3$. The energy-momentum tensor T of the electromagnetic field is defined by:

$$T(x)(\alpha,\beta) = \rho_{\mathfrak{e}}(x)\alpha_0\beta_0 + \alpha_0\vec{\beta}\big(\rho_{\mathfrak{p}}(x)\big) + \beta_0\vec{\alpha}\big(\vec{\jmath}_{\mathfrak{e}}(x)\big) + \vec{\jmath}_{\mathfrak{p}}(x)(\vec{\alpha},\vec{\beta}),$$

for all $x \in \mathbb{R}^4$, $\alpha, \beta \in \mathbb{R}^{4*}$, where $\vec{\jmath}_{\mathfrak{p}}(x) \in \mathbb{R}^3 \otimes \mathbb{R}^3$ is identified with a bilinear form in $\mathbb{R}^{3*}$. Since $\rho_{\mathfrak{p}} = \vec{\jmath}_{\mathfrak{e}}$ and since $\vec{\jmath}_{\mathfrak{p}}(x)$ is symmetric, it follows that $T(x)$ is symmetric. We can write a formula for T directly in terms of the 2-form F:

$$T(x)(\alpha,\beta) = \frac{1}{4\pi}\Big(\frac{1}{4}\langle F(x), F(x)\rangle\langle\alpha,\beta\rangle - \big\langle F(x)(\alpha^\sharp,\cdot), F(x)(\beta^\sharp,\cdot)\big\rangle\Big),$$

for all $x \in \mathbb{R}^4$, $\alpha, \beta \in \mathbb{R}^{4*}$, where, for $\alpha \in \mathbb{R}^{4*}$, $\alpha^\sharp \in \mathbb{R}^4$ is defined by the equality $\langle \alpha^\sharp, \cdot\rangle = \alpha$. Conservation of energy and momentum is expressed by:

$$\langle (\operatorname{div} T)(x), \cdot\rangle = F\big(j(x), \cdot\big),$$

where the divergence $(\operatorname{div} T)(x) \in \mathbb{R}^4$ of T at the point x is defined as the trace of $\mathrm{d}T(x) \in \mathbb{R}^{4*} \otimes \mathbb{R}^4 \otimes \mathbb{R}^4$ in the first two variables and $\mathrm{d}T(x)$ denotes the (ordinary) differential of T at the point x. More precisely:

$$\mathrm{d}T(x)(\dot{x},\alpha,\beta) = \frac{\partial T}{\partial \dot{x}}(x)(\alpha,\beta), \quad x \in \mathbb{R}^4,\ \dot{x} \in \mathbb{R}^4,\ \alpha,\beta \in \mathbb{R}^{4*},$$

with $\frac{\partial T}{\partial \dot{x}}(x)$ the partial derivative of T at the point x, in the direction of the vector $\dot{x}$ and:

$$\beta\big((\operatorname{div} T)(x)\big) = \sum_{i=1}^{4} \mathrm{d}T(x)(e_i, \alpha^i, \beta), \quad x \in \mathbb{R}^4,\ \beta \in \mathbb{R}^{4*},$$

with $(e_i)_{i=1}^4$ an arbitrary basis of $\mathbb{R}^4$ and $(\alpha^i)_{i=1}^4$ its dual basis.

9. Liénard-Wiechert potentials and fields

Let $u : \mathbb{R} \to \mathbb{R}^3$ denote the (smooth) trajectory of a particle, so that

$$\big\{\big(t, u(t)\big) : t \in \mathbb{R}\big\} \subset \mathbb{R}^4$$

denotes its worldline. Assume that $\|u'(t)\| < 1$ for all t and that this particle has charge q. The charge density ρ and current density $\vec{\jmath}$ associated to this moving charged particle are the distributions over $\mathbb{R}^4$ given by:

$$\rho(x) = q\delta_0\big(\vec{x} - u(t)\big), \quad \vec{\jmath}(x) = qu'(t)\delta_0\big(\vec{x} - u(t)\big),$$

where δ_0 denotes the Dirac delta distribution of $\mathbb{R}^3$. More precisely, for a test function $\varphi : \mathbb{R}^4 \to \mathbb{R}$, we have:

