VARIABLES

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ABSTRACT. The use of variables is omnipresent throughout the practice of Mathematics and its various applications. Mathematicians, physicists and other users of Mathematics learn from practice how to handle variables and their usage eventually becomes instinctive and automatical, like talking and moving. Nevertheless, there are certain subtleties and distinctions that people almost never explicitly talk about and the absence of a proper discussion has the potential to create some confusion, specially for students. These notes intend to fill in this gap and most of the material here is meant to be accessible to undergraduate students that have taken a few Calculus courses and that have mild familiarity with more rigorous proof-oriented courses. The use of variables in computer programming languages will not be discussed in these notes.

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1. Introduction

There are two very different notions of variables in the practice of Mathematics and its applications: one of them — more akin to formal Logic — arose from the foundational developments in Mathematics that occurred during the final part of the nineteenth century and the first part of the twentieth century; it became standard in most of the modern rigorous presentations of Mathematics. The other is an old notion of variable that remains largely in use today among physicists and, more generally, among those that study applications of Mathematics. There is also some sort of middle arena consisting of certain areas of Mathematics (specially Calculus, but also parts of Differential Geometry) that are closer to Physics and in which the use of variables tend to be a mixture of the two notions.

The fact that this difference in the usage of variables remains to this day is in part explicable by the existence of a certain degree of isolation between various academic communities. It is also true, however, that the old notion of variable is very convenient for dealing with applications of Mathematics and it is thus to be expected that it will remain in use in more applied fields.

These notes are structured as follows. In Section 2 we summarize how variables are used in modern mathematical practice and we informally explain some terminology that is standard in books on Mathematical Logic. In Sections 3 and 4 we discuss and try to make sense of various notations that are typical in Calculus books. Finally, in Section 5 we discuss the notion of variable that is normally used by physicists and we explain how a rigorous mathematical treatment can be achieved for that notion. While the definitions that we give in Section 5 are not very standard, what we do there is not particularly original: we simply take the notion of random variable that is normally used among probabilists and we remove the "random" part.

2. Variables in modern mathematical practice

Let us start by reviewing some of the more common ways that variables appear in modern presentations of Mathematics. That is the goal of Subsections 2.1, 2.2 and 2.3. We then take some time in Subsections 2.4 and 2.5 to explain some terminology that will be useful for the discussion in the remaining sections.

2.1. **Introduction of a variable by the "let" construction.** This construction works as follows: at some point, the presentation of some mathematical argument states "let x be a real number" (or, more generally[1], "let x satisfy the property $P(x)$"). The idea is that at that point we are being instructed to pick some (arbitrary, unspecified) real number and to put it inside a box labelled "x". In the following lines of text of that presentation,

[1]It would be better to think that here we are combining the construction "let x" with the construction "assume $P(x)$", but you will rarely find just "let x" in a normal presentation of Mathematics, as it doesn't sound good in English.

whenever one talks about x, one will be talking about that (fixed) real number. In other words, inside that block of text, the value of x does not vary, it becomes a *constant*: it always points to the same real number that was picked at the moment of the "let" construction. At some point further in the presentation, the reasoning involving x is concluded and a certain fact $Q(x)$ about x is established. At that point, we are allowed to conclude that the property Q holds for all real numbers, i.e., that "$Q(x)$ holds for every real number x" (more generally, if x were introduced by "let x satisfy $P(x)$", then at this point we would conclude that "for all x satisfying $P(x)$, the property $Q(x)$ holds"). The justification for that conclusion is that, given any real number, we could repeat the reasoning that was just presented by replacing x with that specific real number and conclude that the property Q holds for that real number.

After the desired conclusion is obtained, we normally think that the value of x is discarded (the box labelled "x" becomes empty again), so that now x points to nothing and we are free to use it again later for something else.

2.2. **Bound variables.** Roughly speaking, we say that a variable x appears *free* in a certain formula if that formula is a statement about the value of x so that, typically[2], the formula will be true for some values of x and false for other values of x. For example, in the formula $x^2 = 2x$ the variable x is free: the formula is true if we replace x with 2 and it is false if we replace x with 3. A variable that is not free is called *bound*. The typical way for a variable to become bound is by means of a quantifier. By a *quantifier* we mean the symbols $\forall$ (meaning "for all") and $\exists$ (meaning "there exists"); the symbol $\exists!$ for "there exists precisely one" is also sometimes used as a quantifier. For example, consider the sentence

$$\text{for all } x \in \mathbb{R}, \text{ there exists } y \in \mathbb{R} \text{ such that } y > x$$

or, in symbols:

$$\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ y > x. \tag{1}$$

The variables in this sentence are bound: in order to decide whether (1) is true or false, we don't have to ask for the value of x or for the value of y. Also, we can replace the variables x and y in (1) with other two (distinct) variables and the meaning of the formula will remain exactly the same. We could even state the same thing avoiding variables, by saying "for every real number, there is a greater real number", but as our formulas become complicated, avoiding variables becomes difficult to accomplish in practice and makes the exposition unintelligible. Bound variables are also called *dummy variables*.

[2] But not necessarily. It can happen that a formula in which the variable x appears free turns out to be true for all values of x (such as in $x = x$) or false for all values of x.

A formula can have a mixture of free and bound variables, as in:

$$\exists y \in \mathbb{Q},\ y^2 = x. \tag{2}$$

Here the variable x is free and the variable y is bound, so that the formula makes a statement about the value of x, but not about the value of y. In fact, the same variable x can appear free and bound in one formula, as in:

$$\big(\forall x \in \mathbb{R}, (x+1)^2 = x^2 + 2x + 1\big) \text{ and } x^2 = 2x.$$

In this formula, the occurrences of x before the "and" are bound and the occurrences of x after the "and" are free. What happens is that a quantifier has a limited *scope* within a formula: in this example, the scope of $\forall x \in \mathbb{R}$ is only the expression inside the parenthesis, so the occurrences of x outside of that parenthesis are not bound by that quantifier. Of course, in terms of clarity of exposition, having the same variable occur free and bound inside a formula is not good practice.

When we write a formula in which some variable x appears free, we have normally previously introduced x by a "let x" construction, so that we are talking about that value of x that was fixed at the point where the "let x" appeared. When x appears bound by a quantifier, then we normally haven't introduced it by a "let x" construction (if we did, then the quantifier is normally understood to temporarily override the value assigned by the "let x" construction, but in order to achieve clarity the best option is simply to avoid this).

Notice that a variable that was introduced through a "let" construction has a fixed value throughout an entire argument (that could take several pages of text), while the value of a bound variable is kept fixed only inside one specific formula and, more precisely, only within the scope of the corresponding quantifier. So, if x appears bound by a quantifier in a formula and if we reuse the variable x in the following formula, this new x does not refer to the same thing as the previous x. Even within the same formula, we can use x again in the scope of another quantifier and it does not refer to the same thing. For instance, in

$$\big(\forall x \in \mathbb{R},\ (x+1)^2 = x^2 + 2x + 1\big) \text{ and } (\exists x \in \mathbb{R},\ x^2 = x) \tag{3}$$

the occurrences of x before the "and" have nothing to do with the occurrences of x after the "and". We could replace the occurrences of x before the "and" with y and the sentence would mean exactly the same thing.

