WEAK* TOPOLOGY FOR THE SPACE OF FINITE MEASURES ON A TOPOLOGICAL SPACE

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Contents

1. Introduction

The goal of these notes is to present the basic theory of what probabilists and statisticians call "weak convergence of probability measures" to an audience of readers that has a basic training in Measure Theory, Functional Analysis and General Topology. For the readers convenience, we include six appendices with short summaries of some of the relevant mathematical prerequisites.

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To avoid confusion with the notion of weak convergence on Banach spaces that is used in the context of Functional Analysis we will not use the terminology preferred by probabilists and statisticians and we will use "weak* convergence" instead. Unlike expositions that are more oriented at probability theory, we will state results for arbitrary finite measures and even signed measures whenever possible. Moreover, results will be stated for arbitrary topological spaces (satisfying the necessary assumptions) rather than just metric spaces and we will consider the convergence of arbitrary nets rather than just sequences.

2. Notation and measure-theoretic preliminaires

Let $(X, \mathcal{A})$ be a *measurable space*, i.e., X is a set and $\mathcal{A}$ is a σ-algebra of subsets of X. Elements of $\mathcal{A}$ are called *measurable subsets* of X. A map ϕ between measurable spaces is called *measurable* if the inverse image of measurable subsets under ϕ is measurable. By a *measure* μ on $(X, \mathcal{A})$ we will always mean a (possibly signed) countably additive measure defined on $\mathcal{A}$ satisfying the condition $\mu(\emptyset) = 0$; the latter condition holds automatically unless μ is infinite on every element of $\mathcal{A}$. The *positive part*, *negative part* and *total variation* of a signed measure μ on $(X, \mathcal{A})$ are respectively the nonnegative measures μ^+, μ^- and $|\mu|$ on $(X, \mathcal{A})$ defined by

$$\mu^+(A) = \sup\left\{ \sum_{i=1}^n [\mu(A_i)]^+ : A_1, \ldots, A_n \in \mathcal{A} \text{ disjoint subsets of } A,\ n \geq 1 \right\},$$

$$\mu^-(A) = \sup\left\{ \sum_{i=1}^n [\mu(A_i)]^- : A_1, \ldots, A_n \in \mathcal{A} \text{ disjoint subsets of } A,\ n \geq 1 \right\},$$

$$|\mu|(A) = \sup\left\{ \sum_{i=1}^n |\mu(A_i)| : A_1, \ldots, A_n \in \mathcal{A} \text{ disjoint subsets of } A,\ n \geq 1 \right\},$$

for all $A \in \mathcal{A}$, where as usual for $t \in [-\infty, +\infty]$ we denote by t^+ the maximum between t and 0 and by t^- the maximum between $-t$ and 0. The measure μ, its total variation and positive and negative parts are related by the identities:

$$\mu = \mu^+ - \mu^-, \quad |\mu| = \mu^+ + \mu^-.$$

A signed measure μ on $(X, \mathcal{A})$ always admits a *Hahn decomposition* which is a pair (P, N) of disjoint measurable subsets of X whose union is X and such that μ is nonnegative on measurable subsets of P and nonpositive on measurable subsets of N. The measures μ^+, μ^- and $|\mu|$ are then given by

$$\mu^+(A) = \mu(A \cap P), \quad \mu^-(A) = -\mu(A \cap N), \quad |\mu|(A) = \mu(A \cap P) - \mu(A \cap N),$$

for all $A \in \mathcal{A}$.

Except for Appendices B and C, we will only be interested in finite measures. We denote by $\mathrm{ca}(X, \mathcal{A})$ the space of finite signed measures on $(X, \mathcal{A})$, by $\mathrm{ca}_+(X,\mathcal{A})$ the subset of $\mathrm{ca}(X,\mathcal{A})$ consisting of finite nonnegative measures and by $\mathrm{ca}_+^1(X,\mathcal{A})$ the set of probability measures on $(X,\mathcal{A})$:

$$\mathrm{ca}_+^1(X,\mathcal{A}) = \big\{\mu \in \mathrm{ca}_+(X,\mathcal{A}) : \mu(X) = 1\big\}.$$

We will often use without further comment that, for a finite measure, a collection of pairwise disjoint measurable sets having positive measure must be countable. This follows from the fact that a collection of pairwise disjoint measurable sets with measure greater than some positive constant must be finite.

The space $\mathrm{ca}(X,\mathcal{A})$ will always be endowed with the *total variation norm*

$$\|\mu\| = |\mu|(X).$$

Such norm makes $\mathrm{ca}(X,\mathcal{A})$ into a Banach space and its subsets $\mathrm{ca}_+(X,\mathcal{A})$ and $\mathrm{ca}_+^1(X,\mathcal{A})$ are both convex and closed with respect to the norm topology.

We will use the bilinear pairing notation $\langle\mu, f\rangle$ for the integral of a $[-\infty,+\infty]$-valued measurable function f with respect to a measure μ, i.e., we set

$$\langle\mu, f\rangle = \int_X f \,\mathrm{d}\mu$$

provided that the integral exists. If $\mathcal{M}_{\mathrm{b}}(X,\mathcal{A})$ denotes the space of bounded real-valued measurable functions on X then $\langle\cdot,\cdot\rangle$ defines a bilinear form

$$\mathrm{ca}(X,\mathcal{A}) \times \mathcal{M}_{\mathrm{b}}(X,\mathcal{A}) \ni (\mu, f) \longmapsto \langle\mu, f\rangle \in \mathbb{R} \tag{2.1}$$

such that

$$|\langle\mu, f\rangle| \le \|\mu\|\,\|f\|_{\sup}, \tag{2.2}$$

for all $\mu \in \mathrm{ca}(X,\mathcal{A})$, $f \in \mathcal{M}_{\mathrm{b}}(X,\mathcal{A})$, where

$$\|f\|_{\sup} = \sup_{x\in X} |f(x)|$$

denotes the supremum norm of f. The space $\mathcal{M}_{\mathrm{b}}(X,\mathcal{A})$ will always be endowed with the supremum norm and such norm makes it into a Banach space. It follows easily from the definition of the total variation norm that the bilinear form $\langle\cdot,\cdot\rangle$ defines a linear isometric embedding

$$\mathrm{ca}(X,\mathcal{A}) \ni \mu \longmapsto \langle\mu,\cdot\rangle \in \mathcal{M}_{\mathrm{b}}(X,\mathcal{A})^*$$

of the Banach space $\mathrm{ca}(X,\mathcal{A})$ into the topological dual of $\mathcal{M}_{\mathrm{b}}(X,\mathcal{A})$. The topological dual of a normed space will always be assumed to be endowed with the usual norm (E.3) given by the supremum of the absolute value over the unit ball. For $\mu \in \mathrm{ca}(X,\mathcal{A})$ we will often also denote by $\langle\mu,\cdot\rangle$ the restriction of the linear functional $\mathcal{M}_{\mathrm{b}}(X,\mathcal{A}) \ni f \mapsto \langle\mu, f\rangle \in \mathbb{R}$ to some subspace of $\mathcal{M}_{\mathrm{b}}(X,\mathcal{A})$ and the domain of $\langle\mu,\cdot\rangle$ will be made clear by the context.

The bilinear form (2.1) also defines a linear isometric embedding

$$\mathcal{M}_{\mathrm{b}}(X,\mathcal{A}) \ni f \longmapsto \langle\cdot, f\rangle \in \mathrm{ca}(X,\mathcal{A})^* \tag{2.3}$$

of the Banach space $\mathcal{M}_{\mathrm{b}}(X,\mathcal{A})$ into the topological dual of $\mathrm{ca}(X,\mathcal{A})$. To see this, note that for every $x \in X$ the *Dirac delta* $\delta_x \in \mathrm{ca}^1_+(X,\mathcal{A})$ defined by

$$\delta_x(A) = \chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in X \setminus A \end{cases} \tag{2.4}$$

for all $A \in \mathcal{A}$, has unit norm and satisfies $\langle \delta_x, f\rangle = f(x)$, for all f in $\mathcal{M}_{\mathrm{b}}(X,\mathcal{A})$. The function $\chi_A : X \to \mathbb{R}$ defined in (2.4) will be called the *characteristic function* of the subset A of X (instead of "indicator function" which is the more usual terminology in the Probability Theory community).

Definition 2.1. A linear functional α defined on some vector subspace of the space of all real-valued functions on a set X is called *positive* if $\alpha(f) \geq 0$ for every nonnegative function f that belongs to the domain of α.

Clearly a measure $\mu \in \mathrm{ca}(X,\mathcal{A})$ is nonnegative if and only if the linear functional $\langle \mu, \cdot\rangle \in \mathcal{M}_{\mathrm{b}}(X,\mathcal{A})^*$ is positive, since $\langle \mu, \chi_A\rangle = \mu(A)$ for every $A \in \mathcal{A}$.

If μ is a measure on $(X,\mathcal{A})$ and f is a $[-\infty,+\infty]$-valued measurable function such that $\langle \mu, f\rangle$ exists, we define a measure $f\mu$ on $(X,\mathcal{A})$ by setting

$$(f\mu)(A) = \int_A f \,\mathrm{d}\mu = \langle \mu, f\chi_A\rangle,$$

for all $A \in \mathcal{A}$. The map

$$\mathcal{M}_{\mathrm{b}}(X,\mathcal{A}) \times \mathrm{ca}(X,\mathcal{A}) \ni (f,\mu) \longmapsto f\mu \in \mathrm{ca}(X,\mathcal{A}) \tag{2.5}$$

is bilinear and the identities

$$\langle f\mu, g\rangle = \langle \mu, fg\rangle, \tag{2.6}$$

$$(gf)\mu = g(f\mu) \tag{2.7}$$

hold for all $\mu \in \mathrm{ca}(X,\mathcal{A})$ and all $f, g \in \mathcal{M}_{\mathrm{b}}(X,\mathcal{A})$, so that (2.5) turns $\mathrm{ca}(X,\mathcal{A})$ into a module over the real algebra $\mathcal{M}_{\mathrm{b}}(X,\mathcal{A})$. Moreover, the total variation $|f\mu|$ is equal to $|f||\mu|$, where $|f|$ denotes the (pointwise) absolute value of f. Thus:

$$\|f\mu\| = \langle |\mu|, |f|\rangle \leq \|\mu\| \, \|f\|_{\sup}.$$

If $\phi : X \to X'$ is a measurable map between measurable spaces $(X,\mathcal{A})$, $(X',\mathcal{A}')$ and μ is a measure on $(X,\mathcal{A})$, we denote by $\phi_*\mu$ the *push-forward* of μ by ϕ which is the measure on $(X',\mathcal{A}')$ defined by

$$(\phi_*\mu)(A) = \mu\big(\phi^{-1}[A]\big),$$

for all $A \in \mathcal{A}'$. The total variation of $\phi_*\mu$ is bounded by the total variation of μ, i.e.

$$|\phi_*\mu|(A) \leq (\phi_*|\mu|)(A), \tag{2.8}$$

for every $A \in \mathcal{A}'$. It follows that ϕ_* defines a bounded linear map

$$\phi_* : \mathrm{ca}(X,\mathcal{A}) \ni \mu \longmapsto \phi_*\mu \in \mathrm{ca}(X',\mathcal{A}')$$

whose operator norm $\|\phi_*\|$ is less than or equal to 1. Moreover, the identity

$$\langle \phi_*\mu, f\rangle = \langle \mu, f\circ\phi\rangle \tag{2.9}$$

holds for all $\mu \in \mathrm{ca}(X,\mathcal{A})$ and $f \in \mathcal{M}_{\mathrm{b}}(X',\mathcal{A}')$.

3. Main definition and topological preliminaires

If (X,τ) is a topological space, we denote by $\mathcal{B}(X)$ its *Borel σ-algebra*, i.e., the σ-algebra generated by the topology τ. The topology will be usually fixed by the context, so we just write X instead of (X,τ). We use $\mathrm{ca}(X)$, $\mathrm{ca}_+(X)$, $\mathrm{ca}_+^1(X)$ and $\mathcal{M}_{\mathrm{b}}(X)$ as abbreviations of $\mathrm{ca}\big(X,\mathcal{B}(X)\big)$, $\mathrm{ca}_+\big(X,\mathcal{B}(X)\big)$, $\mathrm{ca}_+^1\big(X,\mathcal{B}(X)\big)$ and $\mathcal{M}_{\mathrm{b}}\big(X,\mathcal{B}(X)\big)$, respectively, and we denote by $C_{\mathrm{b}}(X)$ the closed subspace of the Banach space $\mathcal{M}_{\mathrm{b}}(X)$ consisting of bounded continuous functions. The bilinear pairing (2.1) between $\mathrm{ca}(X)$ and $\mathcal{M}_{\mathrm{b}}(X)$ restricts to a bilinear pairing between $\mathrm{ca}(X)$ and $C_{\mathrm{b}}(X)$ and it induces a bounded linear map

$$\mathrm{ca}(X) \ni \mu \longmapsto \langle \mu, \cdot\rangle \in C_{\mathrm{b}}(X)^* \tag{3.1}$$

taking values in the topological dual $C_{\mathrm{b}}(X)^*$ of the Banach space $C_{\mathrm{b}}(X)$.

Definition 3.1. Let X be a topological space and let the topological dual $C_{\mathrm{b}}(X)^*$ of the Banach space $C_{\mathrm{b}}(X)$ be endowed with its weak* topology. By the *weak* topology* on $\mathrm{ca}(X)$ we mean the topology induced by the linear map (3.1), i.e., the smallest topology that makes (3.1) continuous.

The weak* topology on $\mathrm{ca}(X)$ is simply the topology induced by the linear functionals $\langle\cdot, f\rangle \in \mathrm{ca}(X)^*$ with f varying over $C_{\mathrm{b}}(X)$, i.e., it is the smallest topology that makes all such linear functionals continuous. In the terminology of Appendix E, the weak* topology of $\mathrm{ca}(X)$ is the weak topology induced by the bilinear pairing $\langle\cdot,\cdot\rangle$ of $\mathrm{ca}(X)$ and $C_{\mathrm{b}}(X)$ given by the restriction of (2.1). Such topology is characterized by the fact that a net $(\mu_i)_{i\in I}$ in $\mathrm{ca}(X)$ weak*-converges to a measure $\mu \in \mathrm{ca}(X)$ if and only if $\lim_{i\in I}\langle \mu_i, f\rangle = \langle \mu, f\rangle$ for all $f \in C_{\mathrm{b}}(X)$.

At this level of generality, the weak* topology on $\mathrm{ca}(X)$ can be quite trivial as, for instance, all real-valued continuous functions on X could be constant. The weak* topology always makes $\mathrm{ca}(X)$ into a topological vector space, but it is Hausdorff if and only if the linear map (3.1) is injective (see Remark E.2). For such map to be injective the space X needs to have lots of continuous real-valued functions. So, for instance, it is usually useful to assume that X be *normal*, i.e., that disjoint closed sets can be separated by open sets. Under this assumption, Urysohn's Lemma can be used to construct many real-valued continuous functions. Sometimes it is also useful to assume that X be *perfectly normal*, i.e., that X is normal and every open subset of X is an F_σ. Recall that an *F_σ subset* of a topological space is a subset which is a countable union of closed sets and a *G_δ subset* is a subset that is a countable intersection of open sets.

The condition that every open set is an F_σ (equivalently, that every closed set is a G_δ) is useful to establish regularity conditions for the measures (see Corollary 3.5 below). Probability Theory textbooks tend to focus on the case when X is metrizable or even Polish. Recall that a topological space is called *Polish* if it is completely metrizable (i.e., its topology is induced by a complete metric) and separable. Every metrizable space is perfectly normal.

Although we are not going to put much extra effort to find the absolute most general topological spaces in which the results that we prove are valid, we also think it is silly to put unnecessary assumptions on our statements that are not at all used in the proofs.

Definition 3.2. Let X be a topological space. If $\mu \in \mathrm{ca}_+(X)$ and A is a Borel subset of X, we say that A is *μ-regular* (or *regular with respect to* μ) if it can be μ-approximated internally by closed sets and externally by open sets, i.e., if for every $\varepsilon > 0$ there exists F closed in X and U open in X with $F \subset A \subset U$, $\mu(A \setminus F) < \varepsilon$ and $\mu(U \setminus A) < \varepsilon$. We will say that $\mu \in \mathrm{ca}(X)$ is *regular* if every Borel subset of X is regular with respect to $|\mu|$.

Note that by taking complements one obtains that the condition that every Borel subset of X can be μ-approximated internally by closed sets is equivalent to the condition that every Borel subset of X can be μ-approximated externally by open sets, so that only one of these conditions needs to be checked to prove that a measure $\mu \in \mathrm{ca}_+(X)$ is regular. Moreover, since μ is nonnegative and finite, the condition that a Borel subset A of X is μ-regular is equivalent to the conjunction of the following two equalities:

$$\mu(A) = \inf\big\{\mu(U) : U \subset X \text{ open}, A \subset U\big\}, \tag{3.2}$$

$$\mu(A) = \sup\big\{\mu(F) : F \subset X \text{ closed}, F \subset A\big\}. \tag{3.3}$$

Remark 3.3. The definition of a regular measure lacks complete standardization, exhibiting variations in the literature. For example, some authors will replace "$F \subset X$ closed" in (3.3) with "$F \subset X$ compact". If X is locally compact and Hausdorff then sometimes it is convenient to call "regular" what we call "Radon" in Appendix C. If X is compact and Hausdorff and the measure is finite, all reasonable definitions of regularity agree (see Subsection C.1).

Lemma 3.4. *If X is a topological space and $\mu \in \mathrm{ca}_+(X)$ then the collection of all μ-regular Borel subsets of X is a σ-algebra.*

Proof. If A is μ-regular and for some $\varepsilon > 0$ the open set U and the closed set F witness the μ-regularity condition of A for that ε then the open set $X \setminus F$ and the closed set $X \setminus U$ witness the μ-regularity condition of $X \setminus A$ for the same ε. Moreover, if $(A_n)_{n\geq 1}$ is a sequence of μ-regular Borel subsets and for some $\varepsilon > 0$ the open set U_n and the closed F_n witness the μ-regularity condition of A_n for $\frac{\varepsilon}{2^n}$ then the open set $U = \bigcup_{n=1}^\infty U_n$ and the closed set $F = \bigcup_{n=1}^N F_n$ witness the μ-regularity condition of $A = \bigcup_{n=1}^\infty A_n$ for ε if N is large enough, since $\lim_{N\to+\infty} \mu\big(\bigcup_{n=1}^N F_n\big) = \mu\big(\bigcup_{n=1}^\infty F_n\big)$. □

Corollary 3.5. *If X is a topological space for which every open set is an F_σ then every measure $\mu \in \mathrm{ca}(X)$ is regular.*

Proof. Simply note that if $\mu \in \mathrm{ca}_+(X)$ then every open set that is an F_σ is μ-regular and thus our assumptions and Lemma 3.4 yield that every Borel subset of X is μ-regular. ☐

Proposition 3.6. *If X is a normal topological space and $\mu \in \mathrm{ca}(X)$ is regular then the norm of μ is equal to the norm of the linear functional $\langle \mu, \cdot \rangle \in C_{\mathrm{b}}(X)^*$, i.e.:*

$$\|\mu\| = \sup \big\{ |\langle \mu, f \rangle| : f \in C_{\mathrm{b}}(X),\ \|f\|_{\sup} \le 1 \big\}.$$

In particular, by Corollary 3.5, if X is a perfectly normal topological space then the linear map (3.1) *is an isometric embedding.*

Proof. By (2.2) it is sufficient to show that if $\mu \in \mathrm{ca}(X)$ is regular then for every $\varepsilon > 0$ there exists $f \in C_{\mathrm{b}}(X)$ with $\|f\|_{\sup} \le 1$ and $|\langle \mu, f \rangle| > \|\mu\| - \varepsilon$. If $X = P \cup N$ is a Hahn decomposition for μ then by regularity there exist closed sets $F_+ \subset P$ and $F_- \subset N$ with $|\mu|(P \setminus F_+) < \varepsilon$ and $|\mu|(N \setminus F_-) < \varepsilon$. Moreover, Urysohn's Lemma gives us a continuous function $f : X \to [-1, 1]$ which equals 1 on F_+ and equals -1 on F_-. We have

$$\|\mu\| = \langle \mu, \chi_P - \chi_N \rangle$$

and

$$|f - (\chi_P - \chi_N)| \le 2(\chi_{P \setminus F_+} + \chi_{N \setminus F_-})$$

from which it follows that:

$$\big|\langle \mu, f \rangle - \|\mu\|\big| = \big|\langle \mu, f - (\chi_P - \chi_N) \rangle\big| \le 2|\mu|(P \setminus F_+) + 2|\mu|(N \setminus F_-) < 4\varepsilon. \quad ☐$$

Corollary 3.7. *If X is a perfectly normal topological space then the weak* topology in $\mathrm{ca}(X)$ is Hausdorff.* ☐

Proof. Follows from Proposition 3.6 and Remark E.2. ☐

Example 3.8. Normality of X alone does not ensure even the injectivity of the linear map (3.1). For instance, denote by ω_1 the first uncountable ordinal and let the ordinal segment $X = [0, \omega_1]$ be endowed with the order topology. We have that X is a compact Hausdorff topological space and in particular it is normal. If $\nu \in \mathrm{ca}^1_+(X)$ is the nonregular probability measure defined in Example C.8 then $\langle \nu, f \rangle = \langle \delta_{\omega_1}, f \rangle$ for every continuous function $f : X \to \mathbb{R}$ and therefore (3.1) is not injective on probability measures and the weak* topology on $\mathrm{ca}^1_+(X)$ is not Hausdorff.

Proposition 3.9. *If X is a normal topological space and $\mu \in \mathrm{ca}(X)$ is a regular measure then μ is nonnegative if and only if the linear functional $\langle \mu, \cdot \rangle \in C_{\mathrm{b}}(X)^*$ is positive. In particular, by Corollary 3.5, if X is a perfectly normal topological space then*

$$\mathrm{ca}_+(X) = \big\{ \mu \in \mathrm{ca}(X) : \langle \mu, \cdot \rangle \in C_{\mathrm{b}}(X)^* \text{ is positive} \big\}$$

and hence $\mathrm{ca}_+(X)$ is weak-closed in $\mathrm{ca}(X)$.*

Proof. Assume that $\mu \in \mathrm{ca}(X)$ is regular and that $\langle \mu, \cdot \rangle \in C_{\mathrm{b}}(X)^*$ is positive. If $\mu \notin \mathrm{ca}_+(X)$ and $X = P \cup N$ is a Hahn decomposition for μ then $\mu(N) < 0$. By regularity, for every $\varepsilon > 0$ we obtain $F \subset N$ closed and $U \supset N$ open with $|\mu|(F \setminus N) < \varepsilon$ and $|\mu|(U \setminus N) < \varepsilon$. Now Urisohn's Lemma gives $f : X \to [0, 1]$ continuous which equals 1 on F and vanishes outside of U. This implies $|f - \chi_N| \leq \chi_{U \setminus F}$ and:

$$\big|\langle \mu, f \rangle - \mu(N)\big| = \big|\langle \mu, f - \chi_N \rangle\big| \leq |\mu|(U \setminus F) < 2\varepsilon.$$

But we could have chosen $\varepsilon > 0$ with $\mu(N) + 2\varepsilon \leq 0$ which would yield $\langle \mu, f \rangle < 0$, contradicting our assumptions. □

4. Basic properties of the weak* topology

We start with a very simple result which says that with respect to the weak* topology of $\mathrm{ca}(X)$ the Dirac delta δ_x varies continuously with x in X. This is an important property for practical applications to Statistics as if two points of some sample space are very close and thus "empirically indistinguishable" then probability measures associated to them should also be close. Note that the situation is very different with the norm topology, as $\|\delta_x - \delta_y\| = 2$ for distinct points $x, y \in X$ if the topological space X is Hausdorff (or if X at least satisfies the weakest separation axiom T0). Similarly, with respect to the weak topology of the Banach space $\mathrm{ca}(X)$ we have that $\big\{\mu \in \mathrm{ca}(X) : \mu\big(\{x\}\big) > \frac{1}{2}\big\}$ is an open neighborhood of δ_x that does not contain any δ_y with $y \neq x$, provided that $\{x\}$ is a Borel subset of X. Thus, if X is Hausdorff, the set $\{\delta_x : x \in X\}$ is discrete both in the weak and in the norm topology of the Banach space $\mathrm{ca}(X)$.

Proposition 4.1. *If X is a topological space then the map*

$$\delta : X \ni x \longmapsto \delta_x \in \mathrm{ca}(X)$$

is continuous with respect to the weak topology of* $\mathrm{ca}(X)$.

Proof. Since the weak* topology of $\mathrm{ca}(X)$ is induced by the linear functionals $\langle \cdot, f \rangle$ with $f \in C_{\mathrm{b}}(X)$, it suffices to show that the composition of δ with all such linear functionals is continuous. But this follows trivially from the equality $\langle \delta_x, f \rangle = f(x)$. □

Corollary 4.2. *Let X be a topological space and $f \in \mathcal{M}_{\mathrm{b}}(X)$ be a bounded Borel measurable function. The following statements are equivalent:*

(a) *f is continuous;*
(b) *the map* $\mathrm{ca}(X) \ni \mu \mapsto \langle \mu, f \rangle \in \mathbb{R}$ *is weak*-continuous;*
(c) *the map* $\mathrm{ca}_+(X) \ni \mu \mapsto \langle \mu, f \rangle \in \mathbb{R}$ *is weak*-continuous;*
(d) *the map* $\mathrm{ca}_+^1(X) \ni \mu \mapsto \langle \mu, f \rangle \in \mathbb{R}$ *is weak*-continuous.*

Proof. Obviously (a)⇒(b)⇒(c)⇒(d) and (d)⇒(a) follows by noting that f is the composition of the map $\delta : X \to \mathrm{ca}_+^1(X)$ with the map in (d). □

Corollary 4.3. *Let X be a topological space and B be a Borel subset of X. The following statements are equivalent:*

(a) *B is clopen (i.e., closed and open) in X;*
(b) *the map $\mathrm{ca}(X) \ni \mu \mapsto \mu(B) \in \mathbb{R}$ is weak*-continuous;*
(c) *the map $\mathrm{ca}_+(X) \ni \mu \mapsto \mu(B) \in \mathbb{R}$ is weak*-continuous;*
(d) *the map $\mathrm{ca}^1_+(X) \ni \mu \mapsto \mu(B) \in \mathbb{R}$ is weak*-continuous.*

Proof. Note that $\mu(B) = \langle \mu, \chi_B \rangle$ and that χ_B is continuous if and only if B is clopen. □

Remark 4.4. If X is a topological space and $\mathrm{ca}(X)$ is endowed with the weak* topology then the topology induced on X by the map $\delta : X \to \mathrm{ca}(X)$ is equal to the topology induced by all compositions $\langle \cdot, f \rangle \circ \delta$, $f \in C_{\mathrm{b}}(X)$, and therefore it is equal to the topology induced by all bounded continuous functions $f : X \to \mathbb{R}$. Such topology coincides with the topology of X if and only if X is a completely regular topological space (Definition D.4). In other words, the topology of X coincides with the topology induced by δ from the weak* topology of $\mathrm{ca}(X)$ if and only if X is completely regular. If X is completely regular and Hausdorff then δ is also injective and therefore it is a homeomorphism onto its image endowed with the weak* topology.

Though the evaluation functional $\mu \mapsto \mu(B)$ is weak*-continuous only if B is clopen, we have that its restriction to $\mathrm{ca}_+(X)$ has weak*-semicontinuity properties if B is either open or closed and X is perfectly normal.