$$\langle\!\langle \rho, \varphi\rangle\!\rangle = q\int_{\mathbb{R}} \varphi\big(t, u(t)\big)\,\mathrm{d}t, \quad \langle\!\langle \vec{\jmath}, \varphi\rangle\!\rangle = q\int_{\mathbb{R}} u'(t)\varphi\big(t, u(t)\big)\,\mathrm{d}t,$$

where the bracket $\langle\!\langle \cdot,\cdot\rangle\!\rangle$ denotes evaluation of a distribution at a test function. If we solve the wave equations (3.2) by taking the convolution of ρ and $\vec{\jmath}$ with the fundamental solution of the wave operator given in Section 6, we obtain expressions for the scalar potential ϕ and vector potential $\vec{A}$ that we describe below.

For each $x = (t, \vec{x})$ in $\mathbb{R}^4$, there exists at most one point $\big(t', u(t')\big)$ in the worldline of the particle that belongs to the past lightcone of the point x; the corresponding value of t' is called the *retarded time* for the point x and is denoted by $t_{\rm ret}(x)$. More precisely, $t_{\rm ret}(x) \in \left]-\infty, t\right]$ is defined by the equality:

$$t - t_{\rm ret}(x) = \big\|\vec{x} - u\big(t_{\rm ret}(x)\big)\big\|.$$

The domain of $t_{\rm ret}$ is an open subset of $\mathbb{R}^4$ containing the worldline of the particle. The map $t_{\rm ret}$ is continuous and it is smooth in the complement of the worldline. For $x = (t, \vec{x})$ in the domain of $t_{\rm ret}$, we write:

$$r(x) = t - t_{\rm ret}(x) = \big\|\vec{x} - u\big(t_{\rm ret}(x)\big)\big\|$$

and if x is also outside the worldline of the particle, we set:

$$\vec{n}(x) = \frac{\vec{x} - u\big(t_{\rm ret}(x)\big)}{\|\vec{x} - u\big(t_{\rm ret}(x)\big)\|}.$$

The scalar potential ϕ and vector potential $\vec{A}$ are given by:

$$\phi(x) = \frac{q}{r}\,\frac{1}{1 - \langle u'(t_{\rm ret}), \vec{n}\rangle}, \quad \vec{A}(x) = \phi(x) u'(t_{\rm ret}),$$

for all x in the domain of $t_{\rm ret}$ and outside the worldline of the particle; we have written $t_{\rm ret}$, r and $\vec{n}$ instead of $t_{\rm ret}(x)$, $r(x)$ and $\vec{n}(x)$ for simplicity. For x outside the domain of $t_{\rm ret}$, the potentials ϕ and $\vec{A}$ are equal to zero (the values of ϕ and $\vec{A}$ over the worldline are irrelevant because it has zero Lebesgue measure and we are thinking of ϕ and $\vec{A}$ as distributions).

Using (3.1) we now obtain formulas for the fields E and B. We assume now that the domain of $t_{\rm ret}$ is all of $\mathbb{R}^4$. The electric field is given by:

$$\begin{aligned} E(x) = {} & \frac{q}{r}\,\frac{\vec{n} \wedge \big[\big(\vec{n} - u'(t_{\rm ret})\big) \wedge u''(t_{\rm ret})\big]}{\big(1 - \langle u'(t_{\rm ret}), \vec{n}\rangle\big)^3} \\ & + \frac{q}{r^2}\,\frac{1 - \|u'(t_{\rm ret})\|^2}{\big(1 - \langle u'(t_{\rm ret}), \vec{n}\rangle\big)^3}\,\big(\vec{n} - u'(t_{\rm ret})\big), \end{aligned}$$

and the magnetic field is given by:

$$B(x) = \vec{n} \wedge E(x),$$

for all x outside the worldline of the particle. Again, we have abbreviated $t_{\rm ret}(x)$, $r(x)$ and $\vec{n}(x)$ by $t_{\rm ret}$, r and $\vec{n}$, respectively.