2.3. **Bound variables in the definition of a set and in summation-like constructions.** Not only quantifiers can make a variable bound. For instance, when we define a set by writing

$$\big\{x \in A : P(x)\big\} \tag{4}$$

then the variable x in this expression is bound. This expression is simply a notation for the set of all elements of A satisfying the property P and this set does not depend on the value of x (but it does depend on the value of A, so that A is free in (4)). Another possible notation for defining a set that creates a bound variable is:

$$\big\{f(x) : x \in A\big\}; \tag{5}$$

this set does not depend on the value of x and it is equal to the set:

$$\big\{y : \text{there exists } x \in A \text{ with } y = f(x)\big\}.$$

Bound variables also appear when we write a summation:

$$\sum_{i=1}^{10} i^2. \tag{6}$$

The variable i is bound in this expression. The value of the summation is simply $1^2 + 2^2 + \cdots + 10^2$ and it does not depend on the value of i. There are actually several summation-like constructions in mathematical practice that create bound varibles, such as products $\prod_{i=1}^{10}(2i+3)$, unions $\bigcup_{i\in I} A_i$, and so on.

2.4. **Terms versus formulas.** A *term* is an expression that denotes a *value* like a number, a set, a function, a surface, etc[3]. So, for instance, $3 + 7$ is a term, $xy+z$ is a term, $\sum_{i=1}^{n} x_i$ is a term and $\big\{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\big\}$ is a term. Obviously, when a term has free variables, the value of the term will depend on the values assigned to the free variables. A *formula* is an expression that makes a *statement* and it can be either true or false; it has a *truth-value* (the possible truth-values are "true" and "false"). Of course, just like in the case of terms, the truth-value of a formula will be dependent on the values of the free variables, if any are present. Formulas without free variables are usually called *sentences*. For example, $2 < 3$ is a formula (and a sentence), $x + y = z$ is a formula (but not a sentence), the expression (2) is a formula (but not a sentence), the expressions (1) and (3) are formulas (and sentences) and so on.

We now take advantage of this terminology to make some comments on the notations discussed in the previous subsections. In Subsection 2.2 we were discussing only free and bound occurrences of variables in formulas. A quantifier is something that you can add to a formula to create a new formula; if a variable were free in the original formula, then that variable will become bound in the new formula, if it is under the action of the added quantifier. What we saw in Subsection 2.3 is that the free/bound distinction

[3]Technically, if the axiomatic foundations of mathematics is ZFC (Zermelo–Fraenkel set theory, with the axiom of choice), then everything is a set, though we don't normally think of real numbers as being sets. For instance, the irrational numbers π and e are sets and it makes sense to ask for the intersection $\pi \cap e$, but it is a stupid question about which no one cares. The answer will depend on the specific details chosen for the construction of the real numbers, such as Dedekind cuts or equivalence classes of Cauchy sequences of rational numbers and it will also depend on all sorts of irrelevant conventions. The reason why everything is a set in ZFC is that having a richer ontology would serve no practical purpose and it would only make it harder to formulate the theory.

is also relevant for variables that appear in terms. A summation construction such as (6) takes a term i^2 with a free variable i and it creates a new term $\sum_{i=1}^{10} i^2$ in which the variable i is bound. The construction (4) takes the term A and the formula $P(x)$ (with a free variable x) and it generates a term in which x is bound. Finally, the construction (5) takes the term $f(x)$ (with a free variable x) and the term A and it creates a new term in which x is bound.

2.5. **Syntactic objects versus mathematical objects.** The distinction that we are going to make in this subsection is like the distinction between the word "cat" and a cat. The word "cat" contains three letters and the middle letter is a vowel. The word "cat" cannot scratch your sofa, while a cat can and a cat has no letters. While words are abstract objects and cats are concrete objects made of matter and having a location in space and time, in Mathematics everything is abstract, but nevertheless the distinction between an expression and the thing the expression refers to is relevant. So what I will mean by a *mathematical object* is something like the cat in the previous analogy: it is the type of object that mathematicians talk about, like numbers, sets, functions, surfaces and so on (things that can be the value of a term). On the other hand, *syntactic objects* are like words: they are the symbols and strings of symbols that we use to talk about mathematics[4]. So, for instance, terms and formulas are syntactic objects. Variables, in the sense discussed in this section, are also syntactic objects.

Some comments are in order here to avoid confusion in the future. When we say "let x be a real number" then apparently it will be correct to say that "x is a real number" and it will also be correct to say that "x is a variable". How can that be? The same thing is a real number and a variable? The same problem happens when we say "cat is an animal" and also "cat is a word". How can something be both a word and an animal? It can't: the apparent contradiction arises because we use symbols as names for certain objects and also as names for themselves. In "x is a real number" we are using x as the name of some mathematical object (which is a real number) and in "x is a variable" we are using x as the name of the symbol x. In situations where this might get too confusing, we will use quotation marks, just like in ordinary language: we use "cat" as a name for the word cat, which is not the same thing as a cat. With this convention, we should say then that "cat is an animal" and that ""cat" is a word". Likewise, with this convention, saying "x is a real number" is appropriate, while saying "x is a variable" is not. We should instead say that ""x" is a variable". Since

[4]When you are studying topics like Mathematical Logic and Model Theory, things get a little messier since strings of symbols and formulas become themselves the object of study, so we have to pick certain mathematical objects to represent the symbols and the strings of symbols. This is what is called *Gödelization*, but you don't have to worry about it here.

using so many quotation marks would soon get very annoying, we will only use them when strictly necessary to avoid confusion.

3. Dynamical definition of functions from terms

The perfectly rigorous way to introduce a function in a mathematical presentation is by a statement of the following form: "let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 + x$, for all $x \in \mathbb{R}$". It is not right to simply talk about "the function $x^2 + x$" as this is really the value of the function at the point x. Saying something that is not exactly right is not a big problem if the audience can understand you — sometimes it is even preferable and it makes the exposition more intelligible — but in certain situations this kind of loose talk can generate some ambiguity. For instance, an expression like $x^2y^3 + xy$ can be used to define several different functions. We could think that y is constant and define the function $f_1(x) = x^2y^3 + xy$, we could think that x is constant and define the function $f_2(y) = x^2y^3 + xy$ or we could define the two distinct two-variable functions $f_3(x, y) = x^2y^3 + xy$ and $f_4(y, x) = x^2y^3 + xy$. So, when you talk about "the function $x^2y^3 + xy$", it is not very clear what you are talking about[5].