Lemma 4.5. *Let X be a normal topological space and let $\mathrm{ca}_+(X)$ be endowed with the weak* topology. If U is an open F_σ subset of X then the map $\mathrm{ca}_+(X) \ni \mu \mapsto \mu(U) \in \mathbb{R}$ is lower semicontinuous, i.e., the set*

$$\big\{\mu \in \mathrm{ca}_+(X) : \mu(U) > c\big\} \tag{4.1}$$

is weak-open in $\mathrm{ca}_+(X)$ for every $c \in \mathbb{R}$. Moreover, if F is a closed G_δ subset of X then the map $\mathrm{ca}_+(X) \ni \mu \mapsto \mu(F) \in \mathbb{R}$ is upper semicontinuous, i.e., the set*

$$\big\{\mu \in \mathrm{ca}_+(X) : \mu(F) < c\big\} \tag{4.2}$$

is weak-open in $\mathrm{ca}_+(X)$ for every $c \in \mathbb{R}$.*

In more simple terms: for nonnegative measures on a normal topological space, the measure of an open F_σ set cannot drop a lot if we make a small weak*-perturbation of the measure and the measure of a closed G_δ set cannot increase a lot if we make a small weak*-perturbation of the measure.

Proof. If $\mu(U) > c$, since U is F_σ, we can find $H \subset U$ closed in X with $\mu(H) > c$. Now Urysohn's Lemma yields a $[0, 1]$-valued continuous function f on X which equals 1 on H and that vanishes outside of U. The set $\big\{\nu \in \mathrm{ca}_+(X) : \langle \nu, f \rangle > c\big\}$ is then a weak*-open neighborhood of μ in $\mathrm{ca}_+(X)$ contained in (4.1), since $\nu(H) \le \langle \nu, f \rangle \le \nu(U)$ for $\nu \in \mathrm{ca}_+(X)$. Similarly, if $\mu(F) < c$, since F is G_δ there exists an open set V containing F

with $\mu(V) < c$ and a $[0,1]$-valued continuous function f which equals 1 on F and vanishes outside of V. Hence $\big\{\nu \in \mathrm{ca}_+(X) : \langle \nu, f\rangle < c\big\}$ is a weak*-open neighborhood of μ in $\mathrm{ca}_+(X)$ contained in (4.2). □

Since the characteristic function of a set is lower semicontinuous (resp., upper semicontinuous) if and only if the set is open (resp., the set is closed), one might expect after looking at Lemma 4.5 that for a perfectly normal topological space X the linear functional $\langle\cdot, f\rangle$ restricted to $\mathrm{ca}_+(X)$ will be weak*-semicontinuous if f is semicontinuous. This is indeed correct if f is bounded.

Lemma 4.6. *Let X be a perfectly normal topological space and let $\mathrm{ca}_+(X)$ be endowed with the weak* topology. If $f : X \to\ ]-\infty, +\infty]$ is a lower semicontinuous function bounded from below then the map*

$$\mathrm{ca}_+(X) \ni \mu \longmapsto \langle \mu, f\rangle \in\ ]-\infty, +\infty]$$

is lower semicontinuous, i.e., for every $c \in \mathbb{R}$ the set

$$\big\{\mu \in \mathrm{ca}_+(X) : \langle \mu, f\rangle > c\big\} \tag{4.3}$$

is weak-open in $\mathrm{ca}_+(X)$. Similarly, for every upper semicontinuous function $f : X \to [-\infty, +\infty[$ bounded from above the map*

$$\mathrm{ca}_+(X) \ni \mu \longmapsto \langle \mu, f\rangle \in [-\infty, +\infty[$$

is upper semicontinuous, i.e., for every $c \in \mathbb{R}$ the set

$$\big\{\mu \in \mathrm{ca}_+(X) : \langle \mu, f\rangle < c\big\}$$

is weak-open in $\mathrm{ca}_+(X)$.*

Proof. Obviously the part of the statement for upper semicontinuous functions follows from the part of the statement for lower semicontinuous functions by replacing f with $-f$, so we consider only lower semicontinuous functions. By possibly adding a constant to f we can assume without loss of generality that f is nonnegative. Given $\mu \in \mathrm{ca}_+(X)$ and $c \in \mathbb{R}$ with $\langle \mu, f\rangle > c$, since every open subset of X is an F_σ we can apply Corollary B.6 to get $g \in C_{\mathrm{b}}(X)$ with $0 \le g \le f$ and $\langle \mu, g\rangle > c$. Hence $\big\{\nu \in \mathrm{ca}_+(X) : \langle \nu, g\rangle > c\big\}$ is a weak*-open neighborhood of μ in $\mathrm{ca}_+(X)$ contained in (4.3). □

For nonnegative measures on a perfectly normal topological space there are many interesting characterizations of weak* convergence which we list in Proposition 4.9 below. First we need another simple lemma.

Lemma 4.7. *If $(X, \mathcal{A})$ is a measurable space and $(\mu_i)_{i\in I}$ is a net in $\mathrm{ca}(X, \mathcal{A})$ such that the family $(\mu_i)_{i\ge i_0}$ is bounded for some $i_0 \in I$ then the set*

$$\Big\{f \in \mathcal{M}_{\mathrm{b}}(X, \mathcal{A}) : \lim_{i\in I} \langle \mu_i, f\rangle = \langle \mu, f\rangle\Big\} \tag{4.4}$$

is a closed subspace of $\mathcal{M}_{\mathrm{b}}(X, \mathcal{A})$ with respect to the supremum norm for any $\mu \in \mathrm{ca}(X, \mathcal{A})$.

Proof. Follows directly from (2.2) and Corollary F.3. □

Corollary 4.8. *If $(X, \mathcal{A})$ is a measurable space and $(\mu_i)_{i\in I}$ is a net in $\mathrm{ca}_+(X)$ such that the limit $\lim_{i\in I} \mu_i(X)$ exists and is finite then the set (4.4) is a closed subspace of $\mathcal{M}_{\mathrm{b}}(X, \mathcal{A})$ with respect to the supremum norm for any $\mu \in \mathrm{ca}(X, \mathcal{A})$.*

Proof. Simply note that if the limit $\lim_{i\in I} \mu_i(X)$ exists and is finite then the set $\{\mu_i(X) : i \geq i_0\}$ is bounded for some $i_0 \in I$ and that for nonnegative measures μ_i we have $\|\mu_i\| = \mu_i(X)$. □

In the statement below we talk about the measure of the set of continuity points of a real-valued function on a topological space X and for that to make sense we need such set to be Borel. We recall that this is always the case as the set of continuity points of a map $f : X \to M$ taking values in a metric space M is always a G_δ set. Namely, for every $\varepsilon > 0$ the set of points $x \in X$ for which the oscillation of f is less than ε (i.e., the set of points having a neighborhood V such that $f[V]$ has diameter less than ε) is open and the set of continuity points of f coincides with the set of points for which the oscillation of f is less than $\frac{1}{n}$ for every positive integer n.

Proposition 4.9 (Portmanteau)**.** *Let X be a topological space, $(\mu_i)_{i\in I}$ be a net in $\mathrm{ca}_+(X)$ and $\mu \in \mathrm{ca}_+(X)$ be fixed. If X is perfectly normal then the following conditions are equivalent:*

(a) *the net $(\mu_i)_{i\in I}$ weak*-converges to μ;*

(b) $\lim_{i\in I} \mu_i(X) = \mu(X)$ *and for every open set U in X the inequality*

$$\mu(U) \leq \liminf_{i\in I} \mu_i(U)$$

holds;

(c) $\lim_{i\in I} \mu_i(X) = \mu(X)$ *and for every closed set F in X the inequality*

$$\limsup_{i\in I} \mu_i(F) \leq \mu(F)$$

holds;

(d) *for every lower semicontinuous function $f : X \to \left]-\infty, +\infty\right]$ bounded from below the inequality*

$$\langle \mu, f\rangle \leq \liminf_{i\in I} \langle \mu_i, f\rangle \tag{4.5}$$

holds;

(e) *for every upper semicontinuous function $f : X \to \left[-\infty, +\infty\right[$ bounded from above the inequality*

$$\limsup_{i\in I} \langle \mu_i, f\rangle \leq \langle \mu, f\rangle$$

holds;

(f) *for every Borel subset B of X with $\mu(\partial B) = 0$ the equality*

$$\lim_{i\in I} \mu_i(B) = \mu(B)$$

holds, where ∂B denotes the boundary of B;

(g) *for every bounded Borel measurable function $f : X \to \mathbb{R}$ whose set of discontinuity points has measure zero with respect to μ, the equality*

$$\lim_{i \in I} \langle \mu_i, f \rangle = \langle \mu, f \rangle \tag{4.6}$$

holds.

For an arbitrary topological space X we have:

$$(b) \iff (c) \iff (d) \iff (e) \implies (f) \iff (g) \implies (a).$$

Proof. The equivalence between (b) and (c) is established by taking complements and using Lemma A.4 and the equivalence between (d) and (e) is established by replacing f with $-f$. The fact that (d) implies (b) follows by letting f be the characteristic function of U and by setting $f = \pm 1$ to establish $\lim_{i \in I} \mu_i(X) = \mu(X)$. Now (b) and (c) imply (f) since if we let U be the interior of B and F be the closure of B we obtain:

$$\mu(B) = \mu(U) \le \liminf_{i \in I} \mu_i(U) \le \liminf_{i \in I} \mu_i(B) \\ \le \limsup_{i \in I} \mu_i(B) \le \limsup_{i \in I} \mu_i(F) \le \mu(F) = \mu(B).$$

Trivially (g) implies (f) by letting f be the characteristic function of B and (g) implies (a). If X is perfectly normal, (a) implies (b), (c), (d) and (e) by Lemmas 4.5 and 4.6, keeping in mind Lemma B.4. To see that (b) implies (d) for arbitrary X, note first that by (A.1) the set of those f for which (4.5) holds is closed under finite linear combinations with real nonnegative coefficients and that for nonnegative functions it is also closed under pointwise monotonically increasing limits of sequences by the Monotone Convergence Theorem. Assuming (b), we have that (4.5) holds for characteristic functions of open sets and for constant functions. Since a lower semicontinuous function bounded from below can be made nonnegative by adding a constant, the conclusion follows from Lemma B.5.

To conclude the proof, we prove (g) assuming (f). Using (f) with $B = X$ and Corollary 4.8 we conclude that it is sufficient to show that every f in $\mathcal{M}_{\mathrm{b}}(X)$ whose set of discontinuity points has measure zero with respect to μ can be uniformly approximated by linear combinations of characteristic functions of Borel sets B with $\mu(\partial B) = 0$. To this aim, let $a, b \in \mathbb{R}$ with $a < b$ be such that the image of f is contained in $[a, b[$. Since $\mu\big(f^{-1}(c)\big) > 0$ for at most countably many $c \in \mathbb{R}$, for any $\varepsilon > 0$ we can find a partition $a = t_0 < t_1 < \cdots < t_k = b$ with $\mu\big(f^{-1}(t_j)\big) = 0$ for $j = 1, \ldots, k-1$ and $t_{j+1} - t_j < \varepsilon$ for $j = 0, \ldots, k-1$. Setting $B_j = f^{-1}\big[[t_j, t_{j+1}[\big]$ we have then $\mu(\partial B_j) = 0$ for $j = 0, \ldots, k-1$ and $\|f - g\|_{\sup} < \varepsilon$ for $g = \sum_{j=0}^{k-1} t_j \chi_{B_j}$. □

Example 4.10. Of course one should not expect the equivalence between (a) in Proposition 4.9 and the conditions (b), (c), (d) and (e) involving inequalities to hold for nets of signed measures. For instance, note that the validity of (a) is kept invariant if we multiply all measures by -1, while this is obviously not the case with the conditions involving inequalities. Yet, one could reasonably expect that (a) still implies (f) for nets of signed measures (perhaps requiring in (f) that the boundary of B has measure zero with respect to the total variation of μ). However, this is not true as the following simple counterexample shows. Let $X = \mathbb{R}$, $B =]0, +\infty[$ and consider the sequence $(\mu_n)_{n\geq 1}$ in $\mathrm{ca}(\mathbb{R})$ given by

$$\mu_n = \delta_{\frac{1}{n}} - \delta_{-\frac{1}{n}},$$

for all $n \geq 1$. Clearly $(\mu_n)_{n\geq 1}$ weak*-converges to $\mu = 0$, but $\mu_n(B) = 1$ for all $n \geq 1$.

Example 4.11. Counterexamples to the implications in Proposition 4.9 that are not valid for arbitrary X can be obtained by using the nonregular probability measure ν on the compact Hausdorff space $K = [0, \omega_1]$ defined in Example C.8. Set $X = K$ and let I be an arbitrary directed set. Letting $\mu = \frac{1}{2}\nu + \frac{1}{2}\delta_{\omega_1}$ and $\mu_i = \delta_{\omega_1}$ for all $i \in I$ we obtain an example such that condition (f) in the statement of Proposition 4.9 holds, but condition (b) doesn't. Namely, if B is Borel in K and $\mu(\partial B) = 0$ then $\omega_1 \notin \partial B$. If ω_1 is not in the closure of B we have that B is a countable subset of ω_1 and therefore $\mu_i(B) = \mu(B) = 0$, for all $i \in I$. On the other hand, if ω_1 is an interior point of B then $K \setminus B$ is a countable subset of ω_1 and therefore $\mu_i(B) = \mu(B) = 1$, for all $i \in I$. However, setting $U = \omega_1$ we have that U is open in K, but $\mu(U) = \frac{1}{2}$, while $\mu_i(U) = 0$, for all $i \in I$, contradicting (b). Now if we set $\mu = \nu$ and $\mu_i = \delta_{\omega_1}$ for all $i \in I$ then (a) holds, since $\langle \mu, f \rangle = \langle \mu_i, f \rangle$ for all $i \in I$ and all $f \in C_{\mathrm{b}}(X)$. However, letting $B = \omega_1$ then $\partial B = \{\omega_1\}$, so that $\mu(\partial B) = 0$, and yet $\mu(B) = 1$ and $\mu_i(B) = 0$, for all $i \in I$, contradicting (f).

Proposition 4.9 can be used to obtain convenient bases of open sets and convenient fundamental systems of neighborhoods for the weak* topology on $\mathrm{ca}_+(X)$. Recall that a *subbasis* for a topology τ on a set X is a subset $\mathcal{S}$ of τ such that the collection of all finite intersections of elements of $\mathcal{S}$ is a basis for τ. Given a collection $\mathcal{S}$ of subsets of a set X, we have that there exists a topology on X for which $\mathcal{S}$ is a subbasis if and only if $\mathcal{S}$ is a covering of X.

Corollary 4.12. *Let X be a perfectly normal topological space. The collection of all sets of the form*

$$\begin{gathered}\big\{\mu \in \mathrm{ca}_+(X) : \mu(U) > c\big\}, \quad U \subset X \text{ open, } c \in \mathbb{R},\\ \big\{\mu \in \mathrm{ca}_+(X) : \mu(X) < c\big\}, \quad c \in \mathbb{R}\end{gathered} \tag{4.7}$$

form a subbasis for the weak topology on $\mathrm{ca}_+(X)$. Similarly, the collection of all sets of the form*

$$\begin{gathered}\big\{\mu \in \mathrm{ca}_+(X) : \mu(F) < c\big\}, \quad F \subset X \text{ closed, } c \in \mathbb{R},\\ \big\{\mu \in \mathrm{ca}_+(X) : \mu(X) > c\big\}, \quad c \in \mathbb{R}\end{gathered} \tag{4.8}$$

form a subbasis for the weak topology on $\mathrm{ca}_+(X)$.*

Proof. All the sets (4.7) are weak*-open in $\mathrm{ca}_+(X)$ by Lemma 4.5 and therefore the unique topology τ on $\mathrm{ca}_+(X)$ having (4.7) as subbasis is contained in the weak* topology of $\mathrm{ca}_+(X)$. To prove the reverse inclusion, use the equivalence between (a) and (b) of Proposition 4.9 to establish that τ-convergence of a net in $\mathrm{ca}_+(X)$ implies weak*-convergence of such a net. The proof that (4.8) is a subbasis for the weak* topology is analogous. □

Corollary 4.13. *Let X be a perfectly normal topological space and let μ in $\mathrm{ca}_+(X)$ be fixed. The collection of all finite intersections of sets of the form*

$$\big\{\nu \in \mathrm{ca}_+(X) : |\nu(B) - \mu(B)| < \varepsilon\big\}, \quad B \subset X \text{ Borel with } \mu(\partial B) = 0,\ \varepsilon > 0$$

constitute a fundamental system of (not necessarily open) neighborhoods of μ in $\mathrm{ca}_+(X)$ endowed with the weak topology.*

Proof. Follows from Lemma A.3 and the equivalence between (a) and (f) in Proposition 4.9. □

There are actually many other equivalences for weak* convergence of nets of nonnegative measures on specific classes of topological spaces besides those presented in Proposition 4.9. Our next lemma gives a general recipe for producing other equivalences.

Lemma 4.14. *Let X be a topological space and $\mathcal{F}$ be a collection of continuous $[0,1]$-valued functions on X satisfying the following condition: every open subset U of X is a countable increasing union of closed subsets F of X for which there exists $f \in \mathcal{F}$ that is equal to 1 on F and that vanishes outside of U. Under such assumption, for every net $(\mu_i)_{i\in I}$ in $\mathrm{ca}_+(X)$ and every $\mu \in \mathrm{ca}_+(X)$ we have that $(\mu_i)_{i\in I}$ weak*-converges to μ if and only if $\lim_{i\in I} \mu_i(X) = \mu(X)$ and $\lim_{i\in I}\langle \mu_i, f\rangle = \langle \mu, f\rangle$, for all $f \in \mathcal{F}$. In other words, the weak* topology of $\mathrm{ca}_+(X)$ coincides with the topology induced by all maps of the form $\langle \cdot, f\rangle$ with $f \in \mathcal{F} \cup \{1\}$.*

Proof. Assume that $\lim_{i\in I} \mu_i(X) = \mu(X)$ and $\lim_{i\in I}\langle \mu_i, f\rangle = \langle \mu, f\rangle$ for all $f \in \mathcal{F}$. By Proposition 4.9, to prove that $(\mu_i)_{i\in I}$ weak*-converges to μ it suffices to check that $\mu(U) \le \liminf_{i\in I} \mu_i(U)$ for an arbitrary open subset U of X. Let $(F_n)_{n\ge 1}$ be an increasing sequence of closed subsets of X with $U = \bigcup_{n=1}^{\infty} F_n$ and $(f_n)_{n\ge1}$ be a sequence in $\mathcal{F}$ such that f_n is equal to 1 on F_n and vanishes outside of U. We have

$$\mu(F_n) \le \langle \mu, f_n\rangle = \lim_{i\in I} \langle \mu_i, f_n\rangle \le \liminf_{i\in I} \mu_i(U),$$

for every $n \ge 1$. The conclusion follows by taking the supremum over n noting that $\sup_{n\ge1} \mu(F_n) = \mu(U)$. □

Remark 4.15. Due to Corollary 4.8 we can actually improve Lemma 4.14 by weakening a bit the assumption on $\mathcal{F}$. Namely, it is sufficient to assume that every open subset U of X is a countable increasing union of closed subsets F of X for which there exists $f : X \to [0,1]$ in the closed linear span of $\mathcal{F} \cup \{1\}$ (with respect to the supremum norm) that is equal to 1 on F and vanishes outside of U.

We give two applications of Lemma 4.14.

Proposition 4.16. *If* (X, d) *is a metric space then a net* $(\mu_i)_{i\in I}$ *in* $\mathrm{ca}_+(X)$ *weak*-converges to* $\mu \in \mathrm{ca}_+(X)$ *if and only if* $\lim_{i\in I}\langle \mu_i, f\rangle = \langle \mu, f\rangle$ *for every bounded Lipschitz function* $f : X \to \mathbb{R}$. *In other words, the weak* topology of* $\mathrm{ca}_+(X)$ *coincides with the topology induced by all maps of the form* $\langle \cdot, f\rangle$ *with* $f : X \to \mathbb{R}$ *bounded and Lipschitz.*

Proof. We use Lemma 4.14 with $\mathcal{F}$ equal to the set of $[0, 1]$-valued Lipschitz functions on X. In order to establish the assumptions of Lemma 4.14 we need a Lipschitz version of Urysohn's Lemma. In a metric space (X, d), the function

$$X \ni x \longmapsto \frac{\min\{d(x, F_1), 1\}}{\min\{d(x, F_1), 1\} + d(x, F_2)} \in [0, 1] \tag{4.9}$$

is an Urysohn function for a pair of nonempty disjoint closed subsets F_1 and F_2, i.e., it takes values in $[0, 1]$, it vanishes on F_1 and it is equal to 1 on F_2. Since the product of bounded Lipschitz functions is Lipschitz and the function $\frac{1}{f}$ is Lipschitz if f is a positive Lipschitz function bounded away from zero, we have that (4.9) is Lipschitz if $d(F_1, F_2) > 0$. To conclude the proof, note that an arbitrary open subset U of X can be written as the increasing union of the closed sets F_n given by

$$F_n = \left\{x \in X : d(x, X \setminus U) \geq \tfrac{1}{n}\right\}$$

and that $d(F_n, X \setminus U) > 0$, so that the construction above yields Lipschitz Urysohn functions for the pairs F_n, $X \setminus U$. □

Proposition 4.17. *If* X *is a (Hausdorff, second countable, finite-dimensional) differentiable manifold of class* C^k $(1 \leq k \leq +\infty)$ *then a net* $(\mu_i)_{i\in I}$ *in* $\mathrm{ca}_+(X)$ *weak*-converges to* $\mu \in \mathrm{ca}_+(X)$ *if and only if*

$$\lim_{i\in I} \mu_i(X) = \mu(X)$$

and $\langle \mu, f\rangle = \lim_{i\in I}\langle \mu_i, f\rangle$ *for every function* $f : X \to \mathbb{R}$ *of class* C^k *having compact support.*

Proof. Use Lemma 4.14 with $\mathcal{F}$ equal to the set of $[0, 1]$-valued functions on X of class C^k having compact support. Note that since X is second countable and locally compact, every open set U is a countable increasing union of compact sets K and for every compact subset K of U one obtains an Urysohn function of class C^k with compact support for the pair K, $X \setminus U$ using a partition of unity of class C^k (see, for instance, [4, Proposition 2.26] and note that every compact subset of X is contained in an open relatively compact subset of X). □

Example 4.18. Propositions 4.16 and 4.17 are not in general valid for nets of signed measures, not even for bounded sequences of signed measures. For example, set $X = \mathbb{R}$ and let

$$\mu_n = \delta_{x_n} - \delta_{y_n}, \tag{4.10}$$

for all $n \geq 1$, where $(x_n)_{n\geq 1}$ is a sequence in $\mathbb{R}$ with $\lim_{n\to+\infty} x_n = +\infty$ and $(y_n)_{n\geq 1}$ is a sequence in $\mathbb{R}$ such that $\lim_{n\to+\infty}(x_n - y_n) = 0$ and $x_n \neq y_n$ for all $n \geq 1$. If $f : \mathbb{R} \to \mathbb{R}$ is either Lipschitz or has compact support then

$$\lim_{n\to+\infty} \langle \mu_n, f\rangle = \lim_{n\to+\infty} [f(x_n) - f(y_n)] = 0.$$

But one can easily obtain $f \in C_{\mathrm{b}}(\mathbb{R})$ such that $f(x_n) - f(y_n) = 1$ for infinitely many $n \geq 1$ and $f(x_n) - f(y_n) = 0$ for infinitely many $n \geq 1$, so that $(\mu_n)_{n\geq 1}$ is not weak*-convergent. We note however that it is true that if X is a (Hausdorff, second countable, finite-dimensional) differentiable manifold of class C^k ($1 \leq k \leq +\infty$) then a bounded net $(\mu_i)_{i\in I}$ in $\mathrm{ca}(X)$ weak*-converges to $\mu \in \mathrm{ca}(X)$ if and only if $\langle \mu, f\rangle = \lim_{i\in I}\langle \mu_i, f\rangle$ for every bounded function $f : X \to \mathbb{R}$ of class C^k. This follows from Lemma 4.7 and from the well-known fact that the subspace of $C_{\mathrm{b}}(X)$ consisting of bounded functions of class C^k is dense in $C_{\mathrm{b}}(X)$ with respect to the supremum norm (see [2, Theorem 2.2]). The assumption that the net $(\mu_i)_{i\in I}$ is bounded is really essential here. For example, setting $X = \mathbb{R}$ and

$$\mu_n = \sqrt{n}\,\big(\delta_{\frac{1}{n}} - \delta_0\big), \tag{4.11}$$

for all $n \geq 1$, then

$$\lim_{n\to+\infty} \langle \mu_n, f\rangle = \lim_{n\to+\infty} \sqrt{n}\,\big[f\big(\tfrac{1}{n}\big) - f(0)\big] = 0$$

for a function $f : \mathbb{R} \to \mathbb{R}$ of class C^1 because f is Lipschitz in a neighborhood of zero. However, if $f \in C_{\mathrm{b}}(\mathbb{R})$ is such that $f(x) = \sqrt[4]{|x|}$ for $x \in \mathbb{R}$ near zero then $\lim_{n\to+\infty}\langle \mu_n, f\rangle = +\infty$.

By the very definition of the weak* topology, a fundamental system of weak*-neighborhoods for a measure $\mu \in \mathrm{ca}(X)$ is obtained by considering sets of the form

$$\big\{\nu \in \mathrm{ca}(X) : |\langle \nu, f_i\rangle - \langle \mu, f_i\rangle| < \varepsilon,\ i = 1, \ldots, k\big\},$$

with $f_1, \ldots, f_k \in C_{\mathrm{b}}(X)$, $\varepsilon > 0$ and k a positive integer. Such neighborhoods can be more efficiently written if we aggregate all maps f_i into a single $\mathbb{R}^k$-valued map f. From this observation the following result immediately follows. The space $\mathbb{R}^k$ is assumed to be endowed with some arbitrary fixed norm.

Lemma 4.19. *Let X be a topological space and $\mu \in \mathrm{ca}(X)$. A fundamental system of open neighborhoods of μ in $\mathrm{ca}(X)$ with respect to the weak* topology consists of all sets of the form*

$$\left\{\nu \in \mathrm{ca}(X) : \left\| \int_X f\,\mathrm{d}\nu - \int_X f\,\mathrm{d}\mu \right\| < \varepsilon \right\}, \tag{4.12}$$

where f varies over the set of all bounded $\mathbb{R}^k$-valued continuous functions on X, k varies over all positive integers and ε varies over all positive real numbers. □

We now use Lemma 4.19 to prove that the Dirac deltas span a weak*-dense subspace of $\mathrm{ca}(X)$ and that, moreover, the intersection of such subspace with various interesting subsets of $\mathrm{ca}(X)$ are weak*-dense in the corresponding subset.