9.1. **Explicitly Lorentz invariant description.** If $\gamma : \mathbb{R} \to \mathbb{R}^4$ denotes a positively time-oriented parametrization by arc-length of the worldline of the particle with charge q, we give explicit formulas for the vector field A and the 2-form F over $\mathbb{R}^4$ corresponding to the potentials and fields above (see Section 8 for the relation between A, F and ϕ, $\vec{A}$, E and B). For each

$x \in \mathbb{R}^4$, there exists at most one point $\gamma(s)$ in the worldline of the particle that belongs to the past lightcone of x; the corresponding value of s is called the *retarded proper time* and it is denoted by $s_{\rm ret}(x)$. More precisely, $s_{\rm ret}(x)$ is defined by the requirement that

$$\big\langle x - \gamma\big(s_{\rm ret}(x)\big), x - \gamma\big(s_{\rm ret}(x)\big)\big\rangle = 0$$

and that $\gamma\big(s_{\rm ret}(x)\big)$ be in the past of x (note that, since a time-orientation is fixed, there exists an absolute notion of "y being in the past of x" when the vector $x - y$ is either timelike or lightlike). The maps $s_{\rm ret}$ and $t_{\rm ret}$ have the same domain and both maps share the same regularity properties. Indeed, we have $s_{\rm ret} = \sigma \circ t_{\rm ret}$, where $\sigma : \mathbb{R} \to \mathbb{R}$ is the diffeomorphism defined by $\gamma\big(\sigma(t)\big) = \big(t, u(t)\big)$, for all $t \in \mathbb{R}$. However, while $t_{\rm ret}$ and σ are defined only with respect to a fixed coordinate system in spacetime, $s_{\rm ret}$ is absolute. The vector field A is given by:

$$A(x) = \frac{q\gamma'(s_{\rm ret})}{\langle x - \gamma(s_{\rm ret}), \gamma'(s_{\rm ret})\rangle},$$

for all x in the domain of $s_{\rm ret}$ and not in the worldline of the particle, where we have abbreviated $s_{\rm ret}(x)$ by $s_{\rm ret}$. For x not in the domain of $s_{\rm ret}$, we have $A(x) = 0$. Assuming now that the domain of $s_{\rm ret}$ is $\mathbb{R}^4$, we have:

$$F(x) = \frac{q\big(1 - \langle x - \gamma(s_{\rm ret}), \gamma''(s_{\rm ret})\rangle\big)}{\langle x - \gamma(s_{\rm ret}), \gamma'(s_{\rm ret})\rangle^3}\big(\langle x - \gamma(s_{\rm ret}), \cdot\rangle \wedge \langle \gamma'(s_{\rm ret}), \cdot\rangle\big) + \frac{q\big(\langle x - \gamma(s_{\rm ret}), \cdot\rangle \wedge \langle \gamma''(s_{\rm ret}), \cdot\rangle\big)}{\langle x - \gamma(s_{\rm ret}), \gamma'(s_{\rm ret})\rangle^2},$$

for all $x \in \mathbb{R}^4$ not in the worldline of the particle, where again we have abbreviated $s_{\rm ret}(x)$ by $s_{\rm ret}$ and

$$(\alpha_1 \wedge \alpha_2)(\dot{x}_1, \dot{x}_2) = \alpha_1(\dot{x}_1)\alpha_2(\dot{x}_2) - \alpha_2(\dot{x}_1)\alpha_1(\dot{x}_2),$$

for all $\alpha_1, \alpha_2 \in \mathbb{R}^{4^*}$ and $\dot{x}_1, \dot{x}_2 \in \mathbb{R}^4$.

10. Tubular neighborhood of the trajectory

Let $\gamma : \mathbb{R} \to \mathbb{R}^4$ be a smooth map with $\langle \gamma'(s), \gamma'(s)\rangle = 1$, for all $s \in \mathbb{R}$. We look at the normal bundle

$$\gamma^\perp = \big\{(s_0, z) : s_0 \in \mathbb{R},\ z \in \gamma'(s_0)^\perp\big\} \subset \mathbb{R} \times \mathbb{R}^4$$

of the immersion γ and we define a map $\exp : \gamma^\perp \to \mathbb{R}^4$ by setting:

$$\exp(s_0, z) = \gamma(s_0) + z, \quad (s_0, z) \in \gamma^\perp.$$

The map exp is a smooth diffeomorphism from an open neighborhood U_0 of the zero section of $\gamma^\perp$ onto an open neighborhood U of the image of γ in $\mathbb{R}^4$. If A is a measurable subset of U and $f : A \to \mathbb{R}$ is an integrable function (or a measurable function whose either the positive part or the negative part is integrable), then

$$\int_A f(x)\,\mathrm{d}x = \int_{\mathbb{R}} \Big(\int_{A_{s_0}} f\big(\exp(s_0, z)\big)\big(1 - \langle z, \gamma''(s_0)\rangle\big)\,\mathrm{d}z\Big)\,\mathrm{d}s_0, \tag{10.1}$$

where $A_{s_0} \subset \gamma'(s_0)^\perp$ is defined by:

$$A_{s_0} = \big\{z \in \gamma'(s_0)^\perp \cap U_0 : \gamma(s_0) + z \in A\big\}.$$

In equality (10.1), the $\mathrm{d}x$ integral is taken with respect to the measure in $\mathbb{R}^4$ canonically associated to the Minkowski inner product (which is just the standard Lebesgue measure of $\mathbb{R}^4$), the $\mathrm{d}z$ integral is taken with respect to the measure in $\gamma'(s_0)^\perp$ canonically associated to the restriction to $\gamma'(s_0)^\perp$ of the Minkowski inner product and the $\mathrm{d}s_0$ integral is taken with respect to the Lebesgue measure in $\mathbb{R}$. Note that, for any $s_0 \in \mathbb{R}$, the restriction of the Minkowski inner product $\langle\cdot,\cdot\rangle$ to $\gamma'(s_0)^\perp$ is negative definite; we set:

$$\|z\| = \sqrt{-\langle z, z\rangle},$$

for all $z \in \gamma'(s_0)^\perp$.

If $I \subset \mathbb{R}$ is an open interval then, for any $r > 0$, the tube:

$$\mathrm{tube}(I, r) = \big\{(s_0, z) \in \gamma^\perp : s_0 \in I,\ \|z\| \le r\big\}$$

is a smooth submanifold with boundary of the normal bundle $\gamma^\perp$ and its boundary[3] is given by:

$$\partial\,\mathrm{tube}(I, r) = \big\{(s_0, z) \in \gamma^\perp : s_0 \in I,\ \|z\| = r\big\}.$$

If $\mathrm{tube}(I, r)$ is contained in U_0, then $\exp\big[\,\mathrm{tube}(I, r)\big]$ is a smooth submanifold with boundary of $\mathbb{R}^4$ and its boundary is $\exp\big[\partial\,\mathrm{tube}(I, r)\big]$. Assuming that $\mathrm{tube}(I, r)$ is contained in U_0 then, for all $(s_0, z) \in \partial\,\mathrm{tube}(I, r)$, we have that $\frac{1}{r}z$ is the unit normal vector to $\exp\big[\partial\,\mathrm{tube}(I, r)\big]$ at the point $\exp(s_0, z)$ pointing to the outside of $\exp\big[\,\mathrm{tube}(I, r)\big]$. Thus, if $X : A \to \mathbb{R}^4$ is a vector field defined on a measurable subset A of $\exp\big[\partial\,\mathrm{tube}(I, r)\big]$, then its flow (with respect to the transverse orientation of $\exp\big[\partial\,\mathrm{tube}(I, r)\big]$ pointing to the outside of $\exp\big[\,\mathrm{tube}(I, r)\big]$) is given by:

$$\int_A -\frac{1}{r}\langle X(x), z(x)\rangle\,\mathrm{d}x,$$

where $z(x)$ is defined by $\big(s_0(x), z(x)\big) = (\exp|_{U_0})^{-1}(x)$ and the $\mathrm{d}x$ integral is taken with respect to the measure on the manifold $\exp\big[\partial\,\mathrm{tube}(I, r)\big]$ corresponding to the Lorentzian metric given by the restriction of the Minkowski inner product of $\mathbb{R}^4$. The minus sign inside the integral comes from the fact that $\langle \frac{z(x)}{r}, \frac{z(x)}{r}\rangle = -1$. We note that if $f : A \to \mathbb{R}$ is an integrable function (or a measurable function whose either the positive part or the negative part is integrable), then formula (10.1) still holds if the $\mathrm{d}z$ integral is taken with

[3] Of course, the boundary of $\mathrm{tube}(I, r)$ as a subset of the topological space $\gamma^\perp$ is the union of $\partial\,\mathrm{tube}(I, r)$ with $\big\{(s_0, z) \in \gamma^\perp : s_0 \text{ an endpoint of } I,\ \|z\| \le r\big\}$.

respect to the measure on the sphere $\big\{z \in \gamma'(s_0)^\perp : \|z\| = r\big\}$ corresponding to the Riemannian metric given by the restriction of (minus) the Minkowski inner product of $\mathbb{R}^4$. Finally, note that, if I is bounded, then $\mathrm{tube}(I, r)$ is contained in U_0, for $r > 0$ sufficiently small.