When we are doing Calculus, in a single page of computations we will normally have to deal with dozens of different functions, sometimes more than one per line of text. Namely, each intermediate step of a computation forces us to break a function into pieces and to generate more functions. If we had to formally introduce each function by a "let $f : \mathbb{R} \to \mathbb{R} \ldots$" construction, doing Calculus would be impossible. So we need a notation that allows for the dynamical definition of functions, in such a way that we can quickly use a term with free variables like $x^2y^3 + xy$ to refer to a function. Let us review the various notations of this sort that normally appear in Calculus.

3.1. **Definite integrals.** Given a function $f : I \to \mathbb{R}$ defined over some domain I that contains the interval $[a, b]$, then the definite Riemann (or Lebesgue) integral of f can be denoted simply by

$$\int_a^b f, \tag{7}$$

i.e., we specify the function, the interval and the fact that we are doing an integral. This notation works well if we have a name (like f) for the function that we want to integrate. This name was typically introduced by some "let" construction that came before. Now, if we don't want to go through the trouble of formally introducing every function with a "let"

[5] Ambiguity can get much worse when we are dealing with a function f whose counter-domain is again a set of functions. In this case, f is a function and $f(x)$ is also a function, so confusing a function with its value at a point becomes a more serious problem.

construction, we can use a notation that dynamically creates a function from a term, like this:

$$\int_a^b (x^2y^3 + xy)\,\mathrm{d}x. \tag{8}$$

The expression (8) is a term with a bound variable x (and a free variable y) that was created from a term $x^2y^3 + xy$ with a free variable x (and a free variable y), so in this respect it is the same type of construction used in (6). In the summation (6), the i below the summation symbol $\sum$ indicates that it is the variable i that becomes bound in the summation and in the definite integral (8) it is the x in $\mathrm{d}x$ that indicates the variable that becomes bound, i.e., the *variable of integration*. Historically, the notation $\mathrm{d}x$ was used because the integral was thought of some sort of infinite sum of products $(x^2y^3 + xy)\,\mathrm{d}x$, with $\mathrm{d}x$ an infinitesimal perturbation of x, but in modern Mathematics we normally think of $\mathrm{d}x$ merely as a notation to indicate the variable of integration[6]. The integral (8) is equal to $\int_a^b f$, with $f : [a, b] \to \mathbb{R}$ the function defined by

$$f(x) = x^2y^3 + xy,$$

for all $x \in [a, b]$ (and y has some fixed value that was previously declared by a "let y" construction).

3.2. **Limits.** Like in the case of the definite integral, the usual notation for limits allows us to deal with functions dynamically defined by a term, without the need of formal introduction with a "let" construction. So, for instance, in

$$\lim_{x\to 0} \frac{\sin x}{x} \tag{9}$$

we have a term with a bound variable x obtained from the term $\frac{\sin x}{x}$ with a free variable x. The bound variable is the one that appears below the symbol "lim". In (9) we are taking the limit at 0 of the function f defined by $f(x) = \frac{\sin x}{x}$. Curiously, unlike the case of the definite integral, there is no standard notation for limits which is like (7), i.e., a notation to be used when the function has a name f. Even if the function is named f, we have to write something like $\lim_{x\to a} f(x)$, with the pointless introduction of the new bound variable x.

3.3. **Derivatives.** Given a function $f : I \to \mathbb{R}$ defined, for example, in an interval $I \subset \mathbb{R}$, then the derivative f' is a new function defined at the set of points where f is differentiable. The derivative of f at the point $x \in I$ is then $f'(x)$. Sometimes it is said that this is "the derivative of f with respect to the variable x" or that "f is a function of the variable x" but this is really nonsense: a function is not attached to some variable and "taking a derivative with respect to a certain variable" in this context

[6]The symbol $\mathrm{d}x$ can have a more serious meaning in the theory of differential forms, which we will not discuss in these notes.

is meaningless. In particular, a notation like $\frac{\mathrm{d}f}{\mathrm{d}x}$ in which a variable x is specified is pointless[7]: the function f has a certain derivative at a certain point and there is no need and no use for the specification of a variable. Nevertheless, like in the previous subsections, it is useful to have a notation that allows us to write the derivative of a function being dynamically created from a term (without the need for the formal introduction of f by a "let f" construction) and, in this case, it is necessary to specify a variable so that one can unambiguously obtain a function from the term. A common notation for this purpose is this:

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x^2y + \sin(xy)\big). \tag{10}$$

The expression (10) is a term and it should be understood as being the same as $f'(x)$, with f the function defined by $f(x) = x^2y + \sin(xy)$, for all x, where y has some fixed value (typically fixed by some previous "let y" construction). Notice that, unlike the analogous constructions in the previous subsections, the variable x is *free* in the term (10), as this term is the same thing as $f'(x)$.

There is a small number of authors that use the symbol $\mapsto$ to create functions from terms, so that the function f defined by $f(x) = x^2y+\sin(xy)$ would be denoted by

$$x \longmapsto x^2y + \sin(xy) \tag{11}$$

or

$$\mathbb{R} \ni x \longmapsto x^2y + \sin(xy) \in \mathbb{R}$$

if one wants to specify the domain and the counterdomain. Notice that (11) is the same as f, not as $f(x)$, so that x is a bound variable in the term (11). Using this notation, we obtain the monstrosity

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x^2y + \sin(xy)\big) = \big(x \longmapsto x^2y + \sin(xy)\big)'(x),$$

i.e., we can see (10) as the composition of three operations: forming the function $x \mapsto x^2y+\sin(xy)$ from the term $x^2y+\sin(xy)$, taking the derivative and then evaluating at x. The simple notation (10) is clearly preferable in this case, though the notation $\mapsto$ for creating functions from terms might be helpful in certain situations.

Finally, we note that although the notation $\frac{\mathrm{d}f}{\mathrm{d}x}$ contains a weird meaningless reference to the variable x, the notation

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)\big)$$

is, due to the conventions above, a valid (but unnecessarily complicated) notation for $f'(x)$.

[7]However, see Sections 4 and 5 for further comment.

3.4. **Indefinite integral.** In terms of notation, the situation for indefinite integrals is similar to what we discussed in the previous subsection for derivatives, except for the fact that Calculus books normally take the indefinite integral to (somehow) denote *all* primitives of the given function. For instance, when we write

$$\int (x^2y^3 + xy)\,\mathrm{d}x \tag{12}$$

we mean something like $F(x)$, with F an arbitrary primitive of the function f defined by $f(x) = x^2y^3 + xy$. In (12) the variable x is free, as (12) denotes F evaluated at x, not just F. The insistence that (12) should denote an *arbitrary* primitive of f leads to the odd situation in which (12) must be something like a multi-valued term. Indeed, Calculus books will say that

$$\int (x^2y^3 + xy)\,\mathrm{d}x = \frac{x^3y^3}{3} + \frac{x^2y}{2} + C,$$

with C an "arbitrary constant".

The idea of a multi-valued term is a monstrosity that is normally not used elsewhere in mathematical practice, as it creates all kinds of problems. For instance, if τ_1 and τ_2 are multi-valued terms, then when will a formula like $\tau_1 < \tau_2$ be satisfied? Is it satisfied when the inequality holds for *all* possible values of the given terms or when it holds for *some* of the possible values of the given terms? If a multi-valued term τ appears more than once in an expression, should we understand that all occurrences of τ have the same value or should we understand that different occurrences might have different values? Dealing with multi-valued terms would require a careful formulation of the semantics of the formulas (i.e., a careful explanation of their meaning) and one would then have to carefully check which of the standard rules of deduction continue to hold with this semantics. The standard notation for indefinite integrals does lead to paradoxes if not handled with the appropriate care, as in the following integration by parts

$$\int \frac{1}{x}\,\mathrm{d}x = \int \frac{1}{x}\cdot 1\,\mathrm{d}x = \frac{1}{x}x - \int x\Big(-\frac{1}{x^2}\Big)\,\mathrm{d}x = 1 + \int \frac{1}{x}\,\mathrm{d}x,$$

which leads to $0 = 1$.

There are two ways to handle this: one is to think of the standard notation for indefinite integrals as a calculating tool that is not to be taken very seriously. For rigorous arguments, one should use a notation in which some fixed primitive is considered for the function, for instance one could use $F(x) = \int_{x_0}^{x} f$, which yields the only primitive F of f satisfying $F(x_0) = 0$. The other possibility is to think of an indefinite integral as a set of functions, so that $\int (x^2y^3 + xy)\,\mathrm{d}x$ is not something like a multi-valued term but just a standard single-valued term whose value is a set. For rigorous reasoning, one would have then to take into account the fact that indefinite integrals are sets and treat them accordingly in algebraic manipulations. The sum of two indefinite integrals should be understood as a sum of sets of functions, the sum of a function with an indefinite integral should be understood as a sum of a function with a set of functions and so on[8]. But note that it is important to regard the indefinite integral as the set of functions themselves, not as the set of those functions evaluated at some point. Namely, if $\int (x^2y^3 + xy)\,\mathrm{d}x$ denotes the set

$$\big\{F(x) : F \text{ primitive of } f\big\}$$

with f defined by $f(x) = x^2y^3 + xy$, then it turns out that $\int (x^2y^3 + xy)\,\mathrm{d}x$ is simply the set $\mathbb{R}$ of all real numbers. Taking the set of all possible primitives of a function and evaluating them all at a given point x will always yield the set of all real numbers, so that would not be a very interesting meaning. Thus, the strategy of regarding the indefinite integral as a set of functions contradicts the fact that $\int (x^2y^3 + xy)\,\mathrm{d}x$ is supposed to be a term in which the variable x is free. For this to work, one then has to use a different notion of variable that we will explain in Section 5 (see Remark 5.7).

4. Taking a partial derivative with respect to a variable

Given a function $f : U \to \mathbb{R}$ defined in a (typically open) subset U of $\mathbb{R}^3$, then the partial derivatives of f are normally denoted by

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y} \quad \text{and} \quad \frac{\partial f}{\partial z}, \tag{13}$$

and they are called the "partial derivative of f with respect to the variable x", the "partial derivative of f with respect to the variable y" and the "partial derivative of f with respect to the variable z". But what does this mean really? There is no sense in which f is a "function of the variables x, y and z" and there is no connection between these variables and f. The function f is a rule that takes triples of real numbers as inputs and generates real numbers as outputs. In the standard set-theoretic formalization of mathematics, the function f is a subset of the cartesian product $\mathbb{R}^3 \times \mathbb{R}$. The variables x, y and z are merely symbols and it is not clear what they are supposed to be denoting here.

Notice that, using the notational convention of Subsection 3.3, the partial derivatives of f calculated at a point $(x, y, z) \in U$ can be denoted by

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x,y,z)\big), \quad \frac{\mathrm{d}}{\mathrm{d}y}\big(f(x,y,z)\big) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}z}\big(f(x,y,z)\big),$$

but there is nothing special about the letters x, y and z. One could also say that the partial derivatives of f calculated at a point $(\rho, w, \sigma) \in U$ are denoted by:

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\big(f(\rho,w,\sigma)\big), \quad \frac{\mathrm{d}}{\mathrm{d}w}\big(f(\rho,w,\sigma)\big) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\sigma}\big(f(\rho,w,\sigma)\big).$$

[8] If A and B are sets of functions, the sum $A + B$ should be the set of functions defined by $\big\{f + g : f \in A,\ g \in B\big\}$ and, if f is a function, the sum $f + B$ should be defined as the set of functions $\big\{f + g : g \in B\big\}$. The "arbitrary constant of integration" C could then be understood as a notation for the set of all constant functions.

Let us now review some basic notions of Calculus and then let us discuss a rational foundation for the use of the notation (13).

4.1. **Partial and directional derivatives.** Given a function $f : U \to \mathbb{R}$ defined in a (typically open) subset U of $\mathbb{R}^n$, then for $i = 1, 2, \dots, n$, we define the *i-th partial derivative* of f or *the partial derivative of f with respect to the i-th variable* at a point $x = (x_1, \dots, x_n) \in U$ by setting:

$$(\partial_i f)(x) = g'(x_i),$$

where g is the function defined by

$$g(t) = f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n),$$

for all $t \in \mathbb{R}$ such that $(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n) \in U$. In principle, something like $\partial_i f$ is the only type of notation that makes sense for a partial derivative of f; namely, since the elements in the domain of f are families indexed by the positive integers 1, 2, ..., n, one has merely to specify one of those integers $i \in \{1, 2, \dots, n\}$ and this is all that we need in order to specify a partial derivative of f.

Given a point $x \in U$ and a vector $v \in \mathbb{R}^n$, then the *directional derivative* of f at the point x in the direction of v is defined by[9]:

$$\frac{\partial f}{\partial v}(x) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}.$$

Clearly, the partial derivative $\partial_i f$ is equal to the directional derivative of f in the direction of the i-th vector of the canonical basis of $\mathbb{R}^n$. Thus, if $\{e_1, e_2, \dots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$, then we get this alternative notation for the partial derivatives of f:

$$\partial_i f = \frac{\partial f}{\partial e_i}.$$

4.2. **Going back to the traditional notation.** How can we make sense of the notation (13)? It is really simple: one merely chooses a bijection between the positive integers 1, 2, ..., n and some arbitrary list of n distinct symbols and then we introduce the convention that, for $i = 1, 2, \dots, n$, the partial derivative $\partial_i f$ will be denoted by $\frac{\partial f}{\partial \#}$, with # the symbol that was chosen to correspond to the positive integer i. For instance, if $n = 3$, we could pick the bijection

$$1 \mapsto x, \quad 2 \mapsto y, \quad 3 \mapsto z, \tag{14}$$

and according to this bijection and the convention above, we have:

$$\partial_1 f = \frac{\partial f}{\partial x}, \quad \partial_2 f = \frac{\partial f}{\partial y} \quad \text{and} \quad \partial_3 f = \frac{\partial f}{\partial z}. \tag{15}$$

[9]Though the notation $\frac{\partial f}{\partial v}$ is common, the similarity with the notation (13) for partial derivatives is somewhat annoying: if one is using v as the "name for one of the variables of f", then $\frac{\partial f}{\partial v}$ denotes both a partial derivative of f and the directional derivative in the direction of the vector $v \in \mathbb{R}^n$.

We could choose a different bijection, with different symbols

$$1 \mapsto \rho, \quad 2 \mapsto w, \quad 3 \mapsto \sigma, \tag{16}$$

and then the same partial derivatives of f would be denoted differently:

$$\partial_1 f = \frac{\partial f}{\partial \rho}, \quad \partial_2 f = \frac{\partial f}{\partial w} \quad \text{and} \quad \partial_3 f = \frac{\partial f}{\partial \sigma}. \tag{17}$$

Note that the symbols x, y, z, ρ, w, σ are not variables in any meaningful way, they are merely symbols. But it is common to call them "the names of the variables of f".

In many cases it is convenient to use different bijections for different functions[10] and that is perfectly fine, as long as the bijections are specified in some way. In practice, Calculus books do not explain this very explicitly and they don't present the bijection like we did in (14). A bijection like that is usually implied by the formula used to introduce the function, which will be a formula like:

$$f(x, y, z) = \sin(xy + z^2), \quad \text{for all } (x, y, z) \in \mathbb{R}^3;$$

from this formula, the reader infers the bijection (14) and thus the notation (15) for the partial derivatives of f. If the same function f were introduced by the formula

$$f(\rho, w, \sigma) = \sin(\rho w + \sigma^2), \quad \text{for all } (\rho, w, \sigma) \in \mathbb{R}^3,$$

then the reader should infer the bijection (16) and the notation (17) for the partial derivatives of f.

But what is the point of picking these bijections? Couldn't we just use $\partial_1 f$, $\partial_2 f$ and $\partial_3 f$ for the partial derivatives of f? The point is that in practice we commonly have some concrete meaning in mind for the various variables of f and remembering numbers is not practical. For instance, in a concrete situation, f could be a function of time, volume and pressure and it is easier to remember the association of "time", "volume" and "pressure" with letters like t, V and P than an association with numbers 1, 2 and 3. We want to denote the partial derivative of f with respect to time by something like $\frac{\partial f}{\partial t}$, not $\partial_1 f$; we just don't care to remember if time happens to be the first or the seventh variable of f.

4.3. **How to get rid of the useless variable numbering.** Imagine a practical problem in which we have a function $f : \mathbb{R}^2 \to \mathbb{R}$ which we think of as a function of time $t \in \mathbb{R}$ and some spatial position $x \in \mathbb{R}$, say f is given by:

$$f(t, x) = \cos(3x - 2t), \quad \text{for all } (t, x) \in \mathbb{R}^2. \tag{18}$$

[10] Even if f and g happen to denote exactly the same function, we might want to use one bijection for f and another for g. So it is not just "different bijections for different functions" that we want, but really *different bijections for different function names.*

We would then pick the bijection

$$1 \mapsto t, \quad 2 \mapsto x \tag{19}$$

and denote the partial derivatives $\partial_1 f$ and $\partial_2 f$ by $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial x}$, respectively. Now consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$g(x,t) = \cos(3x - 2t), \quad \text{for all } (x,t) \in \mathbb{R}^2.$$

For the purpose of handling the given practical problem, the function g will be just as good as f. Properly understood, both f and g describe the same physical situation and yet *the functions f and g are different functions.* For instance, we have $f(1,0) = \cos(-2) = \cos 2$ and $g(1,0) = \cos 3 \neq f(1,0)$. For the function g we would prefer to use the bijection

$$1 \mapsto x, \quad 2 \mapsto t$$

so that $\partial_1 g$ is denoted by $\frac{\partial g}{\partial x}$ and $\partial_2 g$ is denoted by $\frac{\partial g}{\partial t}$.

Since we don't care about whether time is going to be the first variable or the second variable, wouldn't it be nice to simply get rid of the useless variable numbering? How can we do this? Simple: forget about $\mathbb{R}^2$, in which the elements are indexed by $\{1,2\}$, and use a more appropriate set for indexation. An element x of $\mathbb{R}^n$ can be thought of as a mapping from $\{1,2,\ldots,n\}$ to $\mathbb{R}$ (the mapping that associates $i \in \{1,2,\ldots,n\}$ to the i-th coordinate x_i of the n-tuple x). We could replace $\{1,2,\ldots,n\}$ by another set with n elements and get something that is not exactly $\mathbb{R}^n$, but it is isomorphic[11] to $\mathbb{R}^n$. So, given a set I, we denote by $\mathbb{R}^I$ the space of all mappings from I to $\mathbb{R}$. If I has n elements, then $\mathbb{R}^I$ is like $\mathbb{R}^n$, but without the numbering of the coordinates. A coordinate of an element $x \in \mathbb{R}^I$ is associated to an element of I, not to a positive integer between 1 and n. So, instead of $\mathbb{R}^n$, we could just use $\mathbb{R}^I$ as the domain of our function f, with I the set containing the n symbols[12] that we would like to associate to the "variables" of f.

I'm not defending that following the strategy proposed in the previous paragraph is desirable, I'm just proposing that as a possibility for those that are annoyed by the necessity of fixing some arbitrary variable ordering when dealing with multivariable functions. In the case of the function (18), one could think that the domain of f is $\mathbb{R}^{\{t,x\}}$ instead of $\mathbb{R}^2$, with t and x being some fixed distinct symbols. One can then denote the partial derivatives of f by $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial x}$ (or simply $\partial_t f$ and $\partial_x f$) and no bijection like (19) has to be specified. If the domain of f is $\mathbb{R}^{\{t,x\}}$, then f does not have a first variable and a second variable, but a t-variable and an x-variable. However,

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[11] As a vector space, for instance.

[12] Strictly speaking, the symbols are syntactic objects, not mathematical objects, so they cannot be elements of the set I. If the axiomatic foundation of mathematics is ZFC, then the elements of I should be sets. But we can simply imagine that we have picked arbitrary (unspecified) sets to correspond to the symbols that we want to be the elements of I.

it is somewhat annoying now that the elements of the domain of f are not ordered pairs, so an expression like $f(3,5)$ will not be understandable. One would have then to use some awkward notation for an element of the domain of f such as $(t \mapsto 3, x \mapsto 5)$ or something like that.

5. Variables as interdependent quantities

In Physics and other applications of Mathematics, we often want to study the behaviour of certain systems and, in particular, we are interested in studying the relation between certain variables that describe properties of a given system. The meaning of "variable" in this context is completely different from the meaning we have discussed in the previous sections. In order to avoid confusion, we will call *dynamical variables* the type of variable that will be discussed in this section and *syntactic variables* the type of variables considered in the previous sections.

For example, suppose that the system we want to study is a (Newtonian) particle moving in three-dimensional space. For each instant of time t, the particle will have a certain position $\vec{r} = (x, y, z)$, represented in terms of coordinates with respect to some given coordinate system in physical space. The particle also has a certain mass m and, for each t, the particle has a vector velocity $\vec{v}$, a scalar velocity v, a vector momentum $\vec{p}$, a scalar momentum p, a kinetic energy K, a total force $\vec{F}$ applied to it and so on. These various dynamical variables can satisfy certain relations that make some of them dependent on the others. For instance, we have the equalities

$$\vec{p} = m\vec{v}, \quad v^2 = \vec{v} \cdot \vec{v}, \quad p^2 = \vec{p} \cdot \vec{p}, \quad K = \frac{1}{2}\, mv^2 = \frac{p^2}{2m},$$

where the dot denotes the scalar product. Moreover, we often talk about taking derivatives of one variable with respect to another variable and we write equalities such as

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}, \quad \vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}, \quad \frac{\mathrm{d}K}{\mathrm{d}t} = \vec{F} \cdot \vec{v},$$

and so on. Instead of taking the derivative of the kinetic energy K with respect to time, we could choose to take, say, the derivative of kinetic energy with respect to the scalar velocity v and this yields:

$$\frac{\mathrm{d}K}{\mathrm{d}v} = mv = p.$$

As we discussed in Subsection 3.3, talking about "taking the derivative of a function with respect to a given variable" does not make sense: a function has a derivative, period, no variables have to be specified. In Section 4 we gave some meaning to the idea of "taking the partial derivative of a function with respect to a variable", but the "variable" there was merely a symbol that served as a mnemonic for a positive integer: we write something like $\frac{\partial f}{\partial t}$ instead of $\partial_2 f$ because the symbol t is easier to remember than the number 2. Also, talking about "taking the derivative of the variable y with respect to the variable x" is meaningless if x and y are syntactic variables. Namely, at a given point in the presentation of a mathematical argument, syntactic variables have just one value (fixed by some previous "let" construction). It might happen that we have a relation like $y = x^2 + x$ (this is the case, for instance, after we say "let $x, y \in \mathbb{R}$ and assume that $y = x^2 + x$"), but no functions are defined by a relation like that since x has only *one* value. If $x = 1$ and $y = 2$, then it is true that $y = x^2 + x$, but it is also true that $y = x^4 + x^5$.

5.1. **A precise formulation for the concept of dynamical variable.** In what follows, we give a rigorous mathematical foundation for the kind of talk that normally appears in Physics books about "a variable being a function of another variable" and "taking the derivative of one variable with respect to another variable". The motivation behind our formulation is the following: in a Physics problem, the system that one wants to study can be in a variety of different *states* and, for each state, a dynamical variable that represents a property of the system will assume a certain value. We then consider a certain set $\mathbb{S}$ which we call the *state space* (i.e., the set of all possible states for the system) and the dynamical variables will be mappings defined in $\mathbb{S}$.

Definition 5.1. Let $\mathbb{S}$ be a set. A *real-valued dynamical variable on the state space* $\mathbb{S}$ is a function $x : \mathbb{S} \to \mathbb{R}$. More generally, given an arbitrary set M, an *M-valued dynamical variable on the state space* $\mathbb{S}$ is a function $x : \mathbb{S} \to M$. For example, we can talk about *vector-valued dynamical variables* by taking $M = \mathbb{R}^n$ or by taking M to be some vector space.

Now we can give precise meaning to the idea that a variable can be a function of another variable.

Definition 5.2. Let $\mathbb{S}$, M and N be sets, $x : \mathbb{S} \to M$ be an M-valued dynamical variable on the state space $\mathbb{S}$ and $f : M \to N$ be a function. The composition $f \circ x : \mathbb{S} \to N$ is then an N-valued dynamical variable on the state space $\mathbb{S}$ and it will be denoted simply by $f(x)$. We say that an N-valued dynamical variable $y : \mathbb{S} \to N$ on the state space $\mathbb{S}$ is a *function of* x if there exists a function $f : M \to N$ such that $y = f(x)$ (i.e., such that $y = f \circ x$).

Note that if y is a function of x in the sense defined above, then there exists a *unique* function f defined on the image of x such that $y = f(x)$. The image of x is, of course, the set of all possible values assumed by x. We can have also $y = f(x)$ for a function f whose domain contains — but it is not necessarily equal to — the image of x, but then the condition $y = f(x)$ says nothing about the values that f assumes outside of the image of x. It is easy to see that y is a function of x if and only if, for every pair of states $\sigma_1, \sigma_2 \in \mathbb{S}$, if $x(\sigma_1) = x(\sigma_2)$, then $y(\sigma_1) = y(\sigma_2)$.

Now, given real-valued dynamical variables x and y on the same state space, we can make sense of the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ of y with respect to x. We choose our definition in such a way that $\frac{\mathrm{d}y}{\mathrm{d}x}$ is also a dynamical variable.

Definition 5.3. Let x and y be real-valued dynamical variables on the same state space $\mathbb{S}$. Assume that the image of x is an interval[13] $I \subset \mathbb{R}$ and that y is a function of x. Let $f : I \to \mathbb{R}$ be the unique function such that $y = f(x)$. If f is differentiable on I, we define the *derivative of y with respect to x* to be the real-valued dynamical variable on the state space $\mathbb{S}$ given by:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = f' \circ x$$

More generally, it could happen that f is not differentiable at all points of I, so that the domain of the derivative f' is a subset of I. The domain of $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$ would then be the subset of $\mathbb{S}$ consisting of those states $\sigma \in \mathbb{S}$ such that f is differentiable at the point $x(\sigma)$. In this case, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is going to be a real-valued dynamical variable in a state space smaller than $\mathbb{S}$.

With this definition of derivative, one readily checks that the following version of the chain rule holds.

Proposition 5.4 (chain rule)**.** *Let x, y and z be real-valued dynamical variables on the same state space. Assume that y is a function of x, z is a function of y and that the derivatives $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}z}{\mathrm{d}y}$ exist. Then z is a function of x, the derivative $\frac{\mathrm{d}z}{\mathrm{d}x}$ exists and:*

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y}\,\frac{\mathrm{d}y}{\mathrm{d}x}. \tag{20}$$

Proof. Note that if $y = g(x)$ and $z = f(y)$, then $z = (f \circ g)(x)$. Now apply the standard chain rule for the composition $f \circ g$. $\square$

In the righthand side of equality (20), the product of the dynamical variables $\frac{\mathrm{d}z}{\mathrm{d}y}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$ is understood in the way that a product of functions is normally understood, namely, the pointwise product. More generally, one can operate with dynamical variables in the normal way that one operates with functions, using pointwise defined operations. Thus, for instance, if x and y are real-valued dynamical variables on the same state space $\mathbb{S}$, one can form the real-valued dynamical variable $x^3y + x\sin(xy^2)$ on $\mathbb{S}$; more explicitly, we have

$$\big(x^3y + x\sin(xy^2)\big)(\sigma) = [x(\sigma)]^3 y(\sigma) + x(\sigma)\sin\big(x(\sigma)[y(\sigma)]^2\big),$$

for all $\sigma \in \mathbb{S}$.

Like the chain rule, other standard rules of Calculus (like the rule for derivatives of sums and products) can be readily shown to hold for dynamical

[13]More generally, it would be sufficient for the image of x to be a subset of $\mathbb{R}$ with no isolated points.

variables. Also, higher order derivatives are readily adaptable to the context of dynamical variables.

We note that when doing Calculus with dynamical variables, the choice of state space $\mathbb{S}$ is largely irrelevant, only the relations between the variables matter. For instance, if you want to formalize the implication

$$y = x^2 + x^3 \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 3x^2$$

using dynamical variables x and y, you could take the state space $\mathbb{S}$ to be the real line $\mathbb{R}$, the real-valued dynamical variable $x : \mathbb{R} \to \mathbb{R}$ to be the identity map and then you would define $y = x^2 + x^3$. You could also set

$$\mathbb{S} = \big\{(t, t^2 + t^3) : t \in \mathbb{R}\big\} \subset \mathbb{R}^2$$

and take $x : \mathbb{S} \to \mathbb{R}$ and $y : \mathbb{S} \to \mathbb{R}$ to be the restriction to $\mathbb{S}$ of the first and second projections of $\mathbb{R}^2$, respectively. This also leads to $y = x^2 + x^3$. The second choice is more elegant and it handles the variables x and y in a more symmetric way than the first choice, but both work just as well. In general, all you need is a sufficiently rich state space $\mathbb{S}$ that can represent all the relevant relations between dynamical variables that you need. One possibility that always works is to take $\mathbb{S}$ to be a subset of some cartesian product, so that each element of $\mathbb{S}$ is a family containing all the possible joint values of the dynamical variables at a state.

Remark 5.5. In terms of the terminology of Subsection 2.5, a dynamical variable is a mathematical object (it is a function) and a syntactic variable is a syntactic object (it is a symbol). However, we talk about dynamical variables like we talk about any object: using words and symbols. When we say "let f be the function defined by ...", we have a syntactic variable f which is a symbol that points to a mathematical object that is a function. Likewise, when we say that "x is a dynamical variable" we have a symbol x which is a syntactic variable that points to the mathematical object that is a dynamical variable. Using the convention discussed in Subsection 2.5 for quotation marks, we should say that "f is a function", while ""f" is a syntactic variable" and that "x is a dynamical variable", while ""x" is a syntactic variable". Note that if x is a real-valued dynamical variable, then the value of the syntactic variable "x" is not a real number, but the dynamical variable x, which is a function. What we normally think as the "value of a dynamical variable x" would be the value $x(\sigma)$ that the function x takes at some state $\sigma \in \mathbb{S}$; that value is indeed a real number.

Remark 5.6. There is another way to make sense of the notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ that does not use dynamical variables. We could take x to be a syntactic variable and y to be a symbol that works as an abbreviation of some term in which the variable x is free. For instance, y could be an abbreviation for $x^2 + x$ or for $f(x)$. Then $\frac{\mathrm{d}y}{\mathrm{d}x}$ could be understood as an instance of the notation (10). Unfortunately, this strategy for interpreting $\frac{\mathrm{d}y}{\mathrm{d}x}$ has some shortcomings when compared to the one that uses dynamical variables. For instance, if x and y

are dynamical variables and if $y = f(x)$ with f an injective function, then $\frac{\mathrm{d}x}{\mathrm{d}y}$ becomes a notation for the derivative of the inverse function of f (composed with y), which is what we would like $\frac{\mathrm{d}x}{\mathrm{d}y}$ to mean. On the other hand, if y is merely an abbreviation for a term containing the syntactic variable x, then $\frac{\mathrm{d}x}{\mathrm{d}y}$ is meaningless.

Remark 5.7. The use of dynamical variables yields a solution to the problems we had in Subsection 3.4 to interpret the standard notation for indefinite integrals. Namely, given dynamical variables x and y such that y is a function of x, we can understand the indefinite integral $\int y\,\mathrm{d}x$ as denoting the following set of dynamical variables

$$\big\{F(x) : F \text{ a primitive of } f\big\},$$

where f is the function defined in the image of x such that $y = f(x)$. This formalism also works nicely to provide a precise meaning for the computations in which one uses substitutions of variables to calculate an indefinite integral.

5.2. **Random variables.** If you know anything about probability theory, then you have probably realized that what we have called dynamical variables is very similar to what is normally called a *random variable*. In probability theory, a *real-valued random variable* in a probability space is a measurable real-valued function defined on that probability space. A probability space is like our state space $\mathbb{S}$, but rather than being just a set it carries a probability measure[14]. We normally think that a probability space models a system in which the states are random, either because the states are going to be obtained as outcomes of independently reproducible experiments exhibiting statistical regularities or because we want to model some situation in which the state is unknown and the probabilities are related to subjective uncertainties. In any case, the randomness of the states contaminates the values of the dynamical variables and then we call them random variables. In probability theory, random variables are usually denoted by uppercase letters such as $X : \mathbb{S} \to \mathbb{R}$ and $Y : \mathbb{S} \to \mathbb{R}$ and a relation like $Y = f \circ X$ is normally denoted as $Y = f(X)$, like we did in Definition 5.2.

[14] A *probability measure* on a set $\mathbb{S}$ is a mapping $P : \mathcal{A} \to [0,1]$ such that $P(\mathbb{S}) = 1$ and $P(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n)$, for every sequence $(E_n)_{n\geq 1}$ of mutually disjoint elements of $\mathcal{A}$. The set $\mathcal{A}$ is a nonempty collection of subsets of $\mathbb{S}$ such that the union of an arbitrary sequence of elements of $\mathcal{A}$ is in $\mathcal{A}$ and the complement of an arbitrary element of $\mathcal{A}$ is in $\mathcal{A}$. The collection $\mathcal{A}$ is then called a *σ-algebra* of subsets of $\mathbb{S}$. A real-valued function defined on $\mathbb{S}$ is said to be *measurable* if the inverse image of every interval belongs to $\mathcal{A}$. For discrete probability theory, one can normally take $\mathcal{A}$ simply to be the collection of all subsets of $\mathbb{S}$, so that every function is measurable. In general, the σ-algebra is important due to technical reasons that you will learn in courses about Lebesgue measure and measure theory.

5.3. **Multivariable calculus and partial derivatives.** Dynamical variables can also easily be used for multivariable calculus. We give now the relevant definitions.

Definition 5.8. Let $x_1 : \mathbb{S} \to M_1$, …, $x_n : \mathbb{S} \to M_n$ be dynamical variables in the same state space $\mathbb{S}$. We denote by $(x_1, \ldots, x_n) : \mathbb{S} \to M_1 \times \cdots \times M_n$ the dynamical variable defined by

$$(x_1, \ldots, x_n)(\sigma) = \big(x_1(\sigma), \ldots, x_n(\sigma)\big) \in M_1 \times \cdots \times M_n,$$

for all $\sigma \in \mathbb{S}$.

For example, if x_1, …, x_n are real-valued dynamical variables, then $(x_1, \ldots, x_n)$ is an $\mathbb{R}^n$-valued dynamical variable.

Definition 5.9. We say that a dynamical variable $y : \mathbb{S} \to N$ is a *function* of the dynamical variables $x_1 : \mathbb{S} \to M_1$, …, $x_n : \mathbb{S} \to M_n$ if y is a function of $(x_1, \ldots, x_n)$, i.e., if there exists a function $f : M_1 \times \cdots \times M_n \to N$ such that:

$$y = f(x_1, \ldots, x_n) = f \circ (x_1, \ldots, x_n).$$

Clearly, if y is a function of x_1, …, x_n, then there exists a unique function f defined in the image of $(x_1, \ldots, x_n)$ such that $y = f(x_1, \ldots, x_n)$. The image of $(x_1, \ldots, x_n)$ is a subset of the product $M_1 \times \cdots \times M_n$. Note that y is a function of x_1, …, x_n if and only if for all $\sigma_1, \sigma_2 \in \mathbb{S}$, if $x_i(\sigma_1) = x_i(\sigma_2)$ for all $i = 1, \ldots, n$, then $y(\sigma_1) = y(\sigma_2)$. It follows that the condition that y be a function of x_1, …, x_n is independent of the chosen ordering for the sequence x_1, …, x_n, though the function f itself does depend on that ordering.

When y is a function of x_1, …, x_n, we can define the partial derivatives $\frac{\partial y}{\partial x_i}$, $i = 1, \ldots, n$, as follows: let f be the function defined in the image of $(x_1, \ldots, x_n)$ such that $y = f(x_1, \ldots, x_n)$ and set

$$\frac{\partial y}{\partial x_i} = (\partial_i f)(x_1, \ldots, x_n), \quad i = 1, \ldots, n, \tag{21}$$

provided that the partial derivative $\partial_i f$ exists[15]. Though f depends on the chosen ordering of x_1, …, x_n, the partial derivative $\frac{\partial y}{\partial x_i}$ doesn't. In Subsection 5.4 below we will give a more general definition of partial derivative for which the independence of ordering is manifest, as no such ordering is mentioned.

We note that in the case of a partial derivative of a dynamical variable y with respect to a dynamical variable x_i, the result does not depend only on

[15]It is also relevant that the domain of f — which is the image of $(x_1, \ldots, x_n)$ — is such that the relevant limit that defines the partial derivative $\partial_i f$ makes sense. This happens, for instance, if such domain is open. The condition that the image of $(x_1, \ldots, x_n)$ is such that the limit involved in the partial derivative $\partial_i f$ makes sense can be understood as the condition that the variables x_1, …, x_n be "sufficiently independent". For example, if $n = 2$ and x_2 is a function of x_1, then the image of (x_1, x_2) will be merely a line in the plane and then the partial derivatives of a function f defined in that line will normally not make sense.

y and x_i, but also on the other dynamical variables x_j involved; namely, the partial derivative $\frac{\partial y}{\partial x_i}$ measures the rate of change of y when x_i varies and the other variables x_j are kept fixed. If you pick another set of variables to be kept fixed, the rate of change of y will be different. For example, assume that $y = x_1 + x_2$ and $\bar{x}_2 = x_1 - x_2$. If the fixed variable is x_2, then the partial derivative $\frac{\partial y}{\partial x_1}$ is equal to 1. On the other hand, we have $y = 2x_1 - \bar{x}_2$, so that if $\bar{x}_2$ is the fixed variable, we obtain that the partial derivative $\frac{\partial y}{\partial x_1}$ is equal to 2. Thus, the standard notation $\frac{\partial y}{\partial x_i}$ is somewhat unfortunate, as it makes the varying variable x_i explicit, but it conceals the fixed variables x_j, $j \neq i$, which are also relevant for the value of the partial derivative. In practice, however, the set of fixed variables is usually clear from context.

We can now state a multivariable version of the chain rule.

Proposition 5.10 (multivariable chain rule)**.** *Let $x_1, \ldots, x_n, y_1, \ldots, y_m, z$ be real-valued dynamical variables on the same state space. Let $j = 1, \ldots, n$ be given. Assume that:*

(i) *y_i is a function of $x_1, \ldots, x_n$, for all $i = 1, \ldots, m$;*
(ii) *the partial derivatives $\frac{\partial y_i}{\partial x_j}$ exist, for all $i = 1, \ldots, m$;*
(iii) *the image of $(y_1, \ldots, y_m)$ is an open subset of $\mathbb{R}^m$;*
(iv) *$z = f(y_1, \ldots, y_m)$, for a differentiable function f defined in the image of $(y_1, \ldots, y_m)$.*

Then z is a function of $x_1, \ldots, x_n$, the partial derivative $\frac{\partial z}{\partial x_j}$ exists and it is given by:

$$\frac{\partial z}{\partial x_j} = \sum_{i=1}^{m} \frac{\partial z}{\partial y_i} \frac{\partial y_i}{\partial x_j}. \tag{22}$$

Proof. We have

$$(y_1, \ldots, y_m) = g(x_1, \ldots, x_n), \tag{23}$$

for a certain function g defined in the image of $(x_1, \ldots, x_n)$ and taking values in $\mathbb{R}^m$. Then:

$$z = (f \circ g)(x_1, \ldots, x_n).$$

Now apply the standard multivariable chain rule for the composition of f with g. □

Unfortunately, the assumptions of Proposition 5.10 will almost never be satisfied if $n < m$. For instance, if $n < m$ and the function g in (23) is of class C^1 then the image of g is never going to be an open subset of $\mathbb{R}^m$ and thus assumption (iii) is not going to be satisfied. The following slight generalization of the statement of the proposition takes care of this problem: we consider a state space $\mathbb{S}$ in which all the dynamical variables in the statement are defined and a subset $\mathbb{S}'$ of $\mathbb{S}$. In the assumptions (i) and (ii), the variables should be restricted to $\mathbb{S}'$; more explicitly, assumptions (i) and (ii) should be replaced by:

- (i') $y_i|_{\mathbb{S}'}$ is a function of $x_1|_{\mathbb{S}'}, \ldots, x_n|_{\mathbb{S}'}$, for all $i = 1, \ldots, m$;
- (ii') the partial derivative of the restriction $y_i|_{\mathbb{S}'}$ with respect to the restriction $x_j|_{\mathbb{S}'}$ exists, for all $i = 1, \ldots, m$.

The new thesis is that $z|_{\mathbb{S}'}$ is a function of $x_1|_{\mathbb{S}'}, \ldots, x_n|_{\mathbb{S}'}$ and that the partial derivative of $z|_{\mathbb{S}'}$ with respect to $x_j|_{\mathbb{S}'}$ exists. Moreover, in equality (22) the appropriate restