PROOF OF ZORN'S LEMMA

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By a *partially ordered set* we mean a set X endowed with a *partial order*, i.e., a binary relation $\leq$ that is reflexive ($x \leq x$, for all $x \in X$), anti-symmetric ($x \leq y$ and $y \leq x$ imply $x = y$, for all $x, y \in X$) and transitive ($x \leq y$ and $y \leq z$ imply $x \leq z$, for all $x, y, z \in X$). A subset of a partially ordered set X will always be considered partially ordered by the restriction of the partial order of X. As usual, we write $x < y$ if both $x \leq y$ and $x \neq y$. Given a subset S of X, we say that $x \in X$ is an *upper bound* (resp., *strict upper bound*) for S if $y \leq x$ (resp., $y < x$) for all $y \in S$. We say that the partial order $\leq$ is *total* if $x \leq y$ or $y \leq x$, for all $x, y \in X$ and that X is *well-ordered* by $\leq$ if every nonempty subset S of X has a *smallest element*, i.e., an element $x \in S$ with $x \leq y$, for all $y \in S$. Obviously, if X is well-ordered then it is also totally ordered. In a well-ordered set, one can do *proofs by transfinite induction*: given a subset P of X, to prove that $P = X$, show that for all $x \in X$, if $(\cdot, x) \subset P$, then $x \in P$, where

$$(\cdot, x) = \big\{y \in X : y < x\big\}.$$

A subset S of X is called an *initial part* of X if for every $x \in S$ we have $(\cdot, x) \subset S$. Obviously, an initial part of an initial part of X is again an initial part of X. If X is well-ordered, then an initial part S of X is either X itself or it is equal to $(\cdot, x)$, for some (automatically unique) $x \in X$. Namely, x is the smallest element of $X \setminus S$. The following result is an immediate consequence of this observation.

Lemma 1. *If X is well-ordered and if S_1 and S_2 are initial parts of X, then either $S_1 \subset S_2$ or $S_2 \subset S_1$.* □

Given partially ordered sets X and Y, we call a map $f : X \to Y$ an *order-isomorphism* if f is bijective and

$$x_1 \leq x_2 \Longleftrightarrow f(x_1) \leq f(x_2),$$

for all $x_1, x_2 \in X$. We say that X and Y are *order-isomorphic* if there exists an order isomorphism from X onto Y.

Lemma 2. *Given well-ordered sets X and Y, then there exists at most one map $f : X \to Y$ whose image $f[X]$ is an initial part of Y and such that $f : X \to f[X]$ is an order-isomorphism.*

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Proof. If f is such a map, then it is easy to see that, for all $x \in X$, $f(x)$ equals the smallest element of $Y \setminus f\big[(\cdot, x)\big]$. It then follows by transfinite induction that if f and g are two such maps then $f(x) = g(x)$, for all $x \in X$. □

Lemma 3. *Given well-ordered sets X and Y, then either X is order-isomorphic to an initial part of Y or Y is order-isomorphic to an initial part of X.*

Proof. Let $\mathcal{F}$ denote the set of all maps that are order-isomorphisms from an initial part of X onto an initial part of Y. It follows directly from Lemmas 1 and 2 that $\mathcal{F}$ is totally ordered by inclusion, i.e., that for $f_1, f_2 \in \mathcal{F}$ we have that either f_2 extends f_1 or f_1 extends f_2. Thus, taking f to be the union of $\mathcal{F}$, we have that $f : X' \to Y'$ is an order-isomorphism from an initial part X' of X onto an initial part Y' of Y. If either $X' = X$ or $Y' = Y$, we are done. Otherwise, let x be the smallest element of $X \setminus X'$ and y be the smallest element of $Y \setminus Y'$. We now obtain $g \in \mathcal{F}$ properly extending f by setting $g(x) = y$, which yields a contradiction. □

An element x of a partially ordered set X is called *maximal* if for every $y \in X$ it is not the case that $x < y$.

Theorem 4 (Zorn)**.** *Let X be a partially ordered set such that every totally ordered subset of X has an upper bound in X. Then X has a maximal element.*

To prove Theorem 4, we start by fixing a map $\phi : \wp(X) \setminus \{\emptyset\} \to X$ such that $\phi(S) \in S$, for every nonempty subset S of X (i.e., a *choice function* for X). We need an auxiliary definition.

Definition 5. A subset S of X will be called *special* (with respect to the choice function ϕ) if S is well-ordered by the (restriction of the) partial order of X and if, for every $x \in S$, we have $x = \phi(U)$, where U is the set of strict upper bounds in X of the set $(\cdot, x) \cap S$. (Note that U is nonempty, since $x \in U$.)

Lemma 6. *If S_1 and S_2 are special subsets of X, then either S_1 is an initial part of S_2 or S_2 is an initial part of S_1.*

Proof. By Lemma 3, either S_1 is order-isomorphic to an initial part of S_2 or S_2 is order-isomorphic to an initial part of S_1. Assume, for instance, that we have a map $f : S_1 \to f[S_1]$ that is an order-isomorphism onto an initial part $f[S_1]$ of S_2. It suffices to show that f must be the inclusion map. We proceed by transfinite induction. Let $x \in S_1$ and assume that $f(y) = y$ for all $y \in (\cdot, x) \cap S_1$. It follows that $f[(\cdot, x) \cap S_1] = (\cdot, x) \cap S_1$ is an initial part of $f[S_1]$ and that $f(x)$ is the smallest element in $f[S_1]$ not in $(\cdot, x) \cap S_1$. Since $f[S_1]$ is an initial part of S_2, we obtain that $(\cdot, x) \cap S_1$ is an initial part of S_2 and that $f(x)$ is the smallest element in S_2 not in $(\cdot, x) \cap S_1$. Thus:

$$(\cdot, x) \cap S_1 = \big(\cdot, f(x)\big) \cap S_2. \tag{1}$$

The fact that S_1 and S_2 are special now implies that both $f(x)$ and x equal $\phi(U)$, where U is the set of strict upper bounds in X of the set on either side of equality (1). □

Proof of Theorem 4. Let S denote the union of all special subsets of X. It follows easily from Lemma 6 that every special subset of X is an initial part of S and then that S is itself special. Let U denote the set of strict upper bounds of S in X. If U were nonempty, then setting $x = \phi(U)$ we would obtain that $S \cup \{x\}$ is a special subset of X properly containing S. Thus S has no strict upper bounds and hence any upper bound of S must be a maximal element of X. □

Remark. We have actually proven Theorem 4 under the weaker assumption that every *well-ordered* subset of X has an upper bound in X.