11. Asymptotic expansions of Liénard-Wiechert fields

We consider the context of Subsection 9.1 and we give an asymptotic expansion for the electromagnetic field F and its corresponding energy-momentum tensor T in a tubular neighborhood of the worldline γ. For $x \in U$ (see Section 10), we set:

$$\big(s_0(x), z(x)\big) = (\exp|_{U_0})^{-1}(x),$$

and we abbreviate $s_0(x)$ by s_0 and $z(x)$ by z. Given $p \in \mathbb{R}$, we denote by $O(\|z\|^p)$ a (possibly vector-valued) function defined in a neighborhood of $\mathrm{Im}(\gamma)$, possibly minus $\mathrm{Im}(\gamma)$ itself, whose norm is bounded by a constant times $\|z\|^p$ in some neighborhood of any compact subset of $\mathrm{Im}(\gamma)$ (the constant might depend on the compact subset, of course). Recall that $\|z\|$ is defined as $\sqrt{-\langle z, z\rangle}$. We write the electromagnetic field F as:

$$F = F^{[-2]} + F^{[-1]} + F^{[0]},$$

with $F^{[p]}$ of order $O(\|z\|^p)$, $p = -2, -1, 0$. Using (A.2) and Taylor expansions one obtains the following explicit formulas for $F^{[p]}$:

$$F^{[-2]}(x) = q\big(1 - \langle z, \gamma''(s_0)\rangle\big)^{-\frac{1}{2}}\|z\|^{-3}\big(\langle z, \cdot\rangle \wedge \langle \gamma'(s_0), \cdot\rangle\big),$$

$$F^{[-1]}(x) = \frac{q}{2}\big(1 - \langle z, \gamma''(s_0)\rangle\big)^{-\frac{3}{2}}\|z\|^{-1}\big(\langle \gamma'(s_0), \cdot\rangle \wedge \langle \gamma''(s_0), \cdot\rangle\big),$$

$$\begin{aligned} F^{[0]}(x) = &-\frac{q}{2}\|z\|^{-1}\big(\langle z, \cdot\rangle \wedge \langle \gamma'''(s_0), \cdot\rangle\big) - \frac{2}{3}q\big(\langle \gamma'(s_0), \cdot\rangle \wedge \langle \gamma'''(s_0), \cdot\rangle\big) \\ &- \frac{q}{8}\langle \gamma''(s_0), \gamma''(s_0)\rangle\|z\|^{-1}\big(\langle z, \cdot\rangle \wedge \langle \gamma'(s_0), \cdot\rangle\big) + O(\|z\|). \end{aligned}$$

In order to write the asymptotic expansion of the energy-momentum tensor, it is convenient to define the symmetric product

$$\dot{x} \vee \dot{y} = \dot{x} \otimes \dot{y} + \dot{y} \otimes \dot{x},$$

of two vectors $\dot{x}, \dot{y} \in \mathbb{R}^4$. We can now write the energy-momentum tensor T as:

$$T = T^{[-4]} + T^{[-3]} + T^{[-2]},$$

with $T^{[p]}$ of order $O(\|z\|^p)$, $p = -4, -3, -2$. The explicit formulas are the following:

$$\begin{aligned} T^{[-4]}(x) = \frac{q^2}{4\pi}\big(1 - \langle z, \gamma''(s_0)\rangle\big)^{-1}\Big(&\|z\|^{-4}\big(\gamma'(s_0) \otimes \gamma'(s_0)\big) - \|z\|^{-6}(z \otimes z) \\ &- \frac{1}{2}\|z\|^{-4}\langle \cdot, \cdot\rangle\Big), \end{aligned}$$

$$T^{[-3]}(x) = \frac{q^2}{4\pi}\big(1 - \langle z, \gamma''(s_0)\rangle\big)^{-1}\|z\|^{-4}\Big(\langle z, \gamma''(s_0)\rangle\big(\gamma'(s_0)\otimes\gamma'(s_0)\big) + \frac{1}{2}\big(z \vee \gamma''(s_0)\big) - \frac{1}{2}\langle z, \gamma''(s_0)\rangle\langle\cdot,\cdot\rangle\Big),$$

$$\begin{aligned} T^{[-2]}(x) = \frac{q^2}{4\pi}\Big[&\langle z, \gamma''(s_0)\rangle\|z\|^{-4}\Big(\langle z, \gamma''(s_0)\rangle\big(\gamma'(s_0)\otimes\gamma'(s_0)\big) \\ &+ \frac{1}{2}\big(z\vee\gamma''(s_0)\big) - \frac{1}{2}\langle z, \gamma''(s_0)\rangle\langle\cdot,\cdot\rangle\Big) + \frac{2}{3}\|z\|^{-3}\langle z, \gamma'''(s_0)\rangle\langle\cdot,\cdot\rangle \\ &- \frac{1}{4}\|z\|^{-2}\langle\gamma''(s_0), \gamma''(s_0)\rangle\langle\cdot,\cdot\rangle - \frac{3}{4}\langle\gamma''(s_0), \gamma''(s_0)\rangle\|z\|^{-4}(z\otimes z) \\ &- \Big(\frac{1}{2}\|z\|^{-4}\langle z, \gamma'''(s_0)\rangle + \frac{2}{3}\|z\|^{-3}\langle\gamma''(s_0), \gamma''(s_0)\rangle\Big)\big(z\vee\gamma'(s_0)\big) \\ &- \frac{2}{3}\|z\|^{-3}\big(z\vee\gamma'''(s_0)\big) - \Big(\frac{4}{3}\|z\|^{-3}\langle z, \gamma'''(s_0)\rangle \\ &+ \frac{1}{2}\|z\|^{-2}\langle\gamma''(s_0), \gamma''(s_0)\rangle\Big)\big(\gamma'(s_0)\otimes\gamma'(s_0)\big) \\ &- \frac{1}{2}\|z\|^{-2}\big(\gamma'(s_0)\vee\gamma'''(s_0)\big) - \frac{1}{4}\|z\|^{-2}\big(\gamma''(s_0)\otimes\gamma''(s_0)\big)\Big] \\ &+ O(\|z\|^{-1}). \end{aligned}$$

11.1. **Turning the energy-momentum tensor into a distribution.** The energy-momentum tensor T of the Liénard-Wiechert field is not a locally integrable function in $\mathbb{R}^4$ and thus it does not in principle define a (Schwartz) distribution over $\mathbb{R}^4$. More precisely, given a test function $\varphi : \mathbb{R}^4 \to \mathbb{R}^{4^*} \otimes \mathbb{R}^{4^*}$ (smooth, with compact support), the integral

$$\int_{\mathbb{R}^4} \langle T(x), \varphi(x)\rangle \,\mathrm{d}x$$

is not always well-defined. The bracket $\langle\cdot,\cdot\rangle$ inside the integral is given by

$$\langle \dot{x}\otimes\dot{y}, \alpha\otimes\beta\rangle = \alpha(\dot{x})\beta(\dot{y}), \quad \dot{x}, \dot{y} \in \mathbb{R}^4,\ \alpha, \beta \in \mathbb{R}^{4^*},$$

and the $\mathrm{d}x$ integral is taken with respect to the measure in $\mathbb{R}^4$ canonically associated to the Minkowski inner product (which is just the standard Lebesgue measure of $\mathbb{R}^4$). We turn T into a distribution, also denoted by T, by taking its value $\langle\!\langle T, \varphi\rangle\!\rangle$ at a test function $\varphi : \mathbb{R}^4 \to \mathbb{R}^{4^*}\otimes\mathbb{R}^{4^*}$ to be the limit:

$$\begin{aligned} (11.1)\quad \langle\!\langle T, \varphi\rangle\!\rangle = \lim_{r\to 0^+}\Big(&\int_{\exp\left[\,\mathrm{tube}(I,r)\right]^{\mathrm{c}}} \langle T(x), \varphi(x)\rangle\,\mathrm{d}x \\ &- \frac{q^2}{r}\int_{\mathbb{R}}\left[\tfrac{1}{2}\varphi\big(\gamma(s_0)\big)\big(\gamma'(s_0), \gamma'(s_0)\big) + \tfrac{1}{6}\operatorname{tr}\big(\varphi\big(\gamma(s_0)\big)|_{\gamma'(s_0)^\perp}\big)\right]\mathrm{d}s_0\Big), \end{aligned}$$

where $I \subset \mathbb{R}$ denotes a bounded open interval containing the inverse image by γ of the support of φ, $\exp\big[\operatorname{tube}(I,r)\big]^{\mathrm{c}}$ denotes the complement in $\mathbb{R}^4$ of $\exp\big[\operatorname{tube}(I,r)\big]$ (see Section 10), elements of $\mathbb{R}^{4*} \otimes \mathbb{R}^{4*}$ are identified with bilinear forms over $\mathbb{R}^4$ as in (5.1) and, for $\mathcal{B} \in \mathbb{R}^{4*} \otimes \mathbb{R}^{4*}$, the restricted trace $\operatorname{tr}(\mathcal{B}|_{\gamma'(s_0)^\perp})$ is defined by:

$$\operatorname{tr}(\mathcal{B}|_{\gamma'(s_0)^\perp}) = \text{trace of } L,$$

with $L : \gamma'(s_0)^\perp \to \gamma'(s_0)^\perp$ the linear operator such that

$$-\langle L(\dot{x}), \dot{y}\rangle = \mathcal{B}(\dot{x}, \dot{y}), \quad \dot{x}, \dot{y} \in \gamma'(s_0)^\perp$$

and $\langle \cdot, \cdot \rangle$ the Minkowski inner product (note that the restriction of $-\langle \cdot, \cdot \rangle$ to $\gamma'(s_0)^\perp$ is positive definite). We observe that the map γ is proper, so that the inverse image of the support of φ by γ is compact. Moreover, if I' is a bounded open interval containing I, then $\exp\big[\operatorname{tube}(I',r)\big] \setminus \exp\big[\operatorname{tube}(I,r)\big]$ is disjoint from the support of φ for $r > 0$ sufficiently small, so that the limit in (11.1) is independent of the choice of I. Using the asymptotic expansion of T and the material from Section 10 one shows that this limit always exists and yields a distribution.

The divergence $\operatorname{div} T$ of T, as a distribution, is an $\mathbb{R}^4$-valued distribution which has to be paired with an $\mathbb{R}^{4*}$-valued test function $\varphi : \mathbb{R}^4 \to \mathbb{R}^{4*}$. We have:

$$\langle\!\langle \operatorname{div} T, \varphi \rangle\!\rangle = -\langle\!\langle T, \mathrm{d}\varphi \rangle\!\rangle,$$

for any test function $\varphi : \mathbb{R}^4 \to \mathbb{R}^{4*}$, where the *standard* (not exterior) differential $\mathrm{d}\varphi$ is the function $\mathrm{d}\varphi : \mathbb{R}^4 \to \mathbb{R}^{4*} \otimes \mathbb{R}^{4*}$ given by:

$$\mathrm{d}\varphi(x)(\dot{x}, \dot{y}) = \frac{\partial \varphi}{\partial \dot{x}}(x)(\dot{y}), \quad x \in \mathbb{R}^4, \ \dot{x}, \dot{y} \in \mathbb{R}^4,$$

with $\frac{\partial \varphi}{\partial \dot{x}}(x) \in \mathbb{R}^{4*}$ the directional derivative of φ at the point x, in the direction of $\dot{x}$. Using the divergence theorem and the asymptotic expansion for T, one obtains the explicit formula:

$$\langle\!\langle \operatorname{div} T, \varphi \rangle\!\rangle = -\frac{2}{3}\, q^2 \int_{\mathbb{R}} \big[\langle \gamma''(s_0), \gamma''(s_0)\rangle \varphi\big(\gamma(s_0)\big)\big(\gamma'(s_0)\big) + \varphi\big(\gamma(s_0)\big)\big(\gamma'''(s_0)\big)\big] \,\mathrm{d}s_0,$$

for any test function $\varphi : \mathbb{R}^4 \to \mathbb{R}^{4*}$.

Appendix A. More on retarded time

Let $f : I \to \mathbb{R}$ be a smooth map defined in a open interval I containing the origin and assume that $f(0) = f'(0) = 0$ and $f''(0) > 0$. We have that $f|_{]0,\varepsilon[}$ has positive derivative for some small $\varepsilon > 0$, so that we can define:

$$g = \big(f|_{[0,\varepsilon[}\big)^{-1}.$$

The map g is continuous on $f\big([0,\varepsilon[\big)$ and smooth on $f\big(]0,\varepsilon[\big)$, but it is not differentiable at zero. Nevertheless, we can write $g(y) = h(\sqrt{y}\,)$, for a smooth map h. Namely, we have $f(x) = x^2 u(x)$, for a smooth map $u : I \to \mathbb{R}$ with $u(0) > 0$ and thus, on some neighborhood of zero in $[0, +\infty[$, the square root of f coincides with the smooth map

$$x \longmapsto x\sqrt{u(x)}.$$

Since the derivative of such map at $x = 0$ is nonzero, it admits a smooth inverse h on a neighborhood of zero. Clearly, we have $g(y) = h(\sqrt{y}\,)$ for $y \geq 0$ close to zero. By calculating explicitly the Taylor polynomial of h up to order 3 in terms of the derivatives of f, one obtains:

$$\text{(A.1)}\quad g(y) = \sqrt{\frac{2}{f''(0)}}\,\sqrt{y} - \frac{1}{3}\frac{f'''(0)}{f''(0)^2}\,y + \Big(\frac{2}{f''(0)}\Big)^{\frac{3}{2}}\Big(\frac{5}{72}\frac{f'''(0)^2}{f''(0)^2} - \frac{1}{24}\frac{f^{(4)}(0)}{f''(0)}\Big)y^{\frac{3}{2}} + O(y^2),$$

where $f^{(4)}$ denotes the fourth derivative of f and $O(y^2)$ denotes a map that is bounded by a constant times y^2 in a neighborhood of zero.

Consider now the context of Subsection 9.1. For x in the neighborhood U of the worldline γ (see Section 10), we will obtain an asymptotic expansion of the retarded proper time $s_{\text{ret}}(x)$ (abbreviated as s_{ret}) in terms of

$$\big(s_0(x), z(x)\big) = (\exp|_{U_0})^{-1}(x).$$

As in Section 11, we abbreviate $s_0(x)$ by s_0 and $z(x)$ by z. We have:

$$\big\langle x - \gamma\big(s_{\text{ret}}(x)\big), x - \gamma\big(s_{\text{ret}}(x)\big)\big\rangle = \big\langle z - \big(\gamma(s_{\text{ret}}) - \gamma(s_0)\big), z - \big(\gamma(s_{\text{ret}}) - \gamma(s_0)\big)\big\rangle = 0$$

and $s_{\text{ret}} \leq s_0$. Thus, for fixed s_0, if we set:

$$f_z(a) = \langle \gamma(s_0 - a) - \gamma(s_0), \gamma(s_0 - a) - \gamma(s_0)\rangle - 2\langle z, \gamma(s_0 - a) - \gamma(s_0)\rangle,$$

then:

$$f_z(s_0 - s_{\text{ret}}) = -\langle z, z\rangle = \|z\|^2.$$

Moreover $f_z(0) = f_z'(0)$ and

$$f_z''(0) = 2\big(1 - \langle z, \gamma''(s_0)\rangle\big) > 0,$$

for small z. If we define g_z from f_z as g was defined from f in the first part of the appendix, we obtain:

$$s_{\text{ret}} = s_0 - g_z(\|z\|^2).$$

Using (A.1) we now get the following asymptotic expansion for s_{ret}:

$$\text{(A.2)}\quad s_{\text{ret}} = s_0 + \big(1 - \langle z, \gamma''(s_0)\rangle\big)^{-\frac{1}{2}}\Big(-\|z\| + \frac{1}{6}\langle z, \gamma'''(s_0)\rangle\|z\|^2 - \frac{1}{24}\langle \gamma''(s_0), \gamma''(s_0)\rangle\|z\|^3\Big) + O(\|z\|^4).$$

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