Proposition 4.20. *If X is a topological space then the linear span of*

$$\{\delta_x : x \in X\} \tag{4.13}$$

is weak-dense in $\mathrm{ca}(X)$. Moreover, the intersection with $\mathcal{S}$ of the linear span of (4.13) is weak*-dense in $\mathcal{S}$ for any of the subsets $\mathcal{S}$ of $\mathrm{ca}(X)$ below:*

(a) $\mathcal{S} = \big\{\mu \in \mathrm{ca}(X) : \|\mu\| \le c\big\}$, *where $c \ge 0$;*

(b) $\mathcal{S} = \big\{\mu \in \mathrm{ca}_+(X) : \mu(X) \in C\big\}$, *where C is a subset of $[0, +\infty[$.*

Proof. Let $\mu \in \mathrm{ca}(X)$ and consider a fundamental weak*-open neighborhood of μ of the form (4.12), with $f : X \to \mathbb{R}^k$ a bounded continuous function and $\varepsilon > 0$. Since the image of f is relatively compact, it can be covered by finitely many open sets with diameter less than $\varepsilon' > 0$, where ε' is chosen with $\varepsilon' \, |\mu|(X) < \varepsilon$. Taking the inverse images of such open sets by f and disjointifying, we obtain a partition $X = \bigcup_{i=1}^n B_i$ of X into nonempty disjoint Borel subsets B_i such that $f[B_i]$ has diameter less than ε'. Choose $x_i \in B_i$ for all $i = 1, \ldots, n$ and set $\nu = \sum_{i=1}^n \mu(B_i)\delta_{x_i}$. We have

$$\begin{aligned}\left\| \int_X f \,\mathrm{d}\mu - \int_X f \,\mathrm{d}\nu \right\| &= \left\| \int_X f \,\mathrm{d}\mu - \sum_{i=1}^n \mu(B_i) f(x_i) \right\| \\ &= \left\| \sum_{i=1}^n \int_{B_i} [f(x) - f(x_i)] \,\mathrm{d}\mu(x) \right\| \le \varepsilon' \, |\mu|(X) < \varepsilon,\end{aligned}$$

so that ν belongs to (4.12). To conclude the proof, note that $\|\nu\| \le \|\mu\|$, $\nu(X) = \mu(X)$ and that ν is nonnegative if μ is nonnegative, so that in any case ν is in $\mathcal{S}$ if μ is in $\mathcal{S}$. □

Corollary 4.21. *If X is a separable topological space then $\mathrm{ca}(X)$ is weak*-separable and any of the subsets $\mathcal{S}$ of $\mathrm{ca}(X)$ appearing in the statement of Proposition 4.20 is weak*-separable.*

Proof. The proof of Proposition 4.20 has shown not only that the intersection of the linear span of (4.13) with $\mathcal{S}$ is weak*-dense in $\mathcal{S}$, but the following fact: there exists a set $A \subset \bigcup_{n=0}^\infty \mathbb{R}^n$ of finite sequences of real numbers such that

$$\bigcup_{n=0}^\infty \left\{ \sum_{i=1}^n a_i \delta_{x_i} : (a_1, \ldots, a_n) \in A_n, \ (x_1, \ldots, x_n) \in X^n \right\} \tag{4.14}$$

is a weak*-dense subset of $\mathcal{S}$, where $A_n = A \cap \mathbb{R}^n$ for all $n \ge 0$. Namely, take $A = \bigcup_{n=0}^\infty A_n$, where A_n is the set of n-tuples of the form $\big(\mu(B_1), \ldots, \mu(B_n)\big)$, with $\mu \in \mathcal{S}$ and $X = \bigcup_{i=1}^n B_i$ a partition of X into nonempty disjoint Borel subsets. We now prove that (4.14) is weak*-separable. Note that (4.14) is the union of the images of the maps

$$A_n \times X^n \ni \big((a_1,\ldots,a_n),(x_1,\ldots,x_n)\big) \longmapsto \sum_{i=1}^{n} a_i\delta_{x_i} \in \mathrm{ca}(X), \tag{4.15}$$

with $n \geq 0$. It follows from Proposition 4.1 and the fact that $\mathrm{ca}(X)$ is a topological vector space that the maps (4.15) are continuous if $\mathrm{ca}(X)$ is endowed with the weak* topology. To conclude the proof, observe that a countable union of separable subsets is separable, that a continuous image of a separable space is separable and that $A_n \times X^n$ is separable for all $n \geq 0$. □

5. Weak*-continuity of operations with measures

We start with two very simple results that hold for the space of all finite signed measures in arbitrary topological spaces.

Proposition 5.1. *If X is a topological space and $f \in C_{\mathrm{b}}(X)$ then the map $\mathrm{ca}(X) \ni \mu \mapsto f\mu \in \mathrm{ca}(X)$ is weak*-continuous.*

Proof. Follows from equality (2.6). □

Proposition 5.2. *If X and Y are topological spaces and $\phi : X \to Y$ is a continuous map then the map $\mathrm{ca}(X) \ni \mu \mapsto \phi_*\mu \in \mathrm{ca}(Y)$ is weak*-continuous.*

Proof. Follows from equality (2.9). □

For nonnegative measures and with reasonable assumptions for the topological spaces much better results can be obtained using Proposition 4.9.

Proposition 5.3. *Let X be a perfectly normal topological space and let $f : X \to [0,+\infty[$ be a nonnegative Borel measurable and bounded function. The map $\mathrm{ca}_+(X) \ni \mu \mapsto f\mu \in \mathrm{ca}_+(X)$ is weak*-continuous at every point $\mu \in \mathrm{ca}_+(X)$ such that the set of discontinuity points of f has measure zero with respect to μ.*

Proof. Follows from equality (2.6) and the equivalence between (a) and (g) in Proposition 4.9. □

We note that in Proposition 5.4 below we cannot talk directly about the measure of the set of discontinuity points of ϕ because at this level of generality we cannot ensure that such set is Borel.

Proposition 5.4. *Let X and Y be topological spaces, with X perfectly normal, and let $\phi : X \to Y$ be a Borel measurable map. The map*

$$\mathrm{ca}_+(X) \ni \mu \longmapsto \phi_*\mu \in \mathrm{ca}_+(Y)$$

is weak-continuous at every point $\mu \in \mathrm{ca}_+(X)$ such that the set of discontinuity points of ϕ is contained in a Borel set which has measure zero with respect to μ.*

Proof. Follows from equality (2.9) and the equivalence between (a) and (g) in Proposition 4.9. □

Let us now investigate the effect of restricting weak* convergent nets of measures to subspaces of the topological space. We start with a more general result from which the more interesting results will follow.

Proposition 5.5. *Let X be a perfectly normal topological space, Y be a set, $\phi : Y \to X$ be a map and let Y be endowed with the topology induced by ϕ. If $\mathrm{ca}_+(X)$ and $\mathrm{ca}_+(Y)$ are endowed with its weak* topologies then $\phi_* : \mathrm{ca}_+(Y) \to \mathrm{ca}_+(X)$ is a homeomorphism onto its image.*

Proof. Since Y has the topology induced by ϕ we have that the Borel σ-algebra of Y coincides with the σ-algebra induced by ϕ from the Borel σ-algebra of X, i.e.:

$$\mathcal{B}(Y) = \big\{\phi^{-1}[B] : B \in \mathcal{B}(X)\big\}.$$

This implies that the map ϕ_* is injective. Moreover, the continuity of ϕ_* follows from the continuity of ϕ, by Proposition 5.2. To prove that the inverse of ϕ_* (defined on its image) is continuous, let $(\mu_i)_{i\in I}$ be a net in $\mathrm{ca}_+(Y)$ and $\mu \in \mathrm{ca}_+(Y)$ be such that $(\phi_*\mu_i)_{i\in I}$ weak*-converges to $\phi_*\mu$. We prove that $(\mu_i)_{i\in I}$ weak*-converges to μ using that (b) implies (a) in Proposition 4.9 for the space Y and that (a) implies (b) for the space X. We have:

$$\lim_{i\in I} \mu_i(Y) = \lim_{i\in I} (\phi_*\mu_i)(X) = (\phi_*\mu)(X) = \mu(Y).$$

Moreover, if U is an open subset of Y then $U = \phi^{-1}[V]$ for some open subset V of X and hence:

$$\mu(U) = (\phi_*\mu)(V) \le \liminf_{i\in I} (\phi_*\mu_i)(V) = \liminf_{i\in I} \mu_i(U). \qquad \square$$

Example 5.6. In the context of the statement of Proposition 5.5, if the image of ϕ is required to be closed, then assuming only that X is normal we can obtain the stronger conclusion that the map $\phi_* : \mathrm{ca}(Y) \to \mathrm{ca}(X)$ is a homeomorphism onto its image. This is a simple consequence of (2.9) and Tietze's Extension Theorem which yields that $C_{\mathrm{b}}(Y) = \big\{f \circ \phi : f \in C_{\mathrm{b}}(X)\big\}$. However, without the assumption that the image of ϕ be closed, one does not have the stronger result that $\phi_* : \mathrm{ca}(Y) \to \mathrm{ca}(X)$ is a homeomorphism onto its image even if $X = \mathbb{R}$. For example, let $X = \mathbb{R}$, $Y =]0, +\infty[$ and $\phi : Y \to X$ be the inclusion map. Consider the sequence $(\mu_n)_{n\ge1}$ in $\mathrm{ca}(Y)$ defined by

$$\mu_n = \delta_{\frac{1}{n}} - \delta_{\frac{1}{n+1}},$$

for all $n \ge 1$. Clearly $(\phi_*\mu_n)_{n\ge1}$ converges to $0 = \phi_*0$ in $\mathrm{ca}(X)$, yet $(\mu_n)_{n\ge1}$ does not converge in $\mathrm{ca}(Y)$. Namely, since the set $\big\{\frac{1}{n} : n \ge 1\big\}$ is discrete and closed in Y, every bounded function $f : \big\{\frac{1}{n} : n \ge 1\big\} \to \mathbb{R}$ extends to a continuous bounded real-valued function on Y and thus we can easily obtain $f \in C_{\mathrm{b}}(Y)$ such that $\lim_{n\to+\infty}\langle \mu_n, f\rangle$ does not exist.

The image of the map ϕ_* in Proposition 5.5 admits a simple description.

Lemma 5.7. *Let $(X, \mathcal{A})$ be a measurable space, Y be a set, $\phi : Y \to X$ be a map and let Y be endowed with the σ-algebra $\big\{\phi^{-1}[A] : A \in \mathcal{A}\big\}$ induced by ϕ (for instance, if $\mathcal{A}$ is the Borel σ-algebra of some topology on X then Y will be endowed with the Borel σ-algebra of the topology induced by ϕ). Given a measure μ on X, the following conditions are equivalent:*

(i) *μ vanishes on every measurable subset of X disjoint from the image of ϕ;*

(ii) *$\mu = \phi_*\nu$ for some measure ν on Y.*

Moreover, the measure ν of item (ii) is unique if it exists and it has the same image as μ. In particular, ν is finite if μ is finite and ν is nonnegative if μ is nonnegative.

Proof. Obviously (ii) implies (i), so assume (i) and let us prove (ii). The measure ν must necessarily be given by

$$\nu\big(\phi^{-1}[A]\big) = \mu(A), \quad A \in \mathcal{A},$$

so that we only have to prove that ν is well-defined by such equality and that it is a measure. First, if $A_1, A_2 \in \mathcal{A}$ are such that $\phi^{-1}[A_1] = \phi^{-1}[A_2]$ then $A_1 \setminus A_2$ and $A_2 \setminus A_1$ are measurable subsets of X disjoint from the image of ϕ and thus (i) yields

$$\begin{aligned}\mu(A_1) = \mu(A_1 \setminus A_2) + \mu(A_1 \cap A_2) &= \mu(A_1 \cap A_2) \\ &= \mu(A_2 \setminus A_1) + \mu(A_1 \cap A_2) = \mu(A_2),\end{aligned}$$

which proves that ν is well-defined. To prove that ν is countably additive, let $(A_n)_{n\geq 1}$ be a sequence in $\mathcal{A}$ such that the sets $\phi^{-1}[A_n]$ are pairwise disjoint. If we set $A_n' = A_n \setminus \bigcup_{m<n} A_m$ then $\phi^{-1}[A_n] = \phi^{-1}[A_n']$ for all n and hence:

$$\nu\Big(\bigcup_{n=1}^{\infty} \phi^{-1}[A_n]\Big) = \mu\Big(\bigcup_{n=1}^{\infty} A_n'\Big) = \sum_{n=1}^{\infty} \mu(A_n') = \sum_{n=1}^{\infty} \nu\big(\phi^{-1}[A_n]\big). \qquad \square$$

If X is a topological space and Y is a Borel subset of X endowed with the induced topology then the Borel σ-algebra of Y consists of the Borel subsets of X that are contained in Y. Thus, given a measure μ on the Borel σ-algebra of X, we obtain a measure on the Borel σ-algebra of Y by simply restricting μ to the Borel σ-algebra of Y. We denote such restriction by $\mu|_Y$ instead of $\mu|_{\mathcal{B}(Y)}$ for simplicity. Note that if $\iota : Y \to X$ denotes the inclusion map then $\iota_* : \mathrm{ca}(Y) \to \mathrm{ca}(X)$ is just the map that extends a measure to zero on subsets of $X \setminus Y$. Thus, given $\mu \in \mathrm{ca}(X)$, we have that μ is on the image of ι_* if and only if $|\mu|(X \setminus Y) = 0$ and in this case we have $\mu = \iota_*(\mu|_Y)$. Moreover, for arbitrary $\mu \in \mathrm{ca}(X)$, we have the equality:

$$\iota_*(\mu|_Y) = \chi_Y \mu.$$

The following result is an immediate consequence of Proposition 5.5 with ϕ equal to the inclusion map of Y in X.

Proposition 5.8. *Let X be a perfectly normal topological space, Y be a Borel subset of X and $(\mu_i)_{i\in I}$ be a net in $\mathrm{ca}_+(X)$ that weak*-converges to some $\mu \in \mathrm{ca}_+(X)$. If $\mu(X \setminus Y) = 0$ and $\mu_i(X \setminus Y) = 0$ for all $i \in I$ then $(\mu_i|_Y)_{i\in I}$ weak*-converges to $\mu|_Y$.* □

We can actually prove something a little better than Proposition 5.8.

Corollary 5.9. *Let X be a perfectly normal topological space, Y be a Borel subset of X and $(\mu_i)_{i\in I}$ be a net in $\mathrm{ca}_+(X)$ that weak*-converges to some $\mu \in \mathrm{ca}_+(X)$. We have that $(\mu_i|_Y)_{i\in I}$ weak*-converges to $\mu|_Y$ if either one of the conditions below hold:*

(a) $\mu(\partial Y) = 0$;

(b) $\big(\mu_i|_{X\setminus Y}\big)_{i\in I}$ *weak*-converges to* $\mu|_{X\setminus Y}$.

Proof. By Proposition 5.8 it is sufficient to prove that $(\chi_Y \mu_i)_{i\in I}$ weak*-converges to $\chi_Y \mu$. Assuming (a), this follows from Proposition 5.3. Assuming (b) and taking the push-forwards by the inclusion of $X \setminus Y$ in X we get that $(\chi_{X\setminus Y}\mu_i)_{i\in I}$ weak*-converges to $\chi_{X\setminus Y}\mu$ and since $\mu_i = \chi_Y\mu_i + \chi_{X\setminus Y}\mu_i$, $\mu = \chi_Y\mu + \chi_{X\setminus Y}\mu$ and $\mathrm{ca}(X)$ endowed with the weak* topology is a topological vector space, we obtain that $(\chi_Y\mu_i)_{i\in I}$ weak*-converges to $\chi_Y\mu$. □

6. Tightness and compactness

Since the weak* topology of $\mathrm{ca}(X)$ is induced from the weak* topology of the topological dual $C_{\mathrm{b}}(X)^*$ of the Banach space $C_{\mathrm{b}}(X)$, one naturally expects that we should be able to establish that certain subsets of $\mathrm{ca}(X)$ are weak*-compact using the Banach–Alaoglu Theorem. Recall that the *Banach–Alaoglu Theorem* states that a closed ball of the topological dual of a normed space, endowed with its weak* topology, is compact.

Maybe we can obtain as a corollary that a closed ball of $\mathrm{ca}(X)$ is weak*-compact as well? The answer is no, because there is a catch: $\mathrm{ca}(X)$ does not correspond to the entire topological dual of $C_{\mathrm{b}}(X)$, but only to a subspace of it and a closed ball of a subspace of the topological dual of a normed space need not be weak*-compact as it is not in general weak*-closed. Another issue is that the linear map

$$\mathrm{ca}(X) \ni \mu \longmapsto \langle \mu, \cdot \rangle \in C_{\mathrm{b}}(X)^* \tag{6.1}$$

that we use to induce the weak* topology on $\mathrm{ca}(X)$ is not necessarily injective for an arbitrary topological space X, so $\mathrm{ca}(X)$ is not truly homeomorphic to a subspace of $C_{\mathrm{b}}(X)^*$. That issue, however, is harmless as the following simple result shows.

Lemma 6.1. *Let $\mathcal{X}$ be a set, $\mathcal{Y}$ be a topological space, $\Phi : \mathcal{X} \to \mathcal{Y}$ be a map and let $\mathcal{X}$ be endowed with the topology induced by Φ. Given a subset $\mathcal{C}$ of $\mathcal{X}$, we have that $\mathcal{C}$ is compact if and only if $\Phi[\mathcal{C}]$ is compact. In particular, if $\mathcal{K}$ is a compact subset of the image of Φ then $\Phi^{-1}[\mathcal{K}]$ is compact.*

Proof. This is a trivial topology exercise. □

Corollary 6.2. *Let $\mathcal{X}$ be a set, $\mathcal{Y}$ be a topological space, $\Phi : \mathcal{X} \to \mathcal{Y}$ be a map and let $\mathcal{X}$ be endowed with the topology induced by Φ. If $\mathcal{Y}$ is Hausdorff then the closure of a compact subset of $\mathcal{X}$ is compact.*

Proof. If $\mathcal{C}$ is a compact subset of $\mathcal{X}$ then Lemma 6.1 yields that $\Phi^{-1}\big[\Phi[C]\big]$ is also compact. But since $\mathcal{Y}$ is Hausdorff, the compact set $\Phi[\mathcal{C}]$ is closed and thus so is $\Phi^{-1}\big[\Phi[C]\big]$. Hence $\mathcal{C}$ is contained in a compact closed set and therefore its closure is compact. □

In view of Lemma 6.1, what is essential in order to establish that certain subsets of $\mathrm{ca}(X)$ are weak*-compact is to find weak*-compact subsets of the image of (6.1). In order to get some insight on such image, it is convenient to identify $C_{\mathrm{b}}(X)^*$ with a space of measures. Such identification depends on two ingredients that are discussed in more detail in Appendices C and D. The first ingredient is that if K is a compact Hausdorff topological space then the topological dual $C(K)^*$ of the space $C(K)$ of continuous real-valued functions on K endowed with the supremum norm can be isometrically identified with the space $M(K)$ of finite signed regular measures on the Borel σ-algebra of K endowed with the total variation norm (see Subsection C.1). In what follows, what we call the *weak* topology* of $M(K)$ is the topology induced by the standard isometric identification $M(K) \ni \nu \mapsto \langle \nu, \cdot \rangle \in C(K)^*$ from the weak* topology of $C(K)^*$.

The second ingredient is the Stone–Čech compactification of a topological space X which allows us to identify the Banach space $C_{\mathrm{b}}(X)$ isometrically with a $C(K)$ space. More explicitly, for an arbitrary topological space X, if $\iota : X \to \beta(X)$ denotes a Stone–Čech compactification of X then $\beta(X)$ is a compact Hausdorff topological space and the composition map

$$\iota^* : C\big(\beta(X)\big) \ni f \longmapsto f \circ \iota \in C_{\mathrm{b}}(X)$$

is a linear isometry between the Banach spaces $C\big(\beta(X)\big)$ and $C_{\mathrm{b}}(X)$. Its adjoint

$$\iota^{**} \stackrel{\text{def}}{=} (\iota^*)^* : C_{\mathrm{b}}(X)^* \longrightarrow C\big(\beta(X)\big)^*$$

is therefore a linear isometry and a weak*-homeomorphism between the topological duals of $C\big(\beta(X)\big)$ and $C_{\mathrm{b}}(X)$. Now we take the composition of (6.1) with ι^{**} and we identify $C\big(\beta(X)\big)^*$ with $M\big(\beta(X)\big)$ in the standard way obtaining a linear map

$$\Phi : \mathrm{ca}(X) \longrightarrow M\big(\beta(X)\big) \tag{6.2}$$

that carries each $\mu \in \mathrm{ca}(X)$ to the unique regular measure $\Phi(\mu) \in M\big(\beta(X)\big)$ on $\beta(X)$ such that:

$$\langle \Phi(\mu), f \rangle = \iota^{**}(\langle \mu, \cdot \rangle)(f) = \langle \mu, \iota^*(f) \rangle = \langle \mu, f \circ \iota \rangle, \tag{6.3}$$

for every $f \in C\big(\beta(X)\big)$. Obviously the weak* topology of $\mathrm{ca}(X)$ is induced by Φ from the weak* topology of $M\big(\beta(X)\big)$, so that what we need now is to find weak*-compact subsets of the image of Φ.

Recall from (2.9) that $\langle \mu, f \circ \iota \rangle = \langle \iota_* \mu, f \rangle$ for all $\mu \in \mathrm{ca}(X)$ and all $f \in C\big(\beta(X)\big)$, so that (6.3) yields:

$$\langle \Phi(\mu), f \rangle = \langle \iota_* \mu, f \rangle. \tag{6.4}$$

One might have the impression that equality (6.4) shows that $\Phi(\mu) = \iota_* \mu$ for all $\mu \in \mathrm{ca}(X)$ but it only does that if $\iota_* \mu$ is regular. Let us formally state this preliminary result.

Lemma 6.3. *For an arbitrary topological space X, we have $\Phi(\mu) = \iota_* \mu$ for all $\mu \in \mathrm{ca}(X)$ for which the measure $\iota_* \mu$ is regular.*

Proof. Follows from (6.4) and the uniqueness of the regular measure representing a bounded linear functional on $C\big(\beta(X)\big)$. □

When the topological space X is completely regular (Definition D.4), Lemmas 6.3 and 5.7 yield a simple sufficient condition for a regular measure on $\beta(X)$ to belong to the image of Φ .

Lemma 6.4. *If X is a completely regular topological space and a regular measure $\nu \in M\big(\beta(X)\big)$ vanishes on every Borel subset of $\beta(X)$ that is disjoint from the image of $\iota : X \to \beta(X)$ then ν belongs to the image of Φ. Moreover, if ν is in addition nonnegative then it belongs to $\Phi[\mathrm{ca}_+(X)]$.*

Proof. Since X is completely regular we have that the topology of X is induced by $\iota : X \to \beta(X)$ (Proposition D.5) and therefore Lemma 5.7 yields $\mu \in \mathrm{ca}(X)$ with $\iota_* \mu = \nu$. Moreover, μ is in $\mathrm{ca}_+(X)$ if ν is nonnegative. Since $\iota_* \mu = \nu$ is regular, it follows from Lemma 6.3 that $\nu = \Phi(\mu)$. □

One might expect that $\iota_* \mu$ is a regular measure on $\beta(X)$ when μ is a regular measure on X, but to prove that $\iota_* \mu$ is regular we need the following additional property for μ.

Definition 6.5. Given a topological space X, a measure $\mu \in \mathrm{ca}(X)$ is called *tight* if for every $\varepsilon > 0$ there exists a Borel compact subset K of X such that $|\mu|(X \setminus K) < \varepsilon$. More generally, a collection $\Lambda \subset \mathrm{ca}(X)$ of measures on X is called *tight* if for every $\varepsilon > 0$ there exists a Borel compact subset K of X such that $|\mu|(X \setminus K) < \varepsilon$ for all $\mu \in \Lambda$.

Since we are stating Definition 6.5 for a completely arbitrary topological space X we need to require explicitly that the compact subset K be Borel in order for the measure of $X \setminus K$ to be defined. Of course, if X is Hausdorff then every compact subset of X is closed and therefore automatically Borel.

Lemma 6.6. *If X is a topological space and $\mu \in \mathrm{ca}(X)$ is a tight regular measure then $\iota_* \mu$ is a regular measure on $\beta(X)$ and therefore by Lemma 6.3 we have $\Phi(\mu) = \iota_* \mu$.*

Proof. We just have to show that given a Borel subset B of $\beta(X)$ and $\varepsilon > 0$ we can find a closed subset of $\beta(X)$ contained in B whose complement in B has measure less than ε with respect to $|\iota_* \mu|$. By the regularity of μ there exists a closed subset F of X contained in the Borel subset $\iota^{-1}[B]$ with $|\mu|\big(\iota^{-1}[B]\setminus F\big)<\varepsilon$ and by the tightness of μ there exists a Borel compact subset K of X with $|\mu|(X\setminus K)<\varepsilon$. Setting $C=F\cap K$ we have that C is a compact subset of $\iota^{-1}[B]$ and:

$$|\mu|\big(\iota^{-1}[B]\setminus C\big)\le|\mu|\big(\iota^{-1}[B]\setminus F\big)+|\mu|(X\setminus K)<2\varepsilon.$$

Now $\iota[C]$ is a closed subset of $\beta(X)$ contained in B and using (2.8) we obtain:

$$\begin{aligned}|\iota_*\mu|\big(B\setminus\iota[C]\big)\le(\iota_*|\mu|)\big(B\setminus\iota[C]\big)&=|\mu|\big(\iota^{-1}[B]\setminus\iota^{-1}[\iota[C]]\big)\\&\le|\mu|\big(\iota^{-1}[B]\setminus C\big)<2\varepsilon.\quad\square\end{aligned}$$

We need now a tool to obtain weak*-closed subsets of $M(K)$.

Lemma 6.7. *If K is a compact Hausdorff topological space and U is an open subset of K then for every $c\ge 0$ the set*

$$\big\{\mu\in M(K):|\mu|(U)\le c\big\}\tag{6.5}$$

is weak-closed in $M(K)$.*

Proof. Given $\mu\in M(K)$, we have that the restriction $\mu|_U$ of μ to the Borel σ-algebra of U is a Radon measure (Definition C.1) on the locally compact Hausdorff space U whose total variation $|\mu|_U|$ is the restriction of the total variation of μ to the Borel σ-algebra of U. Proposition C.2 then says that $\|\mu|_U\|=|\mu|(U)$ is equal to the supremum of $|\langle\mu|_U,f\rangle|$ with f varying over the set of continuous functions $f:U\to\mathbb{R}$ with compact support such that $\|f\|_{\sup}\le 1$. Thus $|\mu|(U)$ is also equal to the supremum of $|\langle\mu,f\rangle|$ with f varying over the set of functions $f\in C(K)$ with support contained in U such that $\|f\|_{\sup}\le 1$. Hence (6.5) is equal to the set of those $\mu\in M(K)$ with $|\langle\mu,f\rangle|\le c$ for every $f\in C(K)$ with support contained in U such that $\|f\|_{\sup}\le 1$. Such set is obviously weak*-closed. $\square$

We now have all the necessary ingredients to establish the core of our weak*-compactness result.

Lemma 6.8. *If X is a completely regular topological space and $\Lambda\subset\mathrm{ca}(X)$ is a tight collection of regular measures then $\Phi[\Lambda]$ is contained in a weak*-closed subset $\mathcal{F}$ of $M\big(\beta(X)\big)$ that is contained in the image of Φ. Moreover, if in addition $\Lambda\subset\mathrm{ca}_+(X)$ then $\mathcal{F}$ can be chosen in such a way that it is contained in $\Phi[\mathrm{ca}_+(X)]$.*

Proof. Using the tightness of Λ, choose for each $n\ge 1$ a compact Borel subset K_n of X such that $|\mu|(X\setminus K_n)\le\frac{1}{n}$ for all $\mu\in\Lambda$. The set

$$\mathcal{F}=\bigcap_{n=1}^{\infty}\big\{\nu\in M\big(\beta(X)\big):|\nu|\big(\beta(X)\setminus\iota[K_n]\big)\le\tfrac{1}{n}\big\}$$

is weak*-closed in $M\big(\beta(X)\big)$ by Lemma 6.7. To see that $\mathcal{F}$ contains $\Phi[\Lambda]$, note that every $\mu\in\Lambda$ is tight and regular so that Lemma 6.6 implies that

$\Phi(\mu) = \iota_*\mu$ and using (2.8) we obtain

$$|\iota_*\mu|\big(\beta(X) \setminus \iota[K_n]\big) \le (\iota_*|\mu|)\big(\beta(X) \setminus \iota[K_n]\big) = |\mu|\big(X \setminus \iota^{-1}[\iota[K_n]]\big) \le |\mu|(X \setminus K_n) \le \tfrac{1}{n},$$

for all $n \ge 1$. Obviously every $\nu \in \mathcal{F}$ vanishes on every Borel subset of $\beta(X)$ disjoint from the image of ι and thus Lemma 6.4 implies that $\mathcal{F}$ is contained in the image of Φ. It remains to check that if Λ is contained in $\mathrm{ca}_+(X)$ then we can choose $\mathcal{F}$ contained in $\Phi\big[\mathrm{ca}_+(X)\big]$. To this aim, simply replace $\mathcal{F}$ with its intersection with the set of nonnegative regular measures on $\beta(X)$. Such set is weak*-closed in $M\big(\beta(X)\big)$ as it corresponds to the set of positive linear functionals on $C\big(\beta(X)\big)$ via the identification $M\big(\beta(X)\big) \equiv C\big(\beta(X)\big)^*$ (Proposition C.3). □

In what follows, a subset of an arbitrary topological space will be called *relatively compact* if its closure is compact (equivalently, if it is contained in a subset that is both compact and closed). For general topological spaces that are not Hausdorff this notion of relative compactness is stronger than the requirement that the set be contained in a compact subset as compact subsets need not have compact closures. However, for the weak* topology of $\mathrm{ca}(X)$ (or the weak* topology of a subset of $\mathrm{ca}(X)$) the situation is nicer and we do have that the closure of a compact subset is compact. Namely, this follows from Corollary 6.2 since the weak* topology of $\mathrm{ca}(X)$ (and also of its subsets) is induced from the weak* topology of $C_{\mathrm{b}}(X)^*$ which is Hausdorff.

Proposition 6.9. *Let X be a completely regular topological space and Λ be a bounded subset of $\mathrm{ca}(X)$. If Λ is tight and every element of Λ is regular then Λ is weak*-relatively compact in $\mathrm{ca}(X)$. If in addition $\Lambda \subset \mathrm{ca}_+(X)$ then Λ is weak*-relatively compact in $\mathrm{ca}_+(X)$.*

Proof. Since Λ is bounded and Φ is a bounded linear map, the set $\Phi[\Lambda]$ is contained in a closed ball B of $M\big(\beta(X)\big)$. Let $\mathcal{F}$ be the weak*-closed subset of $M\big(\beta(X)\big)$ given by Lemma 6.8. The set $\mathcal{F} \cap B$ is weak*-compact by the Banach–Alaoglu Theorem and as the weak* topology of $\mathrm{ca}(X)$ is induced by Φ from the weak* topology of $M\big(\beta(X)\big)$, Lemma 6.1 yields that $\Phi^{-1}[\mathcal{F} \cap B]$ is a (weak*-closed and) weak*-compact subset of $\mathrm{ca}(X)$ containing Λ. To conclude the proof, note that if $\Lambda \subset \mathrm{ca}_+(X)$ then $\mathcal{F}$ can be assumed to be contained in $\Phi[\mathrm{ca}_+(X)]$ and applying Lemma 6.1 to $\Phi|_{\mathrm{ca}_+(X)}$ we obtain that $\Phi^{-1}[\mathcal{F} \cap B] \cap \mathrm{ca}_+(X)$ is (weak*-closed in $\mathrm{ca}_+(X)$ and) weak*-compact. □

Now we are going to study conditions under which a converse of Proposition 6.9 can be proven. First we show that weak*-compact subsets of $\mathrm{ca}(X)$ are bounded under assumptions that make the map (6.1) from $\mathrm{ca}(X)$ to $C_{\mathrm{b}}(X)^*$ an isometric embedding.

Lemma 6.10. *If X is a perfectly normal topological space then every weak*-compact subset of* $\mathrm{ca}(X)$ *is bounded.*

Proof. The image of a weak*-compact subset of $\mathrm{ca}(X)$ under (6.1) is weak*-compact in $C_{\mathrm{b}}(X)^*$ and thus it is a pointwise bounded set of bounded linear functionals on the Banach space $C_{\mathrm{b}}(X)$. The conclusion follows from the Uniform Boundedness Principle and from the fact that (6.1) is an isometric embedding if X is perfectly normal (Proposition 3.6). □

We will only be able to prove tightness of a weak*-compact subset of $\mathrm{ca}_+(X)$ if X is a Polish space. The trick is to use the fact that a totally bounded subset of a complete metric space is relatively compact. Recall that a subset of a metric space is called *totally bounded* if for every $\varepsilon > 0$ it can be covered by finitely many sets of diameter less than ε.

Lemma 6.11. *Let (X, d) be a metric space and Λ be a subset of* $\mathrm{ca}(X)$. *The following conditions are equivalent:*

(a) *for every $\varepsilon > 0$ there exists a Borel subset B of X which is a finite union of sets with diameter less than ε and such that $|\mu|(X \setminus B) < \varepsilon$, for all $\mu \in \Lambda$;*

(b) *for every $\varepsilon > 0$ there exists a totally bounded Borel subset B of X such that $|\mu|(X \setminus B) < \varepsilon$, for all $\mu \in \Lambda$.*

If (X, d) is complete then (a) and (b) are equivalent to:

(c) *Λ is tight.*

Proof. The only nontrivial implication is (a)⇒(b). Given $\varepsilon > 0$, use (a) to find for every $n \geq 1$ a Borel subset B_n of X which is a finite union of sets with diameter less than $\frac{1}{n}$ and for which $|\mu|(X \setminus B_n) < \frac{\varepsilon}{2^n}$, for every $\mu \in \Lambda$. The proof of (b) is concluded by taking $B = \bigcap_{n=1}^{\infty} B_n$. □

Lemma 6.12. *If (X, d) is a separable metric space and Λ is a weak*-compact subset of* $\mathrm{ca}_+(X)$ *then for every $\varepsilon > 0$ there exists a totally bounded Borel subset B of X such that $\mu(X \setminus B) < \varepsilon$, for all $\mu \in \Lambda$.*

Proof. We prove that condition (a) in Lemma 6.11 holds. Given $\varepsilon > 0$, since X is separable we can write X as a countable union of open subsets with diameter less than ε and therefore for every $\mu \in \mathrm{ca}_+(X)$ we can find an open subset U_μ of X which is a finite union of open subsets with diameter less than ε and for which $\mu(X \setminus U_\mu) < \varepsilon$. By Lemma 4.5 the set

$$\big\{\nu \in \mathrm{ca}_+(X) : \nu(X \setminus U_\mu) < \varepsilon\big\} \tag{6.6}$$

is a weak*-open neighborhood of μ in $\mathrm{ca}_+(X)$ and therefore the weak*-compactness of Λ implies that there exists a finite subset F of Λ such that the sets (6.6) with $\mu \in F$ cover Λ. Finally, setting $U = \bigcup_{\mu \in F} U_\mu$ we obtain that U is a finite union of open subsets with diameter less than ε and that $\nu(X \setminus U) < \varepsilon$ for all $\nu \in \Lambda$. □

Corollary 6.13. *If X is a Polish space then every weak*-compact subset of* $\mathrm{ca}_+(X)$ *is tight.* □

All the work done in this section yields now the following characterization of weak*-relatively compact subsets of $\mathrm{ca}_+(X)$ if X is Polish.

Theorem 6.14. *If X is a Polish space then a subset of* $\mathrm{ca}_+(X)$ *is weak*-relatively compact in* $\mathrm{ca}_+(X)$ *if and only if it is bounded and tight.*

Proof. Follows from Proposition 6.9, Corollary 3.5, Lemma 6.10 and Corollary 6.13. □

Corollary 6.15. *If X is a Polish space then a subset of* $\mathrm{ca}^1_+(X)$ *is weak*-relatively compact in* $\mathrm{ca}^1_+(X)$ *if and only if it is tight.*

Proof. Follows from Theorem 6.14 noting that $\mathrm{ca}^1_+(X)$ is weak*-closed in $\mathrm{ca}_+(X)$ and bounded. □

7. The Prokhorov metric

The goal of this section is to show that if X is a separable metric space then the weak* topology of $\mathrm{ca}_+(X)$ is metrizable. This is done by explicitly exhibiting a metric that induces the weak* topology of $\mathrm{ca}_+(X)$.

We start by introducing some notation. Let (X, d) be a metric space. Given a subset A of X and $\varepsilon > 0$ we set:

$$A^\varepsilon = \big\{x \in X : d(x, A) < \varepsilon\big\} = \big\{x \in X : d(x, a) < \varepsilon, \text{ for some } a \in A\big\}.$$

Clearly A^ε is an open subset of X containing A. Moreover,

$$A^\varepsilon \subset A^{\varepsilon'}, \quad \text{if } 0 < \varepsilon \le \varepsilon' \tag{7.1}$$

and

$$(A^\varepsilon)^{\varepsilon'} \subset A^{\varepsilon+\varepsilon'}, \tag{7.2}$$

for all $\varepsilon, \varepsilon' > 0$. Given $\mu, \nu \in \mathrm{ca}_+(X)$, it follows from (7.1) that the set

$$\Pi_{\mu,\nu} = \big\{\varepsilon \in\,]0, +\infty[\, : \mu(A) \le \nu(A^\varepsilon) + \varepsilon, \text{ for every Borel subset } A \text{ of } X\big\}$$

is a right-unbounded interval. We define

$$\pi_0(\mu, \nu) = \inf \Pi_{\mu,\nu}, \tag{7.3}$$

for all $\mu, \nu \in \mathrm{ca}_+(X)$. Let us summarize the properties of π_0.

Lemma 7.1. *If (X, d) is a metric space and π_0 is defined by (7.3) then the following conditions hold for all $\mu, \nu, \rho \in \mathrm{ca}_+(X)$:*

(a) $\pi_0(\mu, \nu) \ge 0$;

(b) $\pi_0(\mu, \nu)$ *is zero if and only if* $\mu(A) \le \nu(A)$ *for every Borel subset A of X;*

(c) $\pi_0(\mu, \rho) \le \pi_0(\mu, \nu) + \pi_0(\nu, \rho)$.

Proof. Condition (a) is obvious. To prove (b), note first that $\pi_0(\mu,\nu)=0$ if and only if

$$\mu(A)\le\nu(A^\varepsilon)+\varepsilon, \tag{7.4}$$

for every Borel subset A of X and every $\varepsilon>0$. Obviously (7.4) holds if $\mu(A)\le\nu(A)$ and, moreover, if A is closed and (7.4) holds for all $\varepsilon>0$ then $\mu(A)\le\nu(A)$, since $\bigcap_{\varepsilon>0}A^\varepsilon$ is the closure of A. To conclude the proof of (b), note that if A is an arbitrary Borel subset of X and $\mu(F)\le\nu(F)$ for every closed subset F of A then $\mu(F)\le\nu(A)$ for every closed subset F of A and thus $\mu(A)\le\nu(A)$ by the regularity of μ (Corollary 3.5). Finally, to prove (c) note that if $\varepsilon\in\Pi_{\mu,\nu}$ and $\varepsilon'\in\Pi_{\nu,\rho}$ then it follows from (7.2) that $\varepsilon+\varepsilon'\in\Pi_{\mu,\rho}$. □

Lemma 7.1 shows that π_0 is close to being a metric and that the ingredient that is missing is the symmetry. So we symmetrize it by setting

$$\pi(\mu,\nu)=\max\{\pi_0(\mu,\nu),\pi_0(\nu,\mu)\}, \tag{7.5}$$

for all $\mu,\nu\in\mathrm{ca}_+(X)$. Note that since $\Pi_{\mu,\nu}$ and $\Pi_{\nu,\mu}$ are right-unbounded intervals, we have:

$$\pi(\mu,\nu)=\inf\,\Pi_{\mu,\nu}\cap\Pi_{\nu,\mu},$$

i.e., $\pi(\mu,\nu)$ is the infimum of the set of those $\varepsilon>0$ such that both inequalities

$$\mu(A)\le\nu(A^\varepsilon)+\varepsilon,\quad \nu(A)\le\mu(A^\varepsilon)+\varepsilon \tag{7.6}$$

hold for every Borel subset A of X.

The fact that π is a metric is an immediate consequence of Lemma 7.1.

Proposition 7.2. *If (X,d) is a metric space then the map π defined in* (7.5) *is a metric on the set* $\mathrm{ca}_+(X)$. □

Definition 7.3. The map π defined in (7.5) is called the *Prokhorov metric*.

Remark 7.4. If $\mu,\nu\in\mathrm{ca}_+(X)$ are such that $\mu(X)=\nu(X)$ then $\Pi_{\mu,\nu}=\Pi_{\nu,\mu}$. Namely, if $\varepsilon>0$ is such that the inequality

$$\mu(A)\le\nu(A^\varepsilon)+\varepsilon \tag{7.7}$$

holds for every Borel subset A of X then replacing A with $X\setminus A^\varepsilon$ in (7.7) and using that $(X\setminus A^\varepsilon)^\varepsilon\subset X\setminus A$ we obtain $\nu(A)\le\mu(A^\varepsilon)+\varepsilon$. It follows from this observation that if $\mu(X)=\nu(X)$ then:

$$\pi_0(\mu,\nu)=\pi_0(\nu,\mu)=\pi(\mu,\nu).$$

In particular, π_0 and π are the same for probability measures.

Let us now study the relationship between the topology induced by the Prokhorov metric π and the weak* topology on $\mathrm{ca}_+(X)$.

Proposition 7.5. *If (X,d) is a metric space then every weak*-open subset of* $\mathrm{ca}_+(X)$ *is open with respect to the topology induced by the Prokhorov metric.*

Proof. It is sufficient to prove that the subbasic weak*-open sets (4.7) are Prokhorov-open. First, given $c \in \mathbb{R}$ and $\mu \in \mathrm{ca}_+(X)$ with $\mu(X) < c$ we show that $\nu(X) < c$ for ν in some Prokhorov-open ball centered at μ. To this aim, pick $\varepsilon > 0$ with $\mu(X) + \varepsilon < c$. If $\nu \in \mathrm{ca}_+(X)$ is such that $\pi(\mu, \nu) < \varepsilon$ then the inequalities (7.6) hold for every Borel subset A of X and in particular setting $A = X$ we get:

$$\nu(X) \leq \mu(X^\varepsilon) + \varepsilon = \mu(X) + \varepsilon < c.$$

Now let $c \in \mathbb{R}$, U be an open subset of X and $\mu \in \mathrm{ca}_+(X)$ be such that $\mu(U) > c$ and let us again show that $\nu(U) > c$ for ν in some Prokhorov-open ball centered at μ. As in the final part of the proof of Proposition 4.16, we can write U as a countable increasing union of closed sets H such that $d(H, X \setminus U) > 0$ and therefore we can choose such a closed set H satisfying $\mu(H) > c$. Now pick $\varepsilon > 0$ small enough so that $\varepsilon \leq d(H, X \setminus U)$ and $\mu(H) > c + \varepsilon$ and let $\nu \in \mathrm{ca}_+(X)$ be such that $\pi(\mu, \nu) < \varepsilon$. Using inequalities (7.6) with $A = H$ we obtain

$$c + \varepsilon < \mu(H) \leq \nu(H^\varepsilon) + \varepsilon \leq \nu(U) + \varepsilon,$$

since $H^\varepsilon \subset U$. Hence $\nu(U) > c$ and we are done. □

Our proof that Prokhorov-open sets in $\mathrm{ca}_+(X)$ are weak*-open when X is a separable metric space will use the weak* fundamental system of neighborhoods given by Corollary 4.13. We first prove two simple lemmas concerning Borel subsets whose boundary has null measure.

Lemma 7.6. *If (X, d) is a metric space and $\mu \in \mathrm{ca}_+(X)$ then for every $x \in X$ we have that the boundary of the open ball $\mathrm{B}(x, r)$ of center x and radius r has measure zero with respect to μ except possibly for countably many $r > 0$.*

Proof. The boundary of the open ball $\mathrm{B}(x, r)$ is contained in the sphere $\big\{y \in X : d(y, x) = r\big\}$ and such spheres are pairwise disjoint, so at most countably many of them can have nonzero measure. □

Lemma 7.7. *If X is a topological space and μ is a nonnegative measure on the Borel σ-algebra of X then the collection of all Borel subsets $B \subset X$ with $\mu(\partial B) = 0$ is an algebra of subsets of X, i.e., it is nonempty, closed under finite unions and complements (and thus also under finite intersections and differences).*

Proof. Simply note that $\partial(B_1 \cup B_2) \subset (\partial B_1) \cup (\partial B_2)$ for any subsets B_1 and B_2 of X and that $\partial(X \setminus B) = X \setminus (\partial B)$ for any subset B of X. □

Theorem 7.8. *If (X, d) is a separable metric space then the weak* topology in $\mathrm{ca}_+(X)$ coincides with the topology induced by the Prokhorov metric.*

Proof. By Proposition 7.5 it is sufficient to prove that for every $\mu \in \mathrm{ca}_+(X)$ and every $\varepsilon > 0$ there exists a weak*-neighborhood of μ in $\mathrm{ca}_+(X)$ contained in the Prokhorov-open ball of center μ and radius ε. Fix $\varepsilon' > 0$ which we will choose later in terms of ε alone. By Lemma 7.6 the space X can be covered by open balls of diameter less than ε' whose boundary has measure zero with respect to μ and by separability such covering can be assumed to be countable. Moreover, using Lemma 7.7 we can disjointify such open balls obtaining a countable partition $X = \bigcup_{n=1}^{\infty} B_n$ of X by Borel subsets B_n with diameter less than ε' and satisfying $\mu(\partial B_n) = 0$. Pick $N \geq 1$ large enough so that $\mu(X \setminus B) < \varepsilon'$, where $B = \bigcup_{n=1}^{N} B_n$. We have that also $\partial(X \setminus B)$ has measure zero with respect to μ and therefore Corollary 4.13 yields that

$$\big\{\nu \in \mathrm{ca}_+(X) : |\mu(B_n) - \nu(B_n)| < \tfrac{\varepsilon'}{N},\ n = 1, \ldots, N \\ \text{and } |\mu(X \setminus B) - \nu(X \setminus B)| < \varepsilon'\big\}$$

is a weak*-neighborhood of μ in $\mathrm{ca}_+(X)$. Let ν in such weak*-neighborhood of μ in $\mathrm{ca}_+(X)$ be fixed and let us estimate the Prokhorov distance $\pi(\mu, \nu)$. Given a Borel subset A of X, let I denote the set of indices $n \in \{1, \ldots, N\}$ such that A intersects B_n. Since all B_n have diameter less than ε', we have that $B_n \subset A^{\varepsilon'}$ for all $n \in I$ and therefore:

$$\text{(7.8)} \quad \mu(A) = \mu(A \setminus B) + \sum_{n \in I} \mu(A \cap B_n) \leq \mu(X \setminus B) + \sum_{n \in I} \mu(B_n) \\ < \varepsilon' + \sum_{n \in I} \big(\nu(B_n) + \tfrac{\varepsilon'}{N}\big) \leq \nu(A^{\varepsilon'}) + 2\varepsilon' \leq \nu(A^{2\varepsilon'}) + 2\varepsilon'.$$

Similarly:

$$\text{(7.9)} \quad \nu(A) = \nu(A \setminus B) + \sum_{n \in I} \nu(A \cap B_n) \leq \nu(X \setminus B) + \sum_{n \in I} \nu(B_n) \\ < \mu(X \setminus B) + \varepsilon' + \sum_{n \in I} \big(\mu(B_n) + \tfrac{\varepsilon'}{N}\big) < \mu(A^{\varepsilon'}) + 3\varepsilon' \leq \mu(A^{3\varepsilon'}) + 3\varepsilon'.$$

From (7.8) and (7.9) we get $\pi(\mu, \nu) \leq 3\varepsilon'$ and we could have chosen $\varepsilon' > 0$ with $3\varepsilon' < \varepsilon$ to now get the desired conclusion that ν is in the Prokhorov-open ball of center μ and radius ε. □

Remark 7.9. We observe that the weak* topology is typically not metrizable in the space $\mathrm{ca}(X)$ of all finite signed measures. In fact, if X is completely regular and Hausdorff then Proposition E.15 and Remark E.16 imply that the weak* topology on $\mathrm{ca}(X)$ is metrizable if and only if X is finite. Even bounded regions of $\mathrm{ca}(X)$ are often not metrizable in the weak* topology. For instace, assuming again that X is completely regular and metrizable, Proposition E.17 and Lemma E.18 imply that the unit ball of $\mathrm{ca}(X)$ is metrizable in the weak* topology if and only if X is compact and metrizable.

We conclude the section by showing that $\mathrm{ca}_+(X)$ is Polish with respect to the weak* topology if X is Polish. The missing ingredient is the completeness of the Prokhorov metric.

Proposition 7.10. *If (X, d) is a separable complete metric space then the Prokhorov metric in $\mathrm{ca}_+(X)$ is complete.*

Proof. Since every compact metric space is complete, it is sufficient to show that every Prokhorov-Cauchy sequence is contained in a Prokhorov-compact subset of $\mathrm{ca}_+(X)$. By Theorem 7.8, Proposition 6.9 and Corollary 3.5 it is then sufficient to prove that if $(\mu_n)_{n\geq 1}$ is Prokhorov-Cauchy in $\mathrm{ca}_+(X)$ then $\{\mu_n : n \geq 1\}$ is bounded (with respect to the total variation norm) and tight. It is readily checked that $|\mu(X) - \nu(X)| \leq \pi(\mu, \nu)$ for all $\mu, \nu \in \mathrm{ca}_+(X)$ and therefore every Prokhorov-bounded sequence in $\mathrm{ca}_+(X)$ is bounded with respect to the norm of $\mathrm{ca}(X)$. To prove that $\{\mu_n : n \geq 1\}$ is tight we check condition (a) of Lemma 6.11. Let $\varepsilon > 0$ be given and fix $\varepsilon' > 0$ which we will choose later in terms of ε alone. Let $n_0 \geq 1$ be such that $\pi(\mu_n, \mu_m) < \varepsilon'$ for all $n, m \geq n_0$.

Since X is separable, it can be covered by countably many open balls $\mathrm{B}(x, \varepsilon')$ of radius ε' and therefore there exists a finite subset $F \subset X$ such that $\mu_n\big(X \setminus \bigcup_{x\in F} \mathrm{B}(x, \varepsilon')\big) < \varepsilon'$ for all $n = 1, \ldots, n_0$. Setting

$$A = X \setminus \bigcup_{x\in F} \mathrm{B}(x, 2\varepsilon'),$$

we have $A^{\varepsilon'} \subset X \setminus \bigcup_{x\in F} \mathrm{B}(x, \varepsilon')$ and since $\pi(\mu_n, \mu_{n_0}) < \varepsilon'$ for $n \geq n_0$ we obtain:

$$\mu_n(A) \leq \mu_{n_0}(A^{\varepsilon'}) + \varepsilon' \leq \mu_{n_0}\left(X \setminus \bigcup_{x\in F} \mathrm{B}(x, \varepsilon')\right) + \varepsilon' < 2\varepsilon'.$$

Since also $\mu_n(A) \leq \mu_n\big(X \setminus \bigcup_{x\in F} \mathrm{B}(x, \varepsilon')\big) < \varepsilon'$ for $n \leq n_0$ we have that the open set $B = \bigcup_{x\in F} \mathrm{B}(x, 2\varepsilon')$ is a finite union of sets of diameter less than or equal to $4\varepsilon'$ and $\mu_n(X \setminus B) = \mu_n(A) < 2\varepsilon'$ for all $n \geq 1$. This concludes the proof as we could have chosen $\varepsilon' > 0$ with $4\varepsilon' < \varepsilon$. □

Theorem 7.11. *If X is a Polish space then $\mathrm{ca}_+(X)$ endowed with the weak* topology is also Polish.*

Proof. Follows from Theorem 7.8, Proposition 7.10 and Corollary 4.21. □

Corollary 7.12. *If X is a Polish space then $\mathrm{ca}_+^1(X)$ endowed with the weak* topology is also Polish.*

Proof. Follows from Theorem 7.11 and the fact that $\mathrm{ca}_+^1(X)$ is weak*-closed in $\mathrm{ca}_+(X)$. □

8. The vague topology

Given a topological space X, we denote by $C_\mathrm{c}(X)$ the space of continuous real-valued functions on X having compact support. The vague topology on $\mathrm{ca}(X)$ is the defined like the weak* topology but replacing $C_\mathrm{b}(X)$ with $C_\mathrm{c}(X)$.

Definition 8.1. Let X be a topological space. The *vague topology* on $\mathrm{ca}(X)$ is the topology induced by the linear functionals $\langle \cdot, f\rangle$ with $f \in C_\mathrm{c}(X)$.

Obviously the weak* topology on $\mathrm{ca}(X)$ is finer than the vague topology so that weak* convergence of a net implies vague convergence. In this section we will always assume that X is locally compact and Hausdorff so that $C_\mathrm{c}(X)$ is "reasonably large". Note, for example, that if X is an infinite-dimensional normed vector space then every compact subset of X has empty interior and therefore $C_\mathrm{c}(X)$ is the null space, so that the vague topology on $\mathrm{ca}(X)$ is trivial.

Recall that a noncompact locally compact Hausdorff space X admits a unique (up to homeomorphism) Hausdorff one-point compactification which is obtained by adding a new point ∞ to the space X and by declaring as open subsets of $X \cup \{\infty\}$ the original open subsets of X and the complements in $X \cup \{\infty\}$ of the compact subsets of X. Since $X \cup \{\infty\}$ is compact Hausdorff, it is normal and therefore Urysohn's Lemma is valid on $X \cup \{\infty\}$. From this observation one readily obtains the following version of Urysohn's Lemma for a locally compact Hausdorff space.

Lemma 8.2. *If X is a locally compact Hausdorff space, K is a compact subset of X and F is a closed subset of X disjoint from K then there exists a continuous function $f : X \to [0, 1]$ that is equal to 1 on K and vanishes on F.*

Proof. Simply note that K is closed in the one-point compactification of X and hence one can apply Urysohn's Lemma to K and the closure of F in the one-point compactification of X. □

Corollary 8.3. *A locally compact Hausdorff topological space is completely regular.* □

Corollary 8.4. *If X is locally compact Hausdorff and K is a compact subset of X contained in an open subset U of X then there exists a continuous function $f : X \to [0, 1]$ that is equal to 1 on K and whose support is compact and contained in U.*

Proof. Apply Lemma 8.2 to K and $F = X \setminus V$, where V is an open subset of X containing K whose closure is compact and contained in U. □

Example 8.5. If X is compact then $C_\mathrm{c}(X) = C_\mathrm{b}(X)$ and therefore the vague and weak* topologies coincide. On the other hand, if X is locally compact Hausdorff but noncompact then there exists a net $(x_i)_{i\in I}$ in X that converges to ∞ in the one-point compactification of X and therefore $\lim_{i\in I}\langle \delta_{x_i}, f\rangle = 0$ for all $f \in C_\mathrm{c}(X)$, so that $(\delta_{x_i})_{i\in I}$ vaguely converges to zero. On the other hand, $\delta_{x_i}(X) = 1$ for all $i \in I$ and thus $(\delta_{x_i})_{i\in I}$ does not weak*-converge to zero.

For an important class of locally compact Hausdorff topological spaces it holds that, for nonnegative measures, weak* convergence is just vague convergence plus convergence of the measure of the entire space, as we show below. Recall that a subset of a topological space is called *σ-compact* if it is a countable union of compact subsets.

Proposition 8.6. *Let X be a locally compact Hausdorff topological space in which every open subset is σ-compact (this happens, for instance, if X is second countable). Given a net $(\mu_i)_{i\in I}$ in $\mathrm{ca}_+(X)$ and $\mu \in \mathrm{ca}_+(X)$ we have that $(\mu_i)_{i\in I}$ weak*-converges to μ if and only if $(\mu_i)_{i\in I}$ vaguely converges to μ and $\lim_{i\in I} \mu_i(X) = \mu(X)$.*

Proof. Apply Lemma 4.14 with $\mathcal{F}$ equal to the set of $[0,1]$-valued continuous functions on X with compact support, keeping in mind Corollary 8.4 and the fact that every open subset of X is an increasing union of a sequence of compact subsets of X. □

Example 8.7. Proposition 8.6 fails for signed measures. For example, if $X = \mathbb{R}$ and $\mu_n = \delta_n - \delta_{n+1}$ for all $n \geq 1$ then the sequence $(\mu_n)_{n\geq 1}$ vaguely converges to zero and $\mu_n(\mathbb{R}) = 0$ for all $n \geq 1$. Yet $(\mu_n)_{n\geq 1}$ is not weak*-convergent as there are bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which the limit $\lim_{n\to+\infty}[f(n) - f(n+1)]$ does not exist.

Example 8.8. Proposition 8.6 fails without the assumption that every open subset is σ-compact. Let ω_1 denote the first uncountable ordinal and let X be the disjoint union of ω_1 and $\mathbb{R}$ endowed with the topology that makes ω_1 and $\mathbb{R}$ open subspaces of X, where $\omega_1 = [0, \omega_1[$ is endowed with the order topology and $\mathbb{R}$ is endowed with the Euclidean topology. Let ν be the nonregular measure on $[0, \omega_1]$ defined in Example C.8 and define $\mu \in \mathrm{ca}_+(X)$ by setting

$$\mu(B) = \nu(B \cap \omega_1),$$

for every Borel subset B of X. The sequence $(\delta_n)_{n\geq 1}$ converges vaguely to μ since for $f \in C_\mathrm{c}(X)$ we have $f(n) = 0$ for n sufficiently large and $\langle \mu, f\rangle = 0$. Also $\delta_n(X) = \mu(X) = 1$, for all $n \geq 1$ and yet $(\delta_n)_{n\geq 1}$ does not weak*-converge to μ. Namely, if f is the characteristic function of the clopen ω_1 then $\langle \delta_n, f\rangle = 0$ for all $n \geq 1$ and $\langle \mu, f\rangle = 1$.

If X is a locally compact Hausdorff topological space then the vague topology on $\mathrm{ca}(X)$ obviously coincides with the topology induced by the linear map

$$\mathrm{ca}(X) \ni \mu \longmapsto \langle \mu, \cdot\rangle \in C_\mathrm{c}(X)^*, \tag{8.1}$$

where the topological dual $C_\mathrm{c}(X)^*$ of $C_\mathrm{c}(X)$ is endowed with the weak* topology. Here we regard $C_\mathrm{c}(X)$ as a normed space endowed with the supremum norm. The map (8.1) is not in general injective (see Example C.8) and therefore the vague topology is not in general Hausdorff. Using the theory developed in Appendix C we obtain a simple sufficient condition for the injectivity of (8.1).

Proposition 8.9. *Let X be a locally compact Hausdorff topological space. The following conditions are equivalent:*

(1) *the linear map* (8.1) *is injective;*
(2) *the vague topology on* $\mathrm{ca}(X)$ *is Hausdorff;*
(3) $\mathrm{ca}_+(X)$ *is vaguely closed in* $\mathrm{ca}(X)$*;*
(4) *the linear map* (8.1) *is an isometry.*

Moreover, the condition

(5) *every open subset of X is σ-compact*

implies (1), (2), (3) and (4).

Proof. We will show that (1), (2), (3) and (4) are all equivalent to the condition that every $\mu \in \mathrm{ca}(X)$ is Radon (see Definition C.1). The proof will then be concluded by observing that Proposition C.7 says that (5) implies that every $\mu \in \mathrm{ca}(X)$ is Radon. Note first that the equivalence between (1) and (2) follows from Remark E.2. Moreover, if every $\mu \in \mathrm{ca}(X)$ is Radon then Corollary C.6 shows that (4) holds and Proposition C.3 shows that (3) holds because the set of bounded positive linear functionals on $C_\mathrm{c}(X)$ is weak*-closed in $C_\mathrm{c}(X)^*$. To see that (1) implies that every $\mu \in \mathrm{ca}(X)$ is Radon note that if $\mu \in \mathrm{ca}(X)$ is not Radon then Theorem C.5 implies that some Radon measure must be mapped under (8.1) to the same linear functional that μ is mapped to. It remains to check that (3) implies (1). To see this, note that if (8.1) is not injective then some nonzero $\mu \in \mathrm{ca}(X)$ is mapped to zero under (8.1) and therefore both μ and $-\mu$ belong to the vague closure of $\{0\} \subset \mathrm{ca}_+(X)$. However, either μ or $-\mu$ is not in $\mathrm{ca}_+(X)$ and hence $\mathrm{ca}_+(X)$ is not vaguely closed. □

We now present adaptations to the vague topology of several results that were proven in the previous sections for the weak* topology. We start with Lemmas 4.5 and 4.6.

Lemma 8.10. *Let X be a locally compact Hausdorff topological space and let $\mathrm{ca}_+(X)$ be endowed with the vague topology. If U is an open σ-compact subset of X then the map $\mathrm{ca}_+(X) \ni \mu \mapsto \mu(U) \in \mathbb{R}$ is lower semicontinuous, i.e., for every $c \in \mathbb{R}$ the set*

$$\big\{\mu \in \mathrm{ca}_+(X) : \mu(U) > c\big\}$$

is vaguely open in $\mathrm{ca}_+(X)$. Moreover, if K is a compact G_δ subset of X then the map $\mathrm{ca}_+(X) \ni \mu \mapsto \mu(K) \in \mathbb{R}$ is upper semicontinuous, i.e., for every $c \in \mathbb{R}$ the set

$$\big\{\mu \in \mathrm{ca}_+(X) : \mu(K) < c\big\}$$

is vaguely open in $\mathrm{ca}_+(X)$.

Proof. Analogous to the proof of Lemma 4.5 using Corollary 8.4 instead of Urysohn's Lemma and noting that H can be chosen to be compact because U is σ-compact. □

Lemma 8.11. *Let X be a locally compact Hausdorff topological space in which every relatively compact open set is an F_σ and let $\mathrm{ca}_+(X)$ be endowed with the vague topology. If $f : X \to \left]-\infty, +\infty\right]$ is a lower semicontinuous function bounded from below with compact support then the map*

$$\mathrm{ca}_+(X) \ni \mu \longmapsto \langle \mu, f\rangle \in \left]-\infty, +\infty\right]$$

is lower semicontinuous, i.e., for every $c \in \mathbb{R}$ the set

$$\big\{\mu \in \mathrm{ca}_+(X) : \langle \mu, f\rangle > c\big\}$$

is vaguely open in $\mathrm{ca}_+(X)$. Similarly, for every upper semicontinuous function $f : X \to \left[-\infty, +\infty\right[$ bounded from above with compact support the map

$$\mathrm{ca}_+(X) \ni \mu \longmapsto \langle \mu, f\rangle \in \left[-\infty, +\infty\right[$$

is upper semicontinuous, i.e., for every $c \in \mathbb{R}$ the set

$$\big\{\mu \in \mathrm{ca}_+(X) : \langle \mu, f\rangle < c\big\}$$

is vaguely open in $\mathrm{ca}_+(X)$.

Proof. A simple adaptation of the proof of Lemma 4.6: to reduce the general case to the case of nonnegative f we add an element of $C_\mathrm{c}(X)$ to f instead of a constant. Since f has compact support, the assumptions in Corollary B.6 can be weakened so that it applies when every relatively compact open subset of X is an F_σ and the space X is only Hausdorff (see Remark B.7). Moreover, the map g given by Corollary B.6 will also have compact support. □

Adapting Proposition 4.9 to the vague topology is a little trickier as we have to remove the condition $\lim_{i\in I} \mu_i(X) = \mu(X)$ from the items in which it appears because such equality is not a consequence of vague convergence. Thus the equivalence between items (b) and (c) is lost and they must be combined in a single item which we call (b-c) to make the comparison between the new and old statements easier.

We need first a slight generalization of Lemma 4.7.

Lemma 8.12. *Let $(X, \mathcal{A})$ be a measurable space, $B \in \mathcal{A}$ be a measurable subset of X and $(\mu_i)_{i\in I}$ be a net in $\mathrm{ca}(X, \mathcal{A})$. If $\sup_{i\geq i_0} |\mu_i|(B) < +\infty$ for some $i_0 \in I$ then the set*

$$\Big\{f \in \mathcal{M}_\mathrm{b}(X, \mathcal{A}) : \lim_{i\in I} \langle \mu_i, f\rangle = \langle \mu, f\rangle \text{ and } f \text{ vanishes outside of } B\Big\}$$

is a closed subspace of $\mathcal{M}_\mathrm{b}(X, \mathcal{A})$ with respect to the supremum norm for any $\mu \in \mathrm{ca}(X, \mathcal{A})$.

Proof. Note that $|\langle \mu, f\rangle| \leq |\mu|(B)\, \|f\|_{\sup}$ if $\mu \in \mathrm{ca}(X, \mathcal{A})$ and $f \in \mathcal{M}_\mathrm{b}(X, \mathcal{A})$ vanishes outside of B and apply Corollary F.3 to the bounded bilinear pairing between the closed subspace of $\mathcal{M}_\mathrm{b}(X, \mathcal{A})$ consisting of functions that vanish outside of B and the space $\mathrm{ca}(X, \mathcal{A})$ endowed with the semi-norm $\mu \mapsto |\mu|(B)$. □

Proposition 8.13. *Let X be a locally compact Hausdorff topological space, $(\mu_i)_{i\in I}$ be a net in $\mathrm{ca}_+(X)$ and $\mu \in \mathrm{ca}_+(X)$ be given. If every relatively compact open subset of X is an F_σ then the following conditions are equivalent:*

(a) *the net $(\mu_i)_{i\in I}$ vaguely converges to μ;*

(b-c) *for every relatively compact open subset U of X and every compact subset K of X the inequalities*
$$\mu(U) \le \liminf_{i\in I} \mu_i(U) \quad \text{and} \quad \limsup_{i\in I} \mu_i(K) \le \mu(K)$$
hold;

(d) *for every lower semicontinuous function $f : X \to\,]-\infty, +\infty]$ bounded from below with compact support the inequality*
$$\langle \mu, f\rangle \le \liminf_{i\in I} \langle \mu_i, f\rangle$$
holds;

(e) *for every upper semicontinuous function $f : X \to [-\infty, +\infty[$ bounded from above with compact support the inequality*
$$\limsup_{i\in I} \langle \mu_i, f\rangle \le \langle \mu, f\rangle$$
holds;

(f) *for every relatively compact Borel subset B of X with $\mu(\partial B) = 0$ the equality*
$$\lim_{i\in I} \mu_i(B) = \mu(B)$$
holds;

(g) *for every bounded Borel measurable function $f : X \to \mathbb{R}$ with compact support whose set of discontinuities has measure zero with respect to μ, the equality*
$$\lim_{i\in I} \langle \mu_i, f\rangle = \langle \mu, f\rangle$$
holds.

If X is an arbitrary locally compact Hausdorff topological space we have:
$$\text{(b-c)} \Longleftrightarrow \text{(d)} \Longleftrightarrow \text{(e)} \Longrightarrow \text{(f)} \Longleftrightarrow \text{(g)} \Longrightarrow \text{(a)}.$$

Proof. We highlight the differences with the proof of Proposition 4.9. If every relatively compact open subset of X is an F_σ, the proof that (a) implies (d) and (e) now uses Lemma 8.11 instead of Lemma 4.6. The proofs of (d)⇔(e)⇒(b-c)⇒(f), (g)⇒(f) and (g)⇒(a) are the same as before. In the proof of (f)⇒(g) we cannot use (f) with $B = X$ to conclude that $\lim_{i\in I} \mu_i(X)$ exists so we use (f) with $B = L$ given by Lemma 8.14 below where K is equal to the support of f. Then we use Lemma 8.12 with B equal to the support of f to conclude that it is sufficient to approximate f uniformly by a function g that vanishes outside of the support of f and that is a linear combination of characteristic functions of relatively compact Borel subsets B with $\mu(\partial B) = 0$. Now the argument can proceed as before, with a slight modification to ensure that the approximating function g vanishes outside of the support of f. This is achieved simply by deleting from the sum $\sum_{j=0}^{k-1} t_j \chi_{B_j}$ defining g the term corresponding to the index j with $0 \in [t_j, t_{j+1}[$.

Finally, the proof of (b-c)⇒(d) also requires an adjustment as we cannot add a constant to make f nonnegative as before. Instead, we add a suitable continuous function with compact support h. As the proof of (b-c)⇒(a) does not go through (d), we can use (a) to conclude that $\lim_{i\in I}\langle \mu_i, h\rangle = \langle \mu, h\rangle$ and we are done. □

Lemma 8.14. *Let X be a locally compact Hausdorff topological space. If $\mu \in \mathrm{ca}_+(X)$ and K is a compact subset of X then there exists a compact subset L of X whose interior contains K and whose boundary has measure zero with respect to μ. Moreover, if K is contained in a given open subset U of X, one can choose L such that $L \subset U$.*

Proof. Pick an Urysohn function f as in Corollary 8.4 and note that since μ is finite and the level sets $f^{-1}(c)$, $c \in \mathbb{R}$, are measurable and disjoint, only countably many of them can have positive measure. We can then find $c \in]0,1[$ such that $f^{-1}(c)$ has null measure and the conclusion is obtained by taking $L = f^{-1}\big[[c, +\infty[\big]$. □

Lemma 4.14 and its consequences also have an adaptation to the vague topology.

Lemma 8.15. *Let X be a locally compact Hausdorff topological space in which every relatively compact open set is an F_σ. Let $\mathcal{F}$ be a collection of continuous $[0,1]$-valued functions on X with compact support satisfying the following condition: for every relatively compact open subset U of X and every compact subset K of U there exists $f \in \mathcal{F}$ that is equal to 1 on K and that vanishes outside of U. Under such assumption, for every net $(\mu_i)_{i\in I}$ in $\mathrm{ca}_+(X)$ and every $\mu \in \mathrm{ca}_+(X)$ we have that $(\mu_i)_{i\in I}$ vaguely converges to μ if and only if $\lim_{i\in I}\langle \mu_i, f\rangle = \langle \mu, f\rangle$ for every $f \in \mathcal{F}$.*

Proof. Assuming $\lim_{i\in I}\langle \mu_i, f\rangle = \langle \mu, f\rangle$ for all $f \in \mathcal{F}$ we prove condition (b-c) of Proposition 8.13. Since a relatively compact open subset U of X is an increasing union of closed (and thus compact) subsets, one proves $\mu(U) \le \liminf_{i\in I} \mu_i(U)$ just like in the proof of Lemma 4.14. Now if K is a compact subset of X, we first observe that K is a decreasing intersection of a sequence of relatively compact open subsets. Namely, since X is locally compact we have that K is contained in a relatively compact open subset U and thus the relatively compact open set $U \setminus K$ is the increasing union of a sequence $(F_n)_{n\ge 1}$ of closed sets. Setting $U_n = U \setminus F_n$, we obtain that K is the decreasing intersection of the sequence $(U_n)_{n\ge1}$ of relatively compact open subsets. Now for each $n \ge 1$ pick $f_n \in \mathcal{F}$ that equals 1 on K and vanishes outside of U_n and note that

$$\limsup_{i\in I} \mu_i(K) \le \lim_{i\in I} \langle \mu_i, f_n\rangle = \langle \mu, f_n\rangle \le \mu(U_n),$$

for all $n \geq 1$. The conclusion follows by taking the infimum over n noting that $\inf_{n\geq 1} \mu(U_n) = \mu(K)$. $\square$

Corollary 8.16. *If (X, d) is a locally compact metric space then a net $(\mu_i)_{i\in I}$ in $\mathrm{ca}_+(X)$ vaguely converges to $\mu \in \mathrm{ca}_+(X)$ if and only if*

$$\lim_{i\in I} \langle \mu_i, f\rangle = \langle \mu, f\rangle$$

for every Lipschitz function $f : X \to \mathbb{R}$ with compact support. In other words, the vague topology of $\mathrm{ca}_+(X)$ coincides with the topology induced by all maps of the form $\langle \cdot, f\rangle$ with $f : X \to \mathbb{R}$ Lipschitz with compact support.

Proof. Apply Lemma 8.15 with $\mathcal{F}$ equal to the set of $[0, 1]$-valued Lipschitz functions on X with compact support and prove the existence of the relevant Lipschitz Urysohn functions like in the proof of Proposition 4.16, keeping in mind that if K is a compact subset of an open set U then $d(K, X\setminus U) > 0$. $\square$

Corollary 8.17. *If X is a (Hausdorff, second countable, finite-dimensional) differentiable manifold of class C^k ($1 \leq k \leq +\infty$) then a net $(\mu_i)_{i\in I}$ in $\mathrm{ca}_+(X)$ vaguely converges to $\mu \in \mathrm{ca}_+(X)$ if and only if $\langle \mu, f\rangle = \lim_{i\in I}\langle \mu_i, f\rangle$ for every function $f : X \to \mathbb{R}$ of class C^k having compact support.*

Proof. Identical to the proof of Proposition 4.17, using Lemma 8.15 instead of Lemma 4.14. $\square$

Let us now present a sufficient condition for the metrizability of the vague topology in $\mathrm{ca}_+(X)$.

Proposition 8.18. *If X is a locally compact Hausdorff and second countable topological space then vague topology on $\mathrm{ca}_+(X)$ is metrizable.*

Proof. Since X is σ-compact and locally compact, we can write it as a countable union $X = \bigcup_{n=1}^{\infty} K_n$ of compact sets K_n such that K_n is contained in the interior of K_{n+1} for all $n \geq 1$. Note that the fact that the interiors of the sets K_n cover X implies that every compact subset of X is contained in some K_n. For each $n \geq 1$, let $f_n : X \to [0, +\infty[$ be a continuous function with compact support that is equal to 1 on K_n (Corollary 8.4) and consider the map:

$$\mathrm{ca}_+(X) \ni \mu \longmapsto (f_n\mu)_{n\geq 1} \in \prod_{n=1}^{\infty} \mathrm{ca}_+(X). \tag{8.2}$$

We endow each factor $\mathrm{ca}_+(X)$ in the product $\prod_{n=1}^{\infty} \mathrm{ca}_+(X)$ with the weak* topology and the product $\prod_{n=1}^{\infty} \mathrm{ca}_+(X)$ with the corresponding product topology. We claim that if the domain of (8.2) is endowed with the vague topology then (8.2) is a homeomorphism onto its image. Since X is metrizable by Urysohn's Metrization Theorem and thus the weak* topology on $\mathrm{ca}_+(X)$ is metrizable by Theorem 7.8, the desired conclusion will follow from the claim and from the fact that a countable product of metrizable spaces is metrizable.

To see that (8.2) is injective note that if $\mu, \nu \in \mathrm{ca}_+(X)$ and $f_n\mu = f_n\nu$ for all $n \geq 1$ then μ and ν agree on every relatively compact Borel subset of X and therefore $\mu = \nu$, as every Borel subset of X is a countable increasing union of relatively compact Borel subsets. Now by (2.6) the topology induced on $\mathrm{ca}_+(X)$ by the map (8.2) coincides with the topology induced by all the linear functionals of the form $\langle \cdot, f_n g\rangle$ with $n \geq 1$ and $g \in C_{\mathrm{b}}(X)$. Since

$$C_{\mathrm{c}}(X) = \big\{ f_n g : n \geq 1,\ g \in C_{\mathrm{b}}(X) \big\}$$

we conclude that the topology induced on $\mathrm{ca}_+(X)$ by the map (8.2) is the vague topology. This proves the claim. □

We observe that the vague topology is typically not metrizable in the space $\mathrm{ca}(X)$ of all finite signed measures. Namely, if X is locally compact and Hausdorff then Proposition E.19 states that $\mathrm{ca}(X)$ is metrizable in the vague topology if and only if X is countable and discrete. On the other hand, the bounded subsets of $\mathrm{ca}(X)$ are metrizable in the vague topology if and only if X is second countable, by Proposition E.20.

Remark 8.19. Let X be a locally compact Hausdorff and second countable topological space. As we have just seen in Proposition 8.18, the vague topology on $\mathrm{ca}_+(X)$ is metrizable and it follows from Corollary 4.21 that it is separable. One then is naturally led to ask if it is also Polish. We will show below in Proposition 8.20 that vaguely closed bounded subsets of $\mathrm{ca}_+(X)$ are vaguely compact and therefore such subsets are Polish, being both compact and metrizable. Moreover, if X is compact then the vague topology coincides with the weak* topology and thus $\mathrm{ca}_+(X)$ is Polish by Theorem 7.11. However, as we show now, if X is not compact then $\mathrm{ca}_+(X)$ cannot be Polish in the vague topology because the thesis of Baire's Category Theorem is not valid in it. To see this, note that for $r \geq 0$ the set

$$\big\{ \mu \in \mathrm{ca}_+(X) : \|\mu\| \leq r \big\} \tag{8.3}$$

is vaguely closed in $\mathrm{ca}_+(X)$ because condition (4) in the statement of Proposition 8.9 holds and a closed ball in $C_{\mathrm{c}}(X)^*$ is weak*-closed. Let us check that (8.3) has empty interior in $\mathrm{ca}_+(X)$ with respect to the vague topology. Let $(x_n)_{n\geq 1}$ be a sequence in X that converges to ∞ in the one-point compactification of X (to obtain such sequence pick K_n as in the proof of Proposition 8.18 and $x_n \in X \setminus K_n$). We have that $(n\delta_{x_n})_{n\geq 1}$ vaguely converges to zero and therefore given μ in (8.3) we have that $(\mu + n\delta_{x_n})_{n\geq 1}$ is a sequence in $\mathrm{ca}_+(X)$ that vaguely converges to μ and that is eventually outside of (8.3). This proves that (8.3) is closed with empty interior in $\mathrm{ca}_+(X)$ endowed with the vague topology and since $\mathrm{ca}_+(X)$ is the union of all sets (8.3) with r a positive integer, we obtain a contradiction with the thesis of Baire's Category Theorem.

We finish the section by discussing compactness with respect to the vague topology. As we have seen in Section 6, weak*-closed bounded subsets of

$\mathrm{ca}_+(X)$ are not in general weak*-compact and obtaining a characterization of weak*-compact subsets of $\mathrm{ca}_+(X)$ (if X is Polish) took us some work. The situation is much simpler with the vague topology.

Proposition 8.20. *If X is a locally compact Hausdorff topological space then for every $r \geq 0$ the sets*

$$\big\{\mu \in \mathrm{ca}(X) : \|\mu\| \leq r\big\}, \tag{8.4}$$

$$\big\{\mu \in \mathrm{ca}_+(X) : \|\mu\| \leq r\big\} \tag{8.5}$$

are vaguely compact. Moreover, every bounded subset of $\mathrm{ca}(X)$ *is vaguely relatively compact in* $\mathrm{ca}(X)$ *and every bounded subset of* $\mathrm{ca}_+(X)$ *is vaguely relatively compact in* $\mathrm{ca}_+(X)$*.*

Proof. Recall that the linear map (8.1) induces the vague topology on $\mathrm{ca}(X)$ from the weak* topology of the topological dual $C_\mathrm{c}(X)^*$ of the space $C_\mathrm{c}(X)$ endowed with the supremum norm and that by the Banach–Alaoglu Theorem the closed balls of $C_\mathrm{c}(X)^*$ are weak*-compact. Moreover, it follows from (2.2) and Corollary C.6 that the image under (8.1) of (8.4) is precisely the closed ball of radius r centered at the origin in $C_\mathrm{c}(X)^*$. Lemma 6.1 then yields that (8.4) is vaguely compact. Using also Proposition C.3 we obtain that the image of (8.5) under (8.1) is the intersection of the closed ball of radius r centered at the origin in $C_\mathrm{c}(X)^*$ with the weak*-closed subset consisting of bounded positive linear functionals. Thus Lemma 6.1 also yields that (8.5) is vaguely compact. To prove the final part of the statement, note that since the weak* topology of $C_\mathrm{c}(X)^*$ is Hausdorff, Corollary 6.2 implies that the vague closure of a vaguely compact subset of $\mathrm{ca}(X)$ is vaguely compact and that the vague closure in $\mathrm{ca}_+(X)$ of a vaguely compact subset of $\mathrm{ca}_+(X)$ is vaguely compact. □

Do we have a converse for Proposition 8.20? Note that the proof of Lemma 6.10 does not work for the vague topology since $C_\mathrm{c}(X)$ is not in general complete with respect to the supremum norm and the Uniform Boundedness Principle does not apply. In fact it is not true that a vaguely compact subset of $\mathrm{ca}(X)$ must be bounded. For example, if there exists a sequence $(x_n)_{n\geq 1}$ in X that converges to ∞ in the one-point compactification of X then $(n\delta_{x_n})_{n\geq 1}$ vaguely converges to zero and therefore $\{n\delta_{x_n} : n \geq 1\} \cup \{0\}$ is vaguely compact. Nevertheless, it holds that vaguely compact subsets of $\mathrm{ca}(X)$ are in some sense "bounded on compact subsets of X" under suitable assumptions on X.

Proposition 8.21. *Let X be a locally compact Hausdorff topological space in which every relatively compact open set is an F_σ. If Λ is a vaguely compact subset of* $\mathrm{ca}(X)$ *then for every compact subset K of X we have that the set* $\{|\mu|(K) : \mu \in \Lambda\}$ *is bounded.*

Proof. Let K be a compact subset of X, U be a relatively compact open subset of X containing K and let K' denote the closure of U. If $C(X; K')$

denotes the subspace of $C_{\rm c}(X)$ consisting of functions with support contained in K' then $C(X;K')$ is a Banach space endowed with the supremum norm and therefore the Uniform Boundedness Principle implies that every weak*-compact subset of the topological dual of $C(X;K')$ is bounded. Since the map

$$\mathrm{ca}(X) \ni \mu \longmapsto \langle \mu, \cdot \rangle \in C(X;K')^* \tag{8.6}$$

is continuous with respect to the vague topology of $\mathrm{ca}(X)$ and the weak* topology of $C(X;K')^*$, we conclude that the image of Λ under (8.6) is weak*-compact and hence bounded. To conclude the proof, it is sufficient to show that for every $\mu \in \mathrm{ca}(X)$ it holds that $|\mu|(K)$ is less than or equal to the norm of the linear functional $\langle \mu, \cdot \rangle \in C(X;K')^*$. To this aim, note first that the norm of $\langle \mu, \cdot \rangle \in C(X;K')^*$ is greater than or equal to the norm of the linear functional $\langle \mu|_U, \cdot \rangle \in C_{\rm c}(U)^*$. Since every relatively compact open subset of X is an F_σ, we have that every open subset of U is σ-compact and thus Proposition C.7 implies that $\mu|_U$ is Radon. Now Proposition C.2 gives us that the norm of the linear functional $\langle \mu|_U, \cdot \rangle \in C_{\rm c}(U)^*$ is equal to $\|\mu|_U\| = |\mu|(U)$, which is greater than or equal to $|\mu|(K)$. □

9. Weak* topology and cumulative distribution functions

In this section we consider only the topological space $X = \mathbb{R}$, endowed with its Euclidean topology.

Definition 9.1. Given $\mu \in \mathrm{ca}(\mathbb{R})$, its *cumulative distribution function* is the function $F_\mu : \mathbb{R} \to \mathbb{R}$ defined by

$$F_\mu(x) = \mu\big(\,]-\infty, x]\,\big),$$

for all $x \in \mathbb{R}$.

Using the continuity properties of measures with respect to monotone limits of sequences of measurable sets, we easily obtain that

$$\lim_{t \to x^+} F_\mu(t) = F_\mu(x), \quad \lim_{t \to x^-} F_\mu(t) = \mu\big(\,]-\infty, x[\,\big),$$

for all $x \in \mathbb{R}$ so that F_μ is right-continuous and admits left limits at every point. In particular

$$\mu\big(\{x\}\big) = F_\mu(x) - \lim_{t \to x^-} F_\mu(t),$$

for all $x \in \mathbb{R}$ so that F_μ is continuous at x if and only if the singleton $\{x\}$ has measure zero with respect to μ. It follows that F_μ has at most countably many discontinuity points. Moreover, we have the equalities:

$$\lim_{t \to -\infty} F_\mu(t) = 0, \quad \lim_{t \to +\infty} F_\mu(t) = \mu(\mathbb{R}).$$

If the measure μ is nonnegative then obviously F_μ is increasing (but not necessarily strictly increasing) and, for arbitrary $\mu \in \mathrm{ca}(\mathbb{R})$, the function F_μ has bounded variation being the difference of the increasing functions F_{μ^+} and F_{μ^-}.

There is a very simple criterion for the weak* convergence and for the vague convergence of nonnegative measures on $\mathbb{R}$ in terms of their cumulative distribution functions.

Proposition 9.2. *Let $(\mu_i)_{i\in I}$ be a net in $\mathrm{ca}_+(\mathbb{R})$ and let $\mu \in \mathrm{ca}_+(\mathbb{R})$ be given. We have that $(\mu_i)_{i\in I}$ weak*-converges to μ if and only if*

$$\lim_{i\in I} \mu_i(\mathbb{R}) = \mu(\mathbb{R})$$

and

$$\lim_{i\in I} F_{\mu_i}(x) = F_\mu(x), \tag{9.1}$$

for every $x \in \mathbb{R}$ that is a continuity point of F_μ. Moreover, $(\mu_i)_{i\in I}$ vaguely converges to μ if and only if

$$\lim_{i\in I} \big(F_{\mu_i}(x) - F_{\mu_i}(y)\big) = F_\mu(x) - F_\mu(y), \tag{9.2}$$

for every $x, y \in \mathbb{R}$ that are continuity points of F_μ.

Proof. If $x \in \mathbb{R}$ is a continuity point of F_μ then the boundary $\{x\}$ of $]-\infty, x]$ has measure zero with respect to μ and thus (9.1) follows if $(\mu_i)_{i\in I}$ weak*-converges to μ from the equivalence between (a) and (f) in Proposition 4.9. Similarly, if $(\mu_i)_{i\in I}$ vaguely converges to μ then the equivalence between (a) and (f) in Proposition 8.13 implies (9.2) for continuity points x, y of F_μ since $\{x, y\}$ is the boundary of $]x, y]$ for $x < y$.

Due to Proposition 8.6, to conclude the proof it is sufficient to show that $(\mu_i)_{i\in I}$ vaguely converges to μ assuming that (9.2) holds for every pair $x, y \in \mathbb{R}$ of continuity points of F_μ. Let $f \in C_{\mathrm{c}}(\mathbb{R})$ be fixed and choose continuity points $a, b \in \mathbb{R}$ of F_μ with $a < b$ such that f vanishes outside of $]a, b]$. Since the limit

$$\lim_{i\in I} \mu_i(]a, b]) = \lim_{i\in I} \big(F_{\mu_i}(b) - F_{\mu_i}(a)\big) = F_\mu(b) - F_\mu(a) = \mu(]a, b]), \tag{9.3}$$

is finite, applying Lemma 8.12 with $B =]a, b]$ we see that in order to show that $\lim_{i\in I}\langle \mu_i, f\rangle = \langle \mu, f\rangle$ it is sufficient to approximate f uniformly by a function $g \in \mathcal{M}_{\mathrm{b}}(\mathbb{R})$ that vanishes outside of $]a, b]$ and for which the equality $\lim_{i\in I}\langle \mu_i, g\rangle = \langle \mu, g\rangle$ holds. Given $\varepsilon > 0$, to obtain such g with $\|f - g\|_{\sup} \le \varepsilon$, pick $\delta > 0$ using the uniform continuity of f on $[a, b]$ such that f oscillates less than ε in subintervals of $[a, b]$ with diameter less than δ and consider a partition

$$a = t_0 < t_1 < \cdots < t_k = b$$

such that all t_j are continuity points of F_μ and $t_{j+1} - t_j < \delta$ for all j. Now define g by setting $g = \sum_{j=0}^{k-1} f(t_{j+1})\chi_{]t_j, t_{j+1}]}$ and use the analogue of (9.3) for the interval $]t_j, t_{j+1}]$ to conclude that $\lim_{i\in I}\langle \mu_i, g\rangle = \langle \mu, g\rangle$. ☐

Remark 9.3. The proof of Proposition 9.2 actually shows that if (9.2) holds for all x, y in a dense subset of $\mathbb{R}$ then the net $(\mu_i)_{i\in I}$ in $\mathrm{ca}_+(\mathbb{R})$ vaguely converges to $\mu \in \mathrm{ca}_+(\mathbb{R})$. If in addition $\lim_{i\in I} \mu_i(\mathbb{R}) = \mu(\mathbb{R})$ then $(\mu_i)_{i\in I}$ weak*-converges to μ.

Example 9.4. Equality (9.1) for continuity points x of F_μ does not follow from the weak*-convergence of $(\mu_i)_{i\in I}$ to μ if the measures are allowed to be signed. For example, setting

$$\mu_n = \delta_0 - \delta_{\frac{1}{n}}$$

for all $n \geq 1$ then obviously $(\mu_n)_{n\geq 1}$ weak*-converges to $\mu = 0$ and yet $F_{\mu_n} = \chi_{[0,\frac{1}{n}[}$ does not converge to $F_\mu = 0$ at the point $x = 0$ even though F_μ is continuous. Equality (9.2) also fails for $x = 0$ and $y \neq 0$.

Example 9.5. Equality (9.1) for continuity points x of F_μ together with $\lim_{i\in I} \mu_i(\mathbb{R}) = \mu(\mathbb{R})$ do not imply that $(\mu_i)_{i\in I}$ weak*-converges to μ if the measures are allowed to be signed. For example, setting $\mu_n = \delta_n - \delta_{n+1}$ for all $n \geq 1$ then $F_{\mu_n} = \chi_{[n,n+1[}$ converges pointwise to $F_0 = 0$ and $\mu_n(\mathbb{R}) = 0$ for all $n \geq 1$ and yet $(\mu_n)_{n\geq 1}$ is not weak*-convergent as there are bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that the limit $\lim_{n\to+\infty}[f(n)-f(n+1)]$ does not exist. In this example $(\mu_n)_{n\geq 1}$ vaguely converges to zero, but using instead

$$\mu_n = n\big(\delta_{\frac{1}{n}} - \delta_{\frac{2}{n}}\big),$$

for all $n \geq 1$ we have that $F_{\mu_n} = n\chi_{[\frac{1}{n},\frac{2}{n}[}$ converges pointwise to zero and $\mu_n(\mathbb{R}) = 0$ for all $n \geq 1$ and yet $(\mu_n)_{n\geq 1}$ does not vaguely converge as picking $f \in C_{\mathrm{c}}(\mathbb{R})$ such that $f(x) = \sqrt{|x|}$ for $x \in \mathbb{R}$ near zero we have that $\lim_{n\to+\infty}\langle \mu_n, f\rangle = -\infty$.

Remark 9.6. For a bounded net $(\mu_i)_{i\in I}$ of signed measures $\mu_i \in \mathrm{ca}(\mathbb{R})$ it is indeed true that $(\mu_i)_{i\in I}$ vaguely converges to $\mu \in \mathrm{ca}(\mathbb{R})$ if (9.2) holds for all continuity points $x, y \in \mathbb{R}$ of F_μ (or for all x, y in a dense subset of $\mathbb{R}$). Namely, note that the argument in the proof of Proposition 9.2 only used the assumption of nonnegativity of the measures to establish that $\sup_{i\geq i_0} |\mu_i|\big(\,]a,b]\,\big) < +\infty$ for some $i_0 \in I$ using (9.3).

10. Characteristic functions of measures

In this section X denotes a real finite-dimensional vector space endowed with its Euclidean topology (i.e., the topology induced by an arbitrary norm).

Definition 10.1. Given a measure $\mu \in \mathrm{ca}(X)$, its *characteristic function* is the complex valued function $\varphi_\mu : X^* \to \mathbb{C}$ on the dual space of X defined by

$$\varphi_\mu(\xi) = \int_X e^{i\xi(x)}\,\mathrm{d}\mu(x), \tag{10.1}$$

for all $\xi \in X^*$.

If $X = \mathbb{R}^n$ we identify X with its dual space using the canonical inner product $\langle\cdot,\cdot\rangle$ and then φ_μ is identified with the function $\varphi_\mu : \mathbb{R}^n \to \mathbb{C}$ given by

$$\varphi_\mu(\xi) = \int_{\mathbb{R}^n} e^{i\langle\xi,x\rangle}\,\mathrm{d}\mu(x),$$

for all $\xi \in \mathbb{R}^n$.

Since the integrand in (10.1) has unit modulus and μ is finite, it follows from the Dominated Convergence Theorem that the function φ_μ is continuous. Moreover, we have

$$\|\varphi_\mu\|_{\sup} \le \|\mu\| \tag{10.2}$$

and thus the mapping $\mu \mapsto \varphi_\mu$ defines a bounded linear map from $\mathrm{ca}(X)$ to the space of bounded continuous complex-valued functions on X^* endowed with the supremum norm. The function φ_μ is in fact uniformly continuous. Namely, we have

$$|\varphi_\mu(\xi_1) - \varphi_\mu(\xi_2)| = \left|\int_X (e^{i(\xi_1-\xi_2)(x)} - 1)e^{i\xi_2(x)}\,\mathrm{d}\mu(x)\right| \le h(\xi_1 - \xi_2),$$

for all $\xi_1, \xi_2 \in X^*$, where $h : X^* \to [0, +\infty[$ is the function defined by

$$h(\xi) = \int_X |e^{i\xi(x)} - 1|\,\mathrm{d}|\mu|(x),$$

for all $\xi \in X^*$. The Dominated Convergence Theorem yields $\lim_{\xi\to 0} h(\xi) = 0$ and the uniform continuity of φ_μ follows.

As it is to be expected, characteristic functions of measures transform naturally under push-forwards by linear maps.

Lemma 10.2. *If X and Y are real-finite dimensional vector spaces and $T : X \to Y$ is a linear map then*

$$\varphi_{T_*\mu} = \varphi_\mu \circ T^*,$$

for all $\mu \in \mathrm{ca}(X)$, where $T^* : Y^* \to X^*$ denotes the adjoint of T.

Proof. Follows from (2.9). □

It should be visible that the notion of characteristic function of a measure is closely related to the notion of Fourier transform. Let us recall some basic definitions and well-known facts. The *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$ is defined as the space of all smooth functions $\phi : \mathbb{R}^n \to \mathbb{C}$ such that any partial derivative of ϕ (of any order) multiplied by an arbitrary polynomial is bounded. The *Fourier transform* of a function $\phi \in \mathcal{S}(\mathbb{R}^n)$ is the map $\hat{\phi} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\hat{\phi}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\langle\xi,x\rangle}\phi(x)\,\mathrm{d}x, \quad \xi \in \mathbb{R}^n$$

and the *inverse Fourier transform* of ϕ is the map $\check{\phi} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\check{\phi}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\langle\xi,x\rangle}\phi(\xi)\,\mathrm{d}\xi, \quad x \in \mathbb{R}^n,$$

where the integrals are taken with respect to the Lebesgue measure and $\langle\cdot,\cdot\rangle$ denotes the canonical inner product of $\mathbb{R}^n$. It is well known that $\phi \mapsto \hat{\phi}$ and $\phi \mapsto \check{\phi}$ are mutually inverse bijective linear maps of $\mathcal{S}(\mathbb{R}^n)$ (see, for instance, [1, 8.26 and 8.28]). Since every $\phi \in \mathcal{S}(\mathbb{R}^n)$ is Lebesgue integrable and a measure $\mu \in \mathrm{ca}(\mathbb{R}^n)$ is finite, a simple application of Fubini's Theorem yields

$$\int_{\mathbb{R}^n} \hat{\phi}(\xi)\,\mathrm{d}\mu(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \varphi_\mu(-x)\phi(x)\,\mathrm{d}x, \tag{10.3}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ and all $\mu \in \mathrm{ca}(\mathbb{R}^n)$.

Remark 10.3. Equality (10.3) means that the Fourier transform of μ regarded as a tempered distribution is the map $\mathbb{R}^n \ni x \mapsto (2\pi)^{-\frac{n}{2}}\varphi_\mu(-x) \in \mathbb{C}$.

As a simple consequence of (10.3) we obtain that two measures with the same characteristic function are equal.

Proposition 10.4. *If X is a real finite-dimensional vector space then the map* $\mathrm{ca}(X) \ni \mu \mapsto \varphi_\mu$ *is injective.*

Proof. Since $\mu \mapsto \varphi_\mu$ is linear, it suffices to prove that $\mu = 0$ if $\varphi_\mu = 0$. Moreover, by Lemma 10.2 we can assume that $X = \mathbb{R}^n$. If $\varphi_\mu = 0$ then equality (10.3) and the fact that the Fourier transform is a bijection of $\mathcal{S}(\mathbb{R}^n)$ imply that $\int_{\mathbb{R}^n} \phi\,\mathrm{d}\mu = 0$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ and in particular $\langle \mu, \phi\rangle = 0$ for every smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ with compact support. It is well-known that every continuous function $f : \mathbb{R}^n \to \mathbb{R}$ with compact support is a uniform limit of smooth functions with compact support (see, for instance, [1, 8.17] or [2, Theorem 2.2]) and thus the continuity of the linear functional $\langle \mu, \cdot\rangle$ with respect to the supremum norm implies that $\langle \mu, \cdot\rangle$ vanishes on $C_{\mathrm{c}}(\mathbb{R}^n)$. To conclude the proof, observe that by Proposition C.7 the measure μ is Radon and hence Proposition C.2 implies that $\mu = 0$. □

Let us now investigate the relationship between characteristic functions of measures and weak*-convergence. We start with a trivial result.

Proposition 10.5. *Let X be a real finite-dimensional vector space endowed with its Euclidean topology. If $(\mu_i)_{i\in I}$ is a net in* $\mathrm{ca}(X)$ *that weak*-converges to some $\mu \in \mathrm{ca}(X)$ then the net of characteristic functions $(\varphi_{\mu_i})_{i\in I}$ converges pointwise to the characteristic function of μ.*

Proof. Simply note that the real and imaginary parts of the value of the characteristic function of a measure at a point $\xi \in X^*$ are obtained by integrating continuous real-valued bounded functions on X with respect to the given measure. □

Now we prove our main result which states that the converse of Proposition 10.5 holds for sequences of nonnegative measures.

Proposition 10.6. *Let X be a real finite-dimensional vector space endowed with its Euclidean topology. Given a sequence $(\mu_k)_{k\geq 1}$ in $\mathrm{ca}_+(X)$ and μ in $\mathrm{ca}_+(X)$ we have that $(\mu_k)_{k\geq 1}$ weak*-converges to μ if and only if the sequence of characteristic functions $(\varphi_{\mu_k})_{k\geq 1}$ converges pointwise to φ_μ.*

Proof. By Proposition 10.5 it is sufficient to prove that if $(\varphi_{\mu_k})_{k\geq 1}$ converges pointwise to φ_μ then $(\mu_k)_{k\geq 1}$ weak*-converges to μ. Moreover, by Lemma 10.2 and Proposition 5.2 we can assume that $X = \mathbb{R}^n$. If $(\varphi_{\mu_k})_{k\geq 1}$ converges pointwise to φ_μ then

$$\lim_{k\to+\infty} \mu_k(X) = \lim_{k\to+\infty} \varphi_{\mu_k}(0) = \varphi_\mu(0) = \mu(X) \tag{10.4}$$

and therefore the sequence $(\mu_k)_{k\geq 1}$ is bounded. It then follows from (10.2) that the sequence $(\varphi_{\mu_k})_{k\geq 1}$ is bounded with respect to the supremum norm. Now equality (10.3) and the Dominated Convergence Theorem yield

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} \hat{\phi}\,\mathrm{d}\mu_k = \int_{\mathbb{R}^n} \hat{\phi}\,\mathrm{d}\mu,$$

for every $\phi \in \mathcal{S}(\mathbb{R}^n)$ because ϕ is Lebesgue integrable over $\mathbb{R}^n$. Since the Fourier transform is a bijection of $\mathcal{S}(\mathbb{R}^n)$ we conclude that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} \phi\,\mathrm{d}\mu_k = \int_{\mathbb{R}^n} \phi\,\mathrm{d}\mu,$$

for every $\phi \in \mathcal{S}(\mathbb{R}^n)$ and in particular the latter equality holds for every $\phi : \mathbb{R}^n \to \mathbb{R}$ that is smooth with compact support. Hence Proposition 4.17 and (10.4) imply that $(\mu_k)_{k\geq 1}$ weak*-converges to μ. □

Example 10.7. Proposition 10.6 fails for signed measures even when the sequence of measures is bounded. Namely, using the Mean Value Inequality we obtain

$$|e^{i\xi x} - e^{i\xi y}| \leq |\xi||x - y|,$$

for all $\xi, x, y \in \mathbb{R}$, so that the maps $\mathbb{R} \ni x \mapsto e^{i\xi x} \in \mathbb{C}$ are Lipschitz for all $\xi \in \mathbb{R}$. Thus, if $(\mu_n)_{n\geq 1}$ is the sequence (4.10) defined in Example 4.18 we obtain

$$\lim_{n\to+\infty} \varphi_{\mu_n}(\xi) = 0,$$

for all $\xi \in \mathbb{R}$ and yet $(\mu_n)_{n\geq 1}$ is not weak*-convergent.

Example 10.8. Proposition 10.6 fails if the sequence $(\mu_k)_{k\geq 1}$ is replaced with an arbitrary net even if we consider only probability measures. Namely, set $X = \mathbb{R}$ and assume by contradiction that Proposition 10.6 holds for nets of probability measures. Denoting by $\mathbb{C}^{\mathbb{R}}$ the set of all complex-valued functions on $\mathbb{R}$ endowed with the product topology (which is the same as the pointwise convergence topology) we conclude that the weak* topology on $\mathrm{ca}^1_+(\mathbb{R})$ coincides with the topology induced by the map:

$$\mathrm{ca}^1_+(\mathbb{R}) \ni \mu \longmapsto \varphi_\mu \in \mathbb{C}^{\mathbb{R}}. \tag{10.5}$$

Remark 4.4 then implies that the Euclidean topology of $\mathbb{R}$ coincides with the topology induced by the map $\Phi : \mathbb{R} \to \mathbb{C}^{\mathbb{R}}$ given by the composition of (10.5) with $\delta : \mathbb{R} \to \mathrm{ca}^1_+(\mathbb{R})$. Clearly

$$\Phi(x)(\xi) = \varphi_{\delta_x}(\xi) = e^{i\xi x},$$

for all $x, \xi \in \mathbb{R}$ and therefore the sets of the form

$$\big\{x \in \mathbb{R} : |e^{i\xi x} - 1| < \varepsilon \text{ for all } \xi \in F\big\} \tag{10.6}$$

with $F = \{\xi_1, \dots, \xi_m\}$ a finite subset of $\mathbb{R}$ and $\varepsilon > 0$ should constitute a fundamental system of neighborhoods of zero in $\mathbb{R}$. But the set (10.6) is necessarily unbounded as an application of Lemma 10.9 below with z equal to $(e^{i\xi_1}, \dots, e^{i\xi_m})$ shows that it must contain infinitely many positive integers.

Lemma 10.9. *Let m be a positive integer and denote by $(S^1)^m$ the m-th power of the unit circle endowed with the group operation of coordinatewise multiplication of complex numbers. Given $z \in (S^1)^m$ and a neighborhood V of the identity in $(S^1)^m$, there are infinitely many positive integers k such that $z^k \in V$.*

Proof. Since $(S^1)^m$ is compact, the sequence $(z^k)_{k\geq 1}$ must contain a convergent subsequence $(z^{k_j})_{j\geq 1}$ and passing to a smaller subsequence we can assume that $\lim_{j\to+\infty}(k_{j+1} - k_j) = +\infty$. Clearly

$$\lim_{j\to+\infty} z^{k_{j+1}-k_j} = \lim_{j\to+\infty} z^{k_{j+1}} \lim_{j\to+\infty} (z^{k_j})^{-1}$$

is equal to the identity and therefore $z^{k_{j+1}-k_j}$ is in V for j sufficiently large. □

Though Example 10.7 shows that Proposition 10.6 fails for signed measures, we have the following weaker result.

Proposition 10.10. *Let X be a real finite-dimensional vector space endowed with its Euclidean topology. Given a bounded sequence $(\mu_k)_{k\geq 1}$ in $\mathrm{ca}(X)$ and $\mu \in \mathrm{ca}(X)$, if the sequence of characteristic functions $(\varphi_{\mu_k})_{k\geq 1}$ converges pointwise to φ_μ then $(\mu_k)_{k\geq 1}$ vaguely converges to μ.*

Proof. As in the proof of Proposition 10.6 we can assume that $X = \mathbb{R}^n$ and we obtain that $\lim_{k\to+\infty}\langle \mu_k, \phi\rangle = \langle \mu, \phi\rangle$ for every $\phi : \mathbb{R}^n \to \mathbb{R}$ that is smooth with compact support. The conclusion then follows from Lemma 4.7 and the fact that a continuous function with compact support $f : \mathbb{R}^n \to \mathbb{R}$ is the uniform limit of a sequence of smooth functions with compact support (see, for instance, [1, 8.17] or [2, Theorem 2.2]). □

Example 10.11. Proposition 10.10 fails if the sequence of measures $(\mu_k)_{k\geq 1}$ is not bounded. For example, set $X = \mathbb{R}$ and let $(\mu_n)_{n\geq 1}$ be the sequence (4.11) defined in Example 4.18. Given $\xi \in \mathbb{R}$ we have

$$\lim_{n\to+\infty} \varphi_{\mu_n}(\xi) = 0$$

because $\mathbb{R} \ni x \mapsto e^{i\xi x} \in \mathbb{C}$ is of class C^1. However, choosing $f \in C_{\rm c}(\mathbb{R})$ with $f(x) = \sqrt[4]{|x|}$ for $x \in \mathbb{R}$ near zero we obtain $\lim_{n\to+\infty}\langle \mu_n, f\rangle = +\infty$ so that $(\mu_n)_{n\ge1}$ is not vaguely convergent.

Appendix A. Nets

We make a very brief summary of the theory of nets and prove a few simple results that are used in the main text.

Definition A.1. A *directed set* is a nonempty set I endowed with a binary relation $\le$ that is reflexive, transitive and such that for all $i_1, i_2 \in I$ there exists $i \in I$ with $i_1 \le i$ and $i_2 \le i$. A *net* is a family $(x_i)_{i\in I}$ indexed by a directed set I. A net $(x_i)_{i\in I}$ on a topological space X is said to *converge* to a point $x \in X$ if for every neighborhood V of x there exists $i_0 \in I$ with $x_i \in V$ for all $i \ge i_0$.

Some authors assume the binary relation $\le$ on a directed set to be also anti-symmetric so that it is indeed a partial order, but the most relevant theorems about nets hold with both definitions of directed set so we adopt the more general definition.

One often writes $\lim_{i\in I} x_i = x$ to mean that a net $(x_i)_{i\in I}$ on a topological space X converges to some $x \in X$ but it should be noted that if X is not Hausdorff then such notation is misleading as a net can converge to more than one point of X, so that the limit $\lim_{i\in I} x_i$ is not well-defined.

For metric spaces (or, more generally, for first countable spaces) the topology can be characterized by the convergence of sequences. In the general case, one needs nets. We state some of the main standard relevant results relating convergence of nets to the topology of a space.

Proposition A.2. *If X is a topological space then the following statements hold:*

(a) *if V is a subset of X and $x \in X$ then V is a neighborhood of x (i.e., x is an interior point of V) if and only if for every net $(x_i)_{i\in I}$ in X converging to x there exists $i_0 \in I$ such that $x_i \in V$ for all $i \ge i_0$;*

(b) *if A is a subset of X and $x \in X$ then x belongs to the closure of A if and only if there exists a net in A converging to x;*

(c) *given a topological space Y, a map $f : X \to Y$ and a point $x \in X$, we have that f is continuous at the point x if and only if $\big(f(x_i)\big)_{i\in I}$ converges to $f(x)$ for every net $(x_i)_{i\in I}$ in X that converges to x.*

Proof. The nontrivial implications in (a) and (c) are proven by contradiction by constructing a suitable net on X converging to x indexed by the directed set of all neighborhoods of x in X partially ordered by reverse inclusion. The proof of the nontrivial implication in (b) is also obtained by constructing such a net. □

The following result is useful for obtaining fundamental systems of neighborhoods from a characterization of convergence of nets.

Lemma A.3. *Let X be a topological space and let $\mathcal{V}$ be a nonempty collection of neighborhoods of a point $x \in X$. Assume that $\mathcal{V}$ characterizes convergence of nets to x, i.e., assume that given a net $(x_i)_{i\in I}$ in X, if for every $V \in \mathcal{V}$ there exists $i_0 \in I$ such that $i \geq i_0$ implies $x_i \in V$ then $(x_i)_{i\in I}$ converges to x. Under such assumption, the collection of finite intersections of elements of $\mathcal{V}$ is a fundamental system of neighborhoods of x in X.*

Proof. Let $\mathcal{V}'$ be the collection of all finite intersections of elements of $\mathcal{V}$. We have that $\mathcal{V}'$ is a directed set partially ordered by reverse inclusion. If W is a neighborhood of x in X which contains no element of $\mathcal{V}'$ we obtain a net $(x_V)_{V\in\mathcal{V}'}$ with $x_V \in V$ and $x_V \notin W$ for all $V \in \mathcal{V}'$. Since $\mathcal{V}$ characterizes convergence of nets to x we get that $(x_V)_{V\in\mathcal{V}'}$ converges to x, which contradicts the fact that $x_V \notin W$ for all $V \in \mathcal{V}'$. □

We finish the section by summarizing the theory of lim inf and lim sup for nets, which is completely analogous to the theory for sequences. We consider nets on the extended real line $[-\infty, +\infty]$ to ensure that lim inf and lim sup always exist. We will assume $[-\infty, +\infty]$ to be endowed with the order, the topology and the operations usually defined in measure theory books. The sum of two elements of $[-\infty, +\infty]$ is well-defined unless it is of the form $(+\infty) + (-\infty)$ or $(-\infty) + (+\infty)$.

A net $(x_i)_{i\in I}$ in $[-\infty, +\infty]$ is called *increasing* (resp., *decreasing*) if for all $i, j \in I$ we have that $i \leq j$ implies $x_i \leq x_j$ (resp., $x_j \leq x_i$). Clearly an increasing (resp., decreasing) net $(x_i)_{i\in I}$ in $[-\infty, +\infty]$ converges to $\sup_{i\in I} x_i$ (resp., to $\inf_{i\in I} x_i$). If $(x_i)_{i\in I}$ is an arbitrary net in $[-\infty, +\infty]$, we set:

$$\liminf_{i\in I} x_i = \lim_{i_0\in I} \inf_{i\geq i_0} x_i = \sup_{i_0\in I} \inf_{i\geq i_0} x_i,$$

$$\limsup_{i\in I} x_i = \lim_{i_0\in I} \sup_{i\geq i_0} x_i = \inf_{i_0\in I} \sup_{i\geq i_0} x_i.$$

Obviously

$$\liminf_{i\in I} (-x_i) = -\limsup_{i\in I} x_i$$

and such equality can be used to reduce proofs of results about lim sup to proves of results about lim inf.

One readily checks that, for $x \in \mathbb{R}$, we have $x \leq \liminf_{i\in I} x_i$ if and only if for every $\varepsilon > 0$ there exists $i_0 \in I$ such that $x < x_i + \varepsilon$ for all $i \geq i_0$. Similarly, for $x \in \mathbb{R}$, we have $x \geq \limsup_{i\in I} x_i$ if and only if for every $\varepsilon > 0$ there exists $i_0 \in I$ such that $x > x_i - \varepsilon$ for all $i \geq i_0$. Moreover, we have

$$\liminf_{i\in I} x_i \leq \limsup_{i\in I} x_i$$

and the equality holds if and only if the net $(x_i)_{i\in I}$ is convergent, in which case $\lim_{i\in I} x_i$ is the common value of $\liminf_{i\in I} x_i$ and $\limsup_{i\in I} x_i$.

Given nets $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ in $[-\infty,+\infty]$, we have that the inequalities

$$\liminf_{i\in I} x_i + \liminf_{i\in I} y_i \le \liminf_{i\in I}\,(x_i+y_i), \tag{A.1}$$

$$\limsup_{i\in I}\,(x_i+y_i) \le \limsup_{i\in I} x_i + \limsup_{i\in I} y_i, \tag{A.2}$$

hold, provided that all sums involved are well-defined. Of course, if the sum $x_i + y_i$ is well-defined only for i larger than or equal to some $i_0 \in I$ then we can just replace I with $\{i \in I : i \ge i_0\}$ as this does not interfere with lim inf and lim sup. Inequalities (A.1) and (A.2) follow from

$$\inf_{i\ge i_0} x_i + \inf_{i\ge i_0} y_i \le \inf_{i\ge i_0}\,(x_i+y_i), \quad \sup_{i\ge i_0}(x_i+y_i) \le \sup_{i\ge i_0} x_i + \sup_{i\ge i_0} y_i$$

by taking the limit with respect to $i_0 \in I$. If any of the nets $(x_i)_{i\in I}$, $(y_i)_{i\in I}$ is actually convergent then inequalities (A.1) and (A.2) become equalities.

Lemma A.4. *If $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ are nets in $[-\infty,+\infty]$ and $(y_i)_{i\in I}$ is convergent in $[-\infty,+\infty]$ then the equalities*

$$\liminf_{i\in I}(x_i+y_i) = \liminf_{i\in I} x_i + \lim_{i\in I} y_i, \tag{A.3}$$

$$\limsup_{i\in I}(x_i+y_i) = \limsup_{i\in I} x_i + \lim_{i\in I} y_i, \tag{A.4}$$

hold provided that all sums involved are well-defined.

Proof. It is sufficient to prove (A.3) as then (A.4) follows by replacing x_i with $-x_i$ and y_i with $-y_i$. Moreover, by (A.1), to prove (A.3) it is sufficient to prove:

$$\liminf_{i\in I}(x_i+y_i) \le \liminf_{i\in I} x_i + \lim_{i\in I} y_i. \tag{A.5}$$

If $\lim_{i\in I} y_i$ is finite then replacing x_i with $x_i + y_i$ and y_i with $-y_i$ in (A.1) we obtain

$$\liminf_{i\in I}(x_i+y_i) + \liminf_{i\in I}(-y_i) \le \liminf_{i\in I}[(x_i+y_i)+(-y_i)] = \liminf_{i\in I} x_i \tag{A.6}$$

from which (A.5) follows. Finally, if $\lim_{i\in I} y_i = +\infty$ then (A.5) holds trivially and if $\lim_{i\in I} y_i = -\infty$ holds and (A.5) fails then (A.6) can be used to conclude that $\liminf_{i\in I} x_i = +\infty$, which contradicts the assumption that the sum $\liminf_{i\in I} x_i + \lim_{i\in I} y_i$ is well-defined. □

Appendix B. Semicontinuous functions

Let us recall the definition and prove some simple facts about semicontinuous functions. We will consider functions taking values in the extended real line $[-\infty,+\infty]$ rather than just real-valued functions.

Definition B.1. Let X be a topological space and $f : X \to [-\infty,+\infty]$ be a function. Given $x \in X$, we say that f is *lower semicontinuous* (resp., *upper semicontinuous*) at the point $x \in X$ if for every $c \in [-\infty,+\infty]$ with $c < f(x)$ (resp., with $c > f(x)$) there exists a neighborhood of x in X in which f is greater than (resp., less than) c. We say that f is *lower* *semicontinuous* (resp., *upper semicontinuous*) if it is lower semicontinuous (resp., upper semicontinuous) at all points of X.

Obviously a function f is automatically lower semicontinuous (resp., upper semicontinuous) at a point x such that $f(x) = -\infty$ (resp., such that $f(x) = +\infty$). Moreover, if we replace "for all $c \in [-\infty, +\infty]$" with "for all $c \in \mathbb{R}$" in Definition B.1 we obtain an equivalent definition.

Note that a function f is lower semicontinuous at a point if and only if $-f$ is upper semicontinuous at that point, so that one can obtain results for upper semicontinuous functions as a corollary of results for lower semicontinuous functions by replacing f with $-f$.

The following characterization of semicontinuous functions is an immediate consequence of the definition.

Lemma B.2. *Let X be a topological space and $f : X \to [-\infty, +\infty]$ be a function. If f is lower semicontinuous (resp., upper semicontinuous) then for every $c \in [-\infty, +\infty]$ the set $\{x \in X : f(x) > c\}$ is open (resp., the set $\{x \in X : f(x) < c\}$ is open). Conversely, if for every $c \in \mathbb{R}$ the set $\{x \in X : f(x) > c\}$ is open (resp., the set $\{x \in X : f(x) < c\}$ is open) then f is lower semicontinuous (resp., upper semicontinuous).* □

Corollary B.3. *The characteristic function of a subset of a topological space is lower semicontinuous (resp., upper semicontinuous) if and only if the subset is open (resp., closed).* □

Let us give a characterization of semicontinuity in terms of nets.

Lemma B.4. *Let X be a topological space and $f : X \to [-\infty, +\infty]$ be a function. Given $x \in X$ we have that f is lower semicontinuous (resp., upper semicontinuous) at x if and only if $f(x) \leq \liminf_{i \in I} f(x_i)$ (resp., $\limsup_{i \in I} f(x_i) \leq f(x)$) for every net $(x_i)_{i \in I}$ in X converging to x.*

Proof. If f is lower semicontinuous at x and $(x_i)_{i \in I}$ converges to x then for every $c \in [-\infty, +\infty]$ with $c < f(x)$ we have that f is larger than c in a neighborhood of x and therefore $f(x_i) > c$ for all $i \geq i_0$, for some $i_0 \in I$. Thus $\liminf_{i \in I} f(x_i) \geq \inf_{i \geq i_0} f(x_i) \geq c$ for all $c \in [-\infty, +\infty]$ less than $f(x)$ and hence $\liminf_{i \in I} f(x_i) \geq f(x)$. To prove the converse, assume f is not lower semicontinuous at x and let $\mathcal{V}$ be the directed set of all neighborhoods of x in X partially ordered by reverse inclusion. We can then pick $c \in [-\infty, +\infty]$ with $c < f(x)$ such that for every $V \in \mathcal{V}$ there exists $x_V \in V$ with $f(x_V) \leq c$ and then $(x_V)_{V \in \mathcal{V}}$ is a net converging to x with $\liminf_{V \in \mathcal{V}} f(x_V) \leq c < f(x)$. The result for upper semicontinuous functions can of course be obtained by replacing f with $-f$. □

In the main text we use the following result concerning approximation of lower semicontinuous functions by linear combinations of characteristic functions of open sets.

Lemma B.5. *Let X be a topological space. If $f : X \to [0, +\infty]$ is a lower semicontinuous function then there exists a sequence $(f_n)_{n\geq 1}$ such that f is the monotonically increasing pointwise limit of $(f_n)_{n\geq 1}$ and each f_n is a finite linear combination of characteristic functions of open subsets of X with real nonnegative coefficients.*

Proof. Set $f_n = \phi_n \circ f$, where $\phi_n : [0, +\infty] \to [0, +\infty[$ is defined by

$$\phi_n = \frac{1}{2^n} \sum_{k=1}^{n2^n} \chi_{U_k}, \quad U_k = \left] \tfrac{k}{2^n}, +\infty \right],$$

for all $n \geq 1$. To conclude the proof, note that $(\phi_n)_{n\geq 1}$ is pointwise monotonically increasing and convergent to the identity map of $[0, +\infty]$. □

Corollary B.6. *Let X be a normal topological space and μ be a nonnegative (not necessarily finite) measure on the Borel σ-algebra of X such that equality* (3.3) *holds for every open subset A of X. If $f : X \to [0, +\infty]$ is a nonnegative lower semicontinuous function then:*

$$\langle \mu, f \rangle = \sup \big\{ \langle \mu, g \rangle : g \in C_{\mathrm{b}}(X),\ 0 \leq g \leq f \big\}.$$

Proof. If f is the characteristic function of an open set A then for every closed subset F of A we use Urysohn's Lemma to get a continuous function $g : X \to [0, 1]$ that is equal to 1 on F and vanishes outside of A. Then $0 \leq g \leq f$, $\langle \mu, g \rangle \geq \mu(F)$ and the conclusion follows from (3.3). From this observation the result follows immediately if f is a finite linear combination of characteristic functions of open sets with nonnegative coefficients. To conclude the proof, pick $(f_n)_{n\geq 1}$ as in Lemma B.5, apply the result already proven to each f_n and note that $\langle \mu, f \rangle = \lim_{n\to+\infty} \langle \mu, f_n \rangle = \sup_{n\geq 1} \langle \mu, f_n \rangle$. □

Remark B.7. If f has compact support then in Corollary B.6 it is sufficient to assume that equality (3.3) holds only for relatively compact open subsets A and the normality assumption can be replaced by the assumption that X is Hausdorff. Namely, the functions f_n given by Lemma B.5 will also have compact support and thus the first part of the proof of Corollary B.6 can assume that the open set A is relatively compact. We can then apply Urysohn's Lemma on the compact Hausdorff space $\overline{A}$ to get a continuous function $g : \overline{A} \to [0, 1]$ the is equal to 1 on F and vanishes outside of A. By letting g be zero outside of $\overline{A}$ we get a continuous extension of g to X.

Appendix C. Radon measures and the Riesz Representation Theorem

In this appendix we recall certain basic facts about representations of linear functionals on spaces of continuous functions by measures. This theory is typically developed on locally compact Hausdorff spaces. We will not supply the proofs of the hardest representation theorems.

Given a topological space X, we will as usual regard it as a measurable space endowed with its Borel σ-algebra, so that by a measure on X we mean a measure on the Borel σ-algebra of X.

Definition C.1. Let X be a locally compact Hausdorff topological space and μ be a nonnegative (not necessarily finite) measure on X. We say that μ is a *Radon measure* if it is finite on compact sets and the following conditions hold

$$\mu(B) = \inf \big\{\mu(U) : U \subset X \text{ open}, \, U \supset B\big\}, \tag{C.1}$$

$$\mu(U) = \sup \big\{\mu(K) : K \subset U \text{ compact}\big\}, \tag{C.2}$$

for every Borel subset B of X and every open subset U of X. If μ is a signed measure on the Borel σ-algebra of X, we say that μ is a *Radon measure* if both its positive part μ^+ and negative part μ^- are Radon.

It is easily seen that if a signed measure μ is Radon then its total variation $|\mu|$ is also Radon, but the converse is not in general true unless μ is finite.

If μ is an arbitrary nonnegative Radon measure on X then condition (C.2) does not hold in general if the open subset U is replaced with an arbitrary Borel subset B. However, for a Borel subset B of X we do have

$$\mu(B) = \sup \big\{\mu(K) : K \subset B \text{ compact}\big\}$$

if $\mu(B)$ is finite (or if B is a countable union of Borel sets of finite measure). This is shown by first approximating B from the outside by an open set V using (C.1) and then approximating V from the inside by a compact subset K using (C.2). The desired compact approximation for B from the inside is then obtained by trimming the excess from K, i.e., subtracting from K an open approximation from the outside for $K \setminus B$.

As in Section 8, if X is a locally compact Hausdorff topological space, we denote by $C_{\rm c}(X)$ the space of real-valued continuous functions on X having compact support and we endow it with the supremum norm. Obviously every measure μ on the Borel σ-algebra of X which is finite over compact sets defines a linear functional

$$C_{\rm c}(X) \ni f \longmapsto \langle \mu, f\rangle \in \mathbb{R}$$

on $C_{\rm c}(X)$ which we denote below simply by $\langle \mu, \cdot\rangle$.

The Radon condition is the appropriate substitute of regularity to obtain versions of Propositions 3.6 and 3.9 in which $C_{\rm b}(X)$ is substituted by $C_{\rm c}(X)$.

Proposition C.2. *Let X be a locally compact Hausdorff topological space and μ a Radon measure on X. We have that the norm of the linear functional $\langle \mu, \cdot\rangle \in C_{\rm c}(X)^*$ is equal to $\|\mu\|$, i.e.:*

$$\|\mu\| = \sup \big\{|\langle \mu, f\rangle| : f \in C_{\rm c}(X), \ \|f\|_{\sup} \le 1\big\}.$$

Proof. If μ is finite, proceed as in the proof of Proposition 3.6 but choosing the closed sets F_+ and F_- to be compact and using Corollary 8.4 instead of

Urysohn's Lemma. If μ is infinite then, say, μ^+ is infinite and μ^- is finite. Since μ^+ is Radon we have:

$$\sup\big\{\mu^+(K) : K \subset X \text{ compact}\big\} = +\infty.$$

To conclude the proof, note that if $f : X \to [0,1]$ is in $C_\mathrm{c}(X)$ and is equal to 1 over a compact subset K then:

$$\langle \mu, f\rangle = \langle \mu^+, f\rangle - \langle \mu^-, f\rangle \geq \mu^+(K) - \mu^-(X). \qquad \square$$

Proposition C.3. *Let X be a locally compact Hausdorff topological space and μ be a Radon measure on X. The linear functional $\langle \mu, \cdot\rangle \in C_\mathrm{c}(X)^*$ is positive if and only if μ is nonnegative.*

Proof. If μ^- is finite, proceed as in the proof of Proposition 3.9 but choosing the closed set F to be compact and using Corollary 8.4 instead of Urysohn's Lemma. If μ^- is infinite note first that if $f : X \to [0,1]$ is in $C_\mathrm{c}(X)$ and is equal to 1 over a compact subset K then

$$\langle \mu, f\rangle = \langle \mu^+, f\rangle - \langle \mu^-, f\rangle \leq \mu^+(X) - \mu^-(K)$$

and obtain the contradiction $\langle \mu, f\rangle < 0$ from the fact that the Radon condition for μ^- yields:

$$\sup\big\{\mu^-(K) : K \subset X \text{ compact}\big\} = +\infty. \qquad \square$$

It turns out that every positive linear functional on $C_\mathrm{c}(X)$ is of the form $\langle \mu, \cdot\rangle$ for a nonnegative measure μ and that the measure μ becomes unique if we require it to be Radon.

Theorem C.4 (Riesz Representation Theorem, positive case)**.** *If X is a locally compact Hausdorff topological space and α is a positive linear functional on $C_\mathrm{c}(X)$ then there exists a unique nonnegative Radon measure μ on X such that α is equal to $\langle \mu, \cdot\rangle$.*

Proof. See [1, 7.2] or [5, Theorem 2.14]. $\square$

Let $C_\mathrm{c}(X)$ be endowed with the supremum norm. Using a standard vector lattice theory trick one easily shows that a bounded linear functional on $C_\mathrm{c}(X)$ can be written as a difference of bounded positive linear functionals and from this fact and Theorem C.4 one obtains the following version of Riesz Representation Theorem for (not necessarily positive) bounded linear functionals.

Theorem C.5 (Riesz Representation Theorem, bounded case)**.** *If X is a locally compact Hausdorff topological space and α is a bounded linear functional on $C_\mathrm{c}(X)$ then there exists a unique (signed) Radon measure μ on X such that α is equal to $\langle \mu, \cdot\rangle$.*

Proof. See [1, 7.15 and 7.17] or [5, Theorem 6.19]. $\square$

It is easily checked that if X is locally compact Hausdorff then the subset of $\mathrm{ca}(X)$ consisting of finite Radon measures is a closed subspace. The results above show that such subspace is linearly isometric to the topological dual of $C_{\mathrm{c}}(X)$.

Corollary C.6. *If X is a locally compact Hausdorff topological space then the map that sends μ to $\langle\mu,\cdot\rangle \in C_{\mathrm{c}}(X)^*$ is a linear isometry between the closed subspace of* $\mathrm{ca}(X)$ *consisting of finite Radon measures and the topological dual of the normed space $C_{\mathrm{c}}(X)$.*

Proof. Follows from Theorem C.5 and Proposition C.2. □

As we have seen in Corollary 3.5, if every open subset of X is an F_σ then every measure $\mu \in \mathrm{ca}(X)$ is regular. We have an analogue of this result in which regularity is replaced with the Radon condition.

Proposition C.7. *If X is a locally compact Hausdorff topological space for which every open subset is σ-compact then every measure μ on X that is finite on compact sets is Radon.*

Proof. We can assume without loss of generality that μ is nonnegative. Condition (C.2) holds trivially for an open subset U of X since U is a countable increasing union of compact subsets. The collection of Borel subsets B for which condition (C.1) holds is easily seen to be closed under countable unions and since X is σ-compact it then suffices to establish (C.1) if B is relatively compact. Given a relatively compact Borel subset B of X, we have that B is contained in a relatively compact open subset A of X, by the local compactness of X. Now A is a topological space in which every open subset is an F_σ and, since $\mu|_A \in \mathrm{ca}(A)$, Corollary 3.5 yields that $\mu|_A$ is regular and hence condition (C.1) holds. □

C.1. **The compact case.** If K is a compact Hausdorff topological space then $C_{\mathrm{c}}(K)$ is just the Banach space $C(K)$ of real-valued continuous functions on K endowed with the supremum norm. In that case the (signed) Radon measures on K are just the (signed) finite regular measures on K and we denote the space of such measures by $M(K)$ as it is usual in the literature of $C(K)$ spaces. In this context all reasonable definitions of regularity for finite measures are equivalent:

(i) $\mu \in \mathrm{ca}(K)$ is regular if Borel subsets can be approximated by open subsets from the outside, i.e., for every Borel subset B of K and every $\varepsilon > 0$ there exists an open subset U of K containing B such that $|\mu|(U \setminus B) < \varepsilon$;

(ii) $\mu \in \mathrm{ca}(K)$ is regular if Borel subsets can be approximated by closed (or compact) subsets from the inside, i.e., for every Borel subset B of K and every $\varepsilon > 0$ there exists a closed (automatically compact) subset F of K contained in B such that $|\mu|(B \setminus F) < \varepsilon$;

(iii) $\mu \in \mathrm{ca}(K)$ is regular if closed subsets can be approximated by open subsets from the outside, i.e., for every closed subset F of K and every $\varepsilon > 0$ there exists an open subset U of K containing F such that $|\mu|(U \setminus F) < \varepsilon$;

(iv) $\mu \in \mathrm{ca}(K)$ is regular if open subsets can be approximated by closed (or compact) subsets from the inside, i.e., for every open subset U of K and every $\varepsilon > 0$ there exists a closed (automatically compact) subset F of K contained in U such that $|\mu|(U \setminus F) < \varepsilon$.

The equivalence between (i) and (ii) and the equivalence between (iii) and (iv) are obtained by taking complements and the equivalence of (i) or (ii) with (iii) or (iv) follows from Lemma 3.4. If $M(K)$ is endowed with the total variation norm then Corollary C.6 just gives the standard linear isometric identification $M(K) \ni \mu \mapsto \langle \mu, \cdot \rangle \in C(K)^*$ between $M(K)$ and the topological dual of $C(K)$.

Example C.8 (a nonregular probability measure on a compact Hausdorff space)**.** Let ω_1 denote the first uncountable ordinal and consider the ordinal segment $K = [0, \omega_1]$ endowed with the order topology, so that K is a compact Hausdorff space. A subset F of $\omega_1 = [0, \omega_1[$ is called a *club* if it is both closed in the order topology of ω_1 and unbounded in ω_1 (equivalently, if it is both closed and uncountable). It is well-known that a countable intersection of clubs is club (see [3, Lemma 6.8]). It follows easily from this that the collection of all subsets B of ω_1 such that either B or $\omega_1 \setminus B$ contain a club is a σ-algebra of subsets of ω_1. In particular, if B is a Borel subset of ω_1 then either B or $\omega_1 \setminus B$ contain a club. Define a probability measure ν on the Borel σ-algebra of K by setting $\nu(B) = 1$ if B is a Borel subset of K such that $\omega_1 \cap B$ contains a club and $\nu(B) = 0$ if B is a Borel subset of K such that $\omega_1 \setminus B$ contains a club. Clearly $\nu(\omega_1) = 1$ and if F is closed in K and contained in ω_1 then F is countable and therefore $\nu(F) = 0$. It follows that ν is not regular. If $f : K \to \mathbb{R}$ is a continuous function then it is easily shown[1] that f must be constant on $[\alpha, \omega_1]$ for some countable ordinal $\alpha \in \omega_1$. It follows that

$$\langle \nu, f \rangle = \langle \delta_{\omega_1}, f \rangle,$$

for all $f \in C(K)$. This means that $\delta_{\omega_1} \in M(K)$ is the regular measure corresponding to the bounded linear functional $\langle \nu, \cdot \rangle \in C(K)^*$ defined by the nonregular measure ν.

Appendix D. Stone–Čech compactification

Let us recall the definition and the standard construction of the Stone–Čech compactification of a topological space.

[1]It is also true that a continuous function $f : \omega_1 \to \mathbb{R}$ must be constant on $[\alpha, \omega_1[$ for some countable ordinal α, but the proof is a little harder.

Definition D.1. Let X be a topological space. A *Stone–Čech compactification* of X consists of a compact Hausdorff topological space $\beta(X)$ and a continuous map $\iota : X \to \beta(X)$ such that the following property holds: for every compact Hausdorff space K and every continuous map $f : X \to K$ there exists a unique continuous map $\tilde{f} : \beta(X) \to K$ such that:

$$\tilde{f} \circ \iota = f.$$

It follows easily from the definition that Stone–Čech compactifications are unique in the sense that if $\iota : X \to \beta(X)$ and $\iota' : X \to \beta(X)'$ are both Stone–Čech compactifications of X then there exists a (unique) homeomorphism $h : \beta(X) \to \beta(X)'$ such that $h \circ \iota = \iota'$.

We recall that every compact Hausdorff space K is homeomorphic to a subspace of some power $[0,1]^\kappa$ of $[0,1]$ endowed with the product topology. To prove this, simply use Urysohn's Lemma to obtain a family $(f_\alpha)_{\alpha\in\kappa}$ of continuous functions $f_\alpha : K \to [0,1]$ that separate the points of K. Such family defines an injective continuous function from K to $[0,1]^\kappa$ which is then a homeomorphism onto its image. This observation yields the following sufficient condition for $\iota : X \to \beta(X)$ to be a Stone–Čech compactification of X.

Lemma D.2. *Let X be a topological space, $\beta(X)$ be a compact Hausdorff space and $\iota : X \to \beta(X)$ be a continuous map. If the image of ι is dense in $\beta(X)$ and if for every continuous function $f : X \to [0,1]$ there exists a continuous function $\tilde{f} : \beta(X) \to [0,1]$ such that $\tilde{f} \circ \iota = f$ then $\iota : X \to \beta(X)$ is a Stone–Čech compactification of X.*

Proof. Given a continuous function $f : X \to K$ taking values on a compact Hausdorff space K, we assume without loss of generality that K is a closed subspace of the product $[0,1]^\kappa$ for some set κ and we apply the property given in the statement to every coordinate of f to obtain a continuous function $\tilde{f} : \beta(X) \to [0,1]^\kappa$ such that $\tilde{f} \circ \iota = f$. The fact that the image of ι is dense in $\beta(X)$ implies that $\tilde{f}$ takes values in K and that $\tilde{f}$ is unique. □

Using Lemma D.2 we obtain the following simple construction of a Stone–Čech compactification of an arbitrary topological space X. Consider the set $\mathcal{C}$ of all $[0,1]$-valued continuous functions on X and the space $[0,1]^{\mathcal{C}}$ of all $[0,1]$-valued maps on the set $\mathcal{C}$ endowed with the product topology. The product $[0,1]^{\mathcal{C}}$ is compact Hausdorff and the map $\iota : X \to [0,1]^{\mathcal{C}}$ defined by

$$\iota(x)(f) = f(x), \quad f \in \mathcal{C},\ x \in X$$

is continuous. Let $\beta(X)$ be the closure of the image of ι. Clearly for every continuous function $f : X \to [0,1]$ the restriction to $\beta(X)$ of the projection onto the f-th coordinate is a continuous function $\tilde{f} : \beta(X) \to [0,1]$ with $\tilde{f} \circ \iota = f$. Hence $\iota : X \to \beta(X)$ is a Stone–Čech compactification of X by Lemma D.2.

We have just proven the following result.

Theorem D.3. *Every topological space X admits a Stone–Čech compactification $\iota : X \to \beta(X)$. Moreover, the image of ι is dense in $\beta(X)$.* □

We note that, despite Theorem D.3, it is not always the case that the map $\iota : X \to \beta(X)$ is a homeomorphism onto its image and in fact it is not even true in general that ι is injective. There is a simple well-known condition which is equivalent to ι being a homeomorphism onto its image.

Definition D.4. A topological space X is called *completely regular* if for every $x \in X$ and every closed subset F of X with $x \notin F$ there exists a continuous function $f : X \to \mathbb{R}$ that vanishes on F and such that $f(x) = 1$.

Note that every normal space is completely regular by Urysohn's Lemma.

Proposition D.5. *Let X be a topological space and $\iota : X \to \beta(X)$ be a Stone–Čech compactification of X. We have that X is completely regular if and only if the topology of X is induced by ι. Moreover, X is completely regular and Hausdorff if and only if ι is a homeomorphism onto its image.*

Proof. If X is completely regular then it is easily checked that the topology of X coincides with the topology induced by all $[0, 1]$-valued continuous functions on X and therefore it follows from the concrete construction of $\iota : X \to \beta(X)$ described above that X has the topology induced by ι. If X is also Hausdorff then $[0, 1]$-valued continuous functions separate the points of X and therefore ι is also injective and hence a homeomorphism onto its image. Conversely, since $\beta(X)$ is compact Hausdorff it is completely regular and from that it follows easily that X is also completely regular (resp., completely regular and Hausdorff) if the topology of X is induced by ι (resp., if ι is a homeomorphism onto its image). □

Appendix E. Bilinear pairings and their induced weak topologies

Given real vector spaces X and Y, by a *bilinear pairing* $\langle\cdot,\cdot\rangle$ between X and Y we mean simply an arbitrary bilinear map

$$X \times Y \ni (x, y) \longmapsto \langle x, y\rangle \in \mathbb{R} \tag{E.1}$$

that associates a real number to an element of X and an element of Y. A bilinear pairing $\langle\cdot,\cdot\rangle$ induces a linear map

$$\Phi : Y \ni y \longmapsto \langle\cdot, y\rangle \in X^* \tag{E.2}$$

from Y to the algebraic dual X^* of X.

Definition E.1. The *weak topology* on X induced by the bilinear pairing (E.1) is defined as the topology induced by the set of all linear functionals $\langle\cdot, y\rangle$, with $y \in Y$, i.e., the topology induced by the image of the map (E.2).

Since the weak topology of X depends only on the image of the map Φ, when proving results about a weak topology one can without loss of generality always assume that Y is a subspace of X^* and that the bilinear pairing $\langle x, y\rangle$ is just given by evaluating the linear functional $y \in X^*$ at $x \in X$. In fact, instead of talking about weak topologies induced by bilinear pairings we could instead talk about weak topologies induced by subspaces of X^*. However, in concrete examples it is often a little easier to define the pairing than the corresponding subspace of X^* so we prefer to formulate the theory in terms of pairings.

Note that the bilinear pairing (E.1) can also be used to define a topology on Y, namely, the topology induced by the linear functionals $\langle x, \cdot\rangle$, with $x \in X$. This topology is also called the *weak topology* induced on Y by the bilinear pairing (E.1) and obviously every theorem about the weak topology of X corresponds to a similar theorem about the weak topology of Y which can be obtained as a corollary by switching the roles of X and Y. We thus choose to state all our results about weak topologies in terms of the weak topology of X alone and ignore the weak topology of Y.

Clearly a fundamental system of open neighborhoods of the origin of X for the weak topology induced by the bilinear pairing (E.1) is obtained by considering all sets of the form

$$V_{F,\delta} = \big\{x \in X : |\langle x, y\rangle| < \delta \text{ for all } y \in F\big\}$$

with F an arbitrary finite subset of Y and δ an arbitrary positive real number.

Remark E.2. The weak topology on X induced by the bilinear pairing (E.1) always makes X into a *topological vector space*, i.e., a vector space endowed with a topology that makes the vector space operations continuous. Moreover, the weak topology is Hausdorff if and only if the linear functionals $\langle \cdot, y\rangle \in X^*$, $y \in Y$, separate the points of X. In other words, the weak topology is Hausdorff if and only if the linear map Φ is injective. Namely, two points of X separated by a weakly continuous linear functional $\langle \cdot, y\rangle$ are separated by weak open sets defined by such functional. Moreover, two points in which Φ take the same value cannot be distinguished by weak open sets (i.e., a weak open set contains one of the points if and only if it contains the other) and thus the weak topology does not even satisfy the weakest separation axiom T0 if Φ is not injective.

In concrete examples the spaces X and Y that are paired by the bilinear pairing (E.1) are often endowed with norms or semi-norms and one wish to consider compatibility conditions between the bilinear pairing and the semi-norms. If X is endowed with a semi-norm $\|\cdot\|$ we will always endow its topological dual (with respect to the semi-norm topology) with the norm defined by

$$\|\alpha\| = \sup_{\|x\|\le 1} |\alpha(x)|, \tag{E.3}$$

for every bounded linear functional $\alpha : X \to \mathbb{R}$. If both X and Y are endowed with semi-norms, which will be both denoted by $\|\cdot\|$, we say that the bilinear pairing (E.1) is *bounded* (with respect to such semi-norms) if it is bounded as a bilinear map, i.e., if there exists a constant $c \geq 0$ such that

$$|\langle x, y\rangle| \leq c\|x\|\|y\|, \tag{E.4}$$

for all $x \in X$ and $y \in Y$. We have that the bilinear pairing $\langle \cdot, \cdot \rangle$ is bounded if and only if the corresponding linear map Φ takes values in the topological dual of X and it is a bounded linear map. When only X is endowed with a semi-norm $\|\cdot\|$ and Φ takes values in the topological dual of X, we can endow Y with the semi-norm *induced by the bilinear pairing* (E.1) which is defined by

$$\|y\| = \|\Phi(y)\| = \sup_{\|x\| \leq 1} |\langle x, y\rangle|, \tag{E.5}$$

for all $y \in Y$. Such semi-norm is a norm if and only if Φ is injective. If Y is endowed with (E.5) then the bilinear pairing is bounded with the constant c in (E.4) equal to 1. Moreover, if Y is endowed with some other semi-norm then the bilinear pairing is bounded if and only if such semi-norm is finer than (E.5).

We want to investigate conditions under which the weak topology induced by a bilinear pairing (E.1) coincides with the topology induced by a set of linear functionals smaller than the image of Φ. We need a lemma.

Lemma E.3. *Let a bilinear pairing* (E.1) *be fixed, F be a finite subset of Y and δ be a positive real number. If a linear functional $\alpha \in X^*$ is bounded on the set $V_{F,\delta}$ then $\alpha = \langle \cdot, y\rangle$ for some y in the linear span of F in Y.*

Proof. Consider the linear map $T : X \to \mathbb{R}^k$ given by

$$T(x) = \big(\langle x, y_1\rangle, \dots, \langle x, y_k\rangle\big),$$

for all $x \in X$, where $F = \{y_1, \dots, y_k\}$. We have that the kernel of T is contained in $V_{F,\delta}$ and therefore α is bounded on $\mathrm{Ker}(T)$, which is only possible if α vanishes on $\mathrm{Ker}(T)$. The conclusion then follows by noting that the annihilator of $\mathrm{Ker}(T)$ is equal to the image of the adjoint $T^* : {\mathbb{R}^k}^* \to X^*$, which coincides with the linear span of $\big\{\langle \cdot, y\rangle : y \in F\big\}$ in X^*. □

Corollary E.4. *A linear functional $\alpha \in X^*$ is continuous with respect to the weak topology induced by a bilinear pairing* (E.1) *if and only if $\alpha = \langle \cdot, y\rangle$ for some $y \in Y$.*

Proof. Simply note that if α is continuous in the weak topology then α is bounded in some fundamental neighborhood $V_{F,\delta}$ of the origin. □

Corollary E.5. *Let a bilinear pairing* (E.1) *be fixed and assume that the linear map Φ defined in* (E.2) *is injective. Given a subset Λ of Y, we have that the topology induced on X by the set of linear functionals $\Phi[\Lambda]$ coincides with the weak topology induced by the bilinear pairing* (E.1) *if and only if Y equals the linear span of Λ.*

Proof. Let Z denote the linear span of Λ in Y, so that $\Phi[Z]$ is the linear span of $\Phi[\Lambda]$ in the algebraic dual of X. Since a linear combination of continuous functions is continuous, the topologies on X that make all elements of $\Phi[\Lambda]$ continuous are the same topologies that make all elements of $\Phi[Z]$ continuous and thus the topology induced on X by the set of linear functionals $\Phi[\Lambda]$ is the same as the topology induced on X by the set of linear functionals $\Phi[Z]$. Hence the statement of the corollary is equivalent to the statement that the weak topology induced by the bilinear pairing $\langle\cdot,\cdot\rangle$ of X with Y coincides with the weak topology induced by the bilinear pairing

$$X \times Z \ni (x, z) \longmapsto \langle x, z\rangle \in \mathbb{R} \tag{E.6}$$

of X with Z if and only if $Y = Z$. To prove the latter statement, simply note that if such weak topologies coincide then, for all $y \in Y$, the linear functional $\langle\cdot, y\rangle$ is continuous with respect to the weak topology induced by the bilinear pairing of X with Z and thus Corollary E.4 yields $z \in Z$ with $\langle\cdot, y\rangle = \langle\cdot, z\rangle$. □

Now we want to investigate conditions under which the weak topology induced by a bilinear pairing (E.1) coincides with the topology induced by a set of linear functionals smaller than the image of Φ in subsets of X that are bounded with respect to a given semi-norm. We need an adaptation of Lemma E.3.

Lemma E.6. *Let a bilinear pairing* (E.1) *be fixed and assume X is endowed with a semi-norm $\|\cdot\|$. Let F be a finite subset of Y and let $\varepsilon > 0$ and $\delta > 0$ be given. If $\alpha \in X^*$ is a linear functional such that $|\alpha(x)| \leq \varepsilon$ for all $x \in V_{F,\delta}$ with $\|x\| \leq 1$ then there exists y in the linear span of F such that $\|\alpha - \langle\cdot, y\rangle\| \leq \varepsilon$.*

Proof. Let T be defined as in the proof of Lemma E.3, so that again $\mathrm{Ker}(T)$ is contained in $V_{F,\delta}$ and our assumptions imply that $\|\alpha|_{\mathrm{Ker}(T)}\| \leq \varepsilon$. The Hahn–Banach Theorem then gives us an extension $\tilde{\alpha} : X \to \mathbb{R}$ of $\alpha|_{\mathrm{Ker}(T)}$ with $\|\tilde{\alpha}\| \leq \varepsilon$. To conclude the proof, note that $\alpha - \tilde{\alpha}$ is in the annihilator of $\mathrm{Ker}(T)$ and hence $\alpha - \tilde{\alpha} = \langle\cdot, y\rangle$ for some y in the linear span of F. □

Corollary E.7. *Let a bilinear pairing* (E.1) *be fixed and assume X is endowed with a semi-norm $\|\cdot\|$ such that the image of the linear map Φ defined in* (E.2) *is contained in the topological dual of X (this happens, for instance, if Y is also endowed with a semi-norm and the bilinear pairing is bounded). Given a linear functional $\alpha \in X^*$, the following conditions are equivalent:*

- (a) *α is bounded and it belongs to the norm-closure of the image of Φ in the topological dual of X;*
- (b) *the restriction of α to any bounded subset of X is continuous with respect to the weak topology;*
- (c) *the restriction of α to the unit ball of X is continuous at the origin with respect to the weak topology.*

Proof. The implication (a)⇒(b) follows by noting that norm convergence on the topological dual of X is equivalent to uniform convergence on bounded subsets of X and that trivially all linear functionals in the image of Φ are continuous with respect to the weak topology. Now assuming (c), for every $\varepsilon > 0$ we obtain a fundamental neighborhood of the origin $V_{F,\delta}$ such that $|\alpha(x)| \leq \varepsilon$ for all $x \in V_{F,\delta}$ with $\|x\| \leq 1$. Hence Lemma E.6 yields $y \in Y$ with $\|\alpha - \Phi(y)\| \leq \varepsilon$. □

Corollary E.8. *Let X and Y be endowed with semi-norms $\|\cdot\|$ and a bounded bilinear pairing* (E.1) *be fixed such that the semi-norm of Y is equivalent to the semi-norm* (E.5) *induced by the bilinear pairing. Given a subset Λ of Y, the following conditions are equivalent:*

(a) *the linear span of Λ is dense in Y with respect to the semi-norm topology;*

(b) *the topology induced on X by the set of linear functionals $\Phi[\Lambda]$ coincides with the weak topology induced by the bilinear pairing* (E.1) *on every bounded subset of X;*

(c) *the topology induced on X by the set of linear functionals $\Phi[\Lambda]$ coincides with the weak topology induced by the bilinear pairing* (E.1) *on the unit ball of X.*

Proof. Let Z denote the linear span of Λ in Y, so that as in the proof of Corollary E.5 the topology induced on X by the set of linear functionals $\Phi[\Lambda]$ coincides with the weak topology induced on X by the bilinear pairing (E.6) of X with Z. If Z is dense in Y with respect to the semi-norm topology then $\Phi[Z]$ is dense in $\Phi[Y]$ with respect to the norm topology of the topological dual of X and thus Corollary E.7 yields that, for every $y \in Y$, the linear functional $\Phi(y) = \langle \cdot, y\rangle$ is continuous when restricted to bounded subsets of X with respect to the weak topology induced by the bilinear pairing of X with Z. This proves (a)⇒(b). Now assuming (c), we have that for all $y \in Y$ the linear functional $\langle \cdot, y\rangle$ is continuous when restricted to the unit ball of X with respect to the weak topology induced by the bilinear pairing of X with Z and thus Corollary E.7 yields that $\Phi(y)$ belongs to the norm-closure of $\Phi[Z]$ in the topological dual of X. Since the semi-norm topology of Y is induced from Φ by the norm topology of the topological dual of X we conclude that y belongs to the closure of Z in the semi-norm topology of Y. This proves (a) and we are done. □

E.1. **Cardinal invariants of weak topologies.** Let us recall some basic definitions from the theory of cardinal invariants of topologies. As usual, when dealing with cardinals, we denote by ω the cardinal of the natural numbers. Cardinal invariants of topological spaces are usually defined by taking the maximum between some cardinal and ω since one is not interested in finite cardinals.

Definition E.9. Let X be a topological space. The *weight* of X is defined as the maximum between ω and the least cardinal of a basis of X and the *density* of X is defined as the maximum between ω and the least cardinal of a dense subset of X. Given a point $x \in X$, the *character of x in X* is defined as the maximum between ω and the least cardinal of a fundamental system of neighborhoods of x in X and the *character of X* is defined as the maximum between ω and the least cardinal κ such that every point of X has a fundamental system of neighborhoods of size less than or equal to κ (equivalently, if X is nonempty, the character of X is the supremum of the characters of all points of X).

The following facts are easy to prove. The weight of a topological space X is equal to the maximum between ω and the least cardinal of a subbasis of X. Thus, the weight of a topology induced by κ real-valued functions is less than or equal to the maximum between κ and ω. The density and the character of a topological space are always less than or equal to its weight. For a metrizable or pseudo-metrizable space (Definition E.11), the weight is equal to the density and the character is always ω. For a topological vector space all points have the same character due to the translation invariance of the topology.

Let us now prove some results about the weight and the character of a weak topology induced by a bilinear pairing.

Proposition E.10. *Let a bilinear pairing* (E.1) *between real vector spaces X and Y be fixed. We have that the character and the weight of the weak topology on X induced by the bilinear pairing coincide. Moreover, if the linear map Φ defined in* (E.2) *is injective then the weight and the character of the weak topology of X are equal to the maximum between the dimension of Y (in the purely algebraic sense) and ω.*

Proof. We can assume without loss of generality that Φ is injective, as a weak topology is always induced by some bilinear pairing with an injective map Φ. First note that if $\mathcal{B}$ is an algebraic basis of Y then, by Corollary E.5, the weak topology of X is induced by $\Phi[\mathcal{B}]$ and thus the weight of X in the weak topology is less than or equal to the maximum between the cardinality of $\mathcal{B}$ (which is the dimension of Y) and ω. To conclude the proof, let κ denote the character of the origin in the weak topology of X and let us show that the dimension of Y is less than or equal to κ. We have that every fundamental system of neighborhoods of the origin contains a fundamental system of neighborhoods of the origin with cardinality less than or equal to κ and therefore theree exists a subset Λ of Y with cardinality less than or equal to κ such that the collection of all $V_{F,\delta}$ with $F \subset \Lambda$ finite and $\delta > 0$ constitute a fundamental system of neighborhoods of the origin in the weak topology. Now Lemma E.3 implies that $\Phi[Y]$ is equal to the linear span of $\Phi[\Lambda]$ and hence Y is equal to the linear span of Λ. $\square$

Definition E.11. A topological space is called *pseudo-metrizable* if its topology is induced by some pseudo-metric (the definition of a pseudo-metric is the same as the definition of a metric, but removing the requirement that distinct points have a nonzero distance).

Obviously, a topology is metrizable if and only if it is pseudo-metrizable and Hausdorff (alternatively, if and only if it is pseudo-metrizable and T0).

Corollary E.12. *Let a bilinear pairing* (E.1) *between real vector spaces* X *and* Y *be fixed and assume that the linear map* Φ *defined in* (E.2) *is injective. The following conditions are equivalent:*

(a) *the weak topology of* X *is second countable;*

(b) *the weak topology of* X *is first countable;*

(c) *the weak topology of* X *is pseudo-metrizable;*

(d) *the dimension of* Y *(in the purely algebraic sense) is countable.*

Proof. The equivalence between (a), (b) and (d) follows directly from Proposition E.10. Now (c) obviously implies (b) and from the proof of Proposition E.10 we see that (d) implies (c) as a topology induced by a countable set of real-valued functions is pseudo-metrizable. □

Let us now look at the weight and the character of the weak topology on bounded subsets.

Proposition E.13. *Let a bilinear pairing* (E.1) *between real vector spaces* X *and* Y *be fixed and let* X *be endowed with a semi-norm such that the image of the linear map* Φ *defined in* (E.2) *is contained in the topological dual of* X*. Let also* Y *be endowed with a semi-norm equivalent to the semi-norm* (E.5) *induced by the bilinear pairing and with the corresponding semi-norm topology. If* B *denotes the unit ball of* X *endowed with the weak topology then the character of the origin in* B*, the character of* B *and the weight of* B *are all equal to the density of* Y*.*

Proof. If follows from Corollary E.8 that the weak topology of B is induced by a set of real-valued functions whose cardinality is less than or equal to the cardinality of a dense subset of Y and thus the weight of B is less than or equal to the density of Y. To conclude the proof, let κ denote the character of the origin in B and let us check that the density of Y is less than or equal to κ. As in the proof of Proposition E.10, we obtain a subset Λ of Y with cardinality less than or equal to κ such that the collection of all $V_{F,\delta} \cap B$ with $F \subset \Lambda$ finite and $\delta > 0$ constitute a fundamental system of neighborhoods for the origin in B. Now Lemma E.6 implies that the linear span of $\Phi[\Lambda]$ is dense in $\Phi[Y]$ and thus the linear span of Λ is dense in Y, as the topology of Y is induced by Φ. Hence the $\mathbb{Q}$-linear span of Λ is a dense subset of Y whose cardinality is less than or equal to κ. □

Corollary E.14. *Let a bilinear pairing* (E.1) *between real vector spaces X and Y be fixed and let X be endowed with a semi-norm such that the image of the linear map Φ defined in* (E.2) *is contained in the topological dual of X. Let also Y be endowed with a semi-norm equivalent to the semi-norm* (E.5) *induced by the bilinear pairing and with the corresponding semi-norm topology. The following conditions are equivalent:*

(a) *the unit ball of X is second countable in the weak topology;*

(a') *every bounded subset of X is second countable in the weak topology;*

(b) *the unit ball of X is first countable in the weak topology;*

(b') *every bounded subset of X is first countable in the weak topology;*

(c) *the unit ball of X is pseudo-metrizable in the weak topology;*

(c') *every bounded subset of X is pseudo-metrizable in the weak topology;*

(d) *Y is separable in the semi-norm topology.*

Proof. Assuming (d), Corolary E.8 yields that the weak topology on a bounded subset of X is induced by a countable set of real-valued functions and thus (a'), (b') and (c') follow. Moreover, it is obvious that (a'), (b') and (c') imply (a), (b) and (c), respectively. Finally, (c) implies (b) and the conditions (a), (b) and (d) are all equivalent by Proposition E.13. □

Corollaries E.12 and E.14 have some interesting consequences for the weak* topology on $\mathrm{ca}(X)$, where X denotes now an arbitrary topological space. Recalling that the weak* topology of $\mathrm{ca}(X)$ is the weak topology induced by the bilinear pairing of $\mathrm{ca}(X)$ with $C_{\mathrm{b}}(X)$ given by the restriction of (2.1) and that the linear map (2.3) (and thus also its restriction to $C_{\mathrm{b}}(X)$) is an isometric embedding, we obtain the following results.

Proposition E.15. *For an arbitrary topological space X, the following conditions are equivalent:*

(a) *the weak* topology of* $\mathrm{ca}(X)$ *is second countable;*

(b) *the weak* topology of* $\mathrm{ca}(X)$ *is first countable;*

(c) *the weak* topology of* $\mathrm{ca}(X)$ *is pseudo-metrizable;*

(d) $C_{\mathrm{b}}(X)$ *is finite-dimensional.*

Proof. Follows from Corollary E.12 keeping in mind that $C_{\mathrm{b}}(X)$ is a Banach space and that by the Baire Category Theorem an infinite dimensional Banach space cannot have infinite countable dimension. □

Remark E.16. For a "reasonable" infinite topological space X the space $C_{\mathrm{b}}(X)$ is never going to be finite-dimensional. For example, if X is an infinite Hausdorff completely regular topological space then $C_{\mathrm{b}}(X)$ is always infinite-dimensional. Namely, if $(x_n)_{n\geq 1}$ is a sequence of distinct points of X and $f_n : X \to [0,1]$ is a continuous function with $f_n(x_n) = 1$ and $f_n(x_k) = 0$ for $k < n$ then the sequence $(f_n)_{n\geq 1}$ in $C_{\mathrm{b}}(X)$ is linearly independent.

Proposition E.17. *For an arbitrary topological space X, the following conditions are equivalent:*

(a) *the unit ball of* $\mathrm{ca}(X)$ *is second countable in the weak* topology;*

(a') *every bounded subset of* $\mathrm{ca}(X)$ *is second countable in the weak* topology;*

(b) *the unit ball of* $\mathrm{ca}(X)$ *is first countable in the weak* topology;*

(b') *every bounded subset of* $\mathrm{ca}(X)$ *is first countable in the weak* topology;*

(c) *the unit ball of* $\mathrm{ca}(X)$ *is pseudo-metrizable in the weak* topology;*

(c') *every bounded subset of* $\mathrm{ca}(X)$ *is pseudo-metrizable in the weak* topology;*

(d) *the Stone–Čech compactification* $\beta(X)$ *is metrizable.*

Proof. Corollary E.14 says that (a), (a'), (b), (b'), (c) and (c') are all equivalent to $C_{\mathrm{b}}(X)$ being separable. The conclusion follows by noting that $C_{\mathrm{b}}(X)$ is isometric to $C\big(\beta(X)\big)$ and that for a compact Hausdorff space K the space $C(K)$ is separable if and only if K is metrizable. □

Among completely regular Hausdorff topological spaces we can easily characterize those with a metrizable Stone–Čech copactification.

Lemma E.18. *If X is a completely regular Hausdorff topological space then its Stone–Čech compactification $\beta(X)$ is metrizable if and only if X is compact and metrizable.*

Proof. If X is compact and metrizable then $\beta(X)$ is homeomorphic to X, so obviously it is compact and metrizable. Since X is completely regular and Hausdorff, the map $\iota : X \to \beta(X)$ is a homeomorphism onto its image and therefore we can identify X with $\iota[X]$, which is a dense subspace of $\beta(X)$. Assuming that $\beta(X)$ is metrizable, we prove that $X = \beta(X)$, so that X is compact and metrizable. Assume by contradiction that there exists a point $p \in \beta(X)$ not in X. Since X is dense in $\beta(X)$ and $\beta(X)$ is metrizable, there exists a sequence $(x_n)_{n\geq 1}$ of distinct points of X converging to p. The set $\big\{x_n : n \geq 1\big\}$ has the discrete topology and it is closed in X and therefore by Tietze's Extension Theorem every bounded real-valued function defined on $\big\{x_n : n \geq 1\big\}$ admits a bounded continuous extension to X. In particular we can obtain a bounded continuous function $f : X \to \mathbb{R}$ such that the limit $\lim_{n\to+\infty} f(x_n)$ does not exist and such function does not admit a continuous extension to $\beta(X)$, contradicting the fact that $\iota : X \to \beta(X)$ is the Stone–Čech compactification of X. □

We can now apply Corollaries E.12 and E.14 to obtain results for the vague topology of $\mathrm{ca}(X)$, with X a locally compact Hausdorff topological space. The vague topology of $\mathrm{ca}(X)$ is obviously the weak topology induced by the bilinear pairing of $\mathrm{ca}(X)$ with $C_{\mathrm{c}}(X)$ given by the restriction of (2.1).

Proposition E.19. *For a locally compact Hausdorff topological space X, the following conditions are equivalent:*

(a) *the vague topology of* $\mathrm{ca}(X)$ *is second countable;*
(b) *the vague topology of* $\mathrm{ca}(X)$ *is first countable;*
(c) *the vague topology of* $\mathrm{ca}(X)$ *is pseudo-metrizable;*
(d) *the vague topology of* $\mathrm{ca}(X)$ *is metrizable;*
(e) X *is countable and discrete.*

Proof. By Corollary E.12, (a), (b) and (c) are equivalent to $C_c(X)$ having countable dimension and by Proposition 8.9 the conjunction of (c) and (e) implies (d). It thus remains to show that $C_c(X)$ has countable dimension if and only if X is countable and discrete. First, if X is discrete then the characteristic functions of the singletons of X generate $C_c(X)$ and thus $C_c(X)$ has countable dimension if X is countable. Conversely, assume that $C_c(X)$ has countable dimension. We claim that if K is a compact subset of X then the restriction map

$$C_c(X) \ni f \longmapsto f|_K \in C(K)$$

is surjective. Namely, applying Tietze's Extension Theorem in the one-point compactification of X we obtain that every real-valued continuous function on K admits a continuous extension to X and we can make such an extension have compact support by multiplying it by an element of $C_c(X)$ that is equal to 1 over K (Corollary 8.4). This proves the claim. Now, since $C_c(X)$ has countable dimension, the Banach space $C(K)$ will also have countable dimension and therefore it must be finite dimensional. But this is only possible if K is finite (Remark E.16) and therefore we have proven that every compact subset of X is finite. Since X is locally compact, this implies that every point of X has a finite neighborhood and then the fact that X is Hausdorff implies that it must be discrete. To conclude the proof, note that if X is discrete then the characteristic functions of the singletons of X constitute a linearly independent family in $C_c(X)$ and hence X must be countable. □

Proposition E.20. *For a locally compact Hausdorff topological space X, the following conditions are equivalent:*

(a) *the unit ball of* $\mathrm{ca}(X)$ *is second countable in the vague topology;*
(a') *every bounded subset of* $\mathrm{ca}(X)$ *is second countable in the vague topology;*
(b) *the unit ball of* $\mathrm{ca}(X)$ *is first countable in the vague topology;*
(b') *every bounded subset of* $\mathrm{ca}(X)$ *is first countable in the vague topology;*
(c) *the unit ball of* $\mathrm{ca}(X)$ *is pseudo-metrizable in the vague topology;*
(c') *every bounded subset of* $\mathrm{ca}(X)$ *is pseudo-metrizable in the vague topology;*
(d) *the unit ball of* $\mathrm{ca}(X)$ *is metrizable in the vague topology;*

(d') *every bounded subset of* $\mathrm{ca}(X)$ *is metrizable in the vague topology;*

(e) X *is second countable.*

Proof. Corollary E.14 says that (a), (a'), (b), (b'), (c) and (c') are all equivalent to $C_{\mathrm{c}}(X)$ being separable and by Proposition 8.9 the conjunction of (c) and (e) (resp., of (c') and (e)) implies (d) and (e) (resp., (d') and (e)). It thus remains to show that $C_{\mathrm{c}}(X)$ is separable if and only if X is second countable. If X is compact then $C_{\mathrm{c}}(X) = C(X)$ and $C(X)$ is separable if and only if X is second countable. If X is not compact, we consider its one-point compactification $X \cup \{\infty\}$ and we identify $C_{\mathrm{c}}(X)$ isometrically with the subspace of $C\big(X \cup \{\infty\}\big)$ consisting of maps that vanish on a neighborhood of the point ∞. The closure of such subspace is the hyperplane consisting of maps that vanish at ∞ and therefore $C_{\mathrm{c}}(X)$ is separable if and only if $C\big(X \cup \{\infty\}\big)$ is separable. To conclude the proof, note that $C\big(X \cup \{\infty\}\big)$ is separable if and only if $X \cup \{\infty\}$ is second countable and that this holds if and only if X is second countable[2]. □

Appendix F. Equicontinuity

In this short section we recall a very elementary but very useful lemma about the set of points in which an equicontinuous net of functions converges.

Definition F.1. Let X be a topological space and (M, d) be a pseudo-metric space. A set $\mathcal{F}$ of maps $f : X \to M$ is said to be *equicontinuous at a point* $x \in X$ if for every $\varepsilon > 0$ there exists a neighborhood V of x in X such that $d\big(f(x), f(y)\big) < \varepsilon$ for all $y \in V$ and all $f \in \mathcal{F}$. We say that $\mathcal{F}$ is *equicontinuous* if it is equicontinuous at every point of X.

Lemma F.2. *Let* X *be a topological space,* (M, d) *be a pseudo-metric space,* $(f_i)_{i\in I}$ *be a net of maps from* X *to* M *and* $f : X \to M$ *be a continuous map. If the set* $\{f_i : i \geq i_0\}$ *is equicontinuous for some* $i_0 \in I$ *then the set*

$$\big\{x \in X : \big(f_i(x)\big)_{i\in I} \text{ converges to } f(x)\big\} \tag{F.1}$$

is closed in X.

Proof. Replacing $(f_i)_{i\in I}$ with $(f_i)_{i\geq i_0}$ does not alter the set (F.1), so we might just assume that $\{f_i : i \in I\}$ is equicontinuous. Let $x \in X$ belong to the closure of (F.1) and let us check that x belongs to (F.1). Given $\varepsilon > 0$, pick a neighborhood V of x such that $d\big(g(x), g(y)\big) < \varepsilon$ for all $y \in V$ and all $g \in \{f_i : i \in I\} \cup \{f\}$. Now choose y in the intersection between V and (F.1) and $i_0 \in I$ with $d\big(f_i(y), f(y)\big) < \varepsilon$ for all $i \geq i_0$. We then obtain $d\big(f_i(x), f(x)\big) < 3\varepsilon$ for all $i \geq i_0$, concluding the proof. □

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[2]To see that if X is second countable then $X \cup \{\infty\}$ is second countable note that if K_n are compact sets like in the proof of Proposition 8.18 then the complements in $X \cup \{\infty\}$ of the sets K_n constitute a countable fundamental system of neighborhoods for ∞.

If the topology of X is induced by a pseudo-metric and if $\mathcal{F}$ is a set of maps $f : X \to M$ which are Lipschitz with a common Lipschitz constant then $\mathcal{F}$ is obviously equicontinuous. In particular, a collection of linear maps between semi-normed vector spaces that is bounded in the operator norm is equicontinuous. This observation yields the following corollary.

Corollary F.3. *Let X and Y be semi-normed real vector spaces and $\langle \cdot, \cdot \rangle$ be a bounded bilinear pairing between X and Y. If $(y_i)_{i \in I}$ is a net in Y and if the family $(y_i)_{i \geq i_0}$ is bounded for some $i_0 \in I$ then the set*

$$\big\{ x \in X : \big(\langle x, y_i \rangle \big)_{i \in I} \text{ converges to } \langle x, y \rangle \big\}$$

is a closed subspace of X with respect to the semi-norm topology. □

References

[1] G. B. Folland, Real Analysis, second edition, John Wiley & Sons, 1999.
[2] M. W. Hirsch, Differential Topology, Springer–Verlag, 1976.
[3] K. Kunen, Set Theory, North–Holand, 1980.
[4] J. M. Lee, Introduction to Smooth Manifolds, Springer, 2003.
[5] W. Rudin, Real and Complex Analysis, third edition, McGraw–Hill, 1987.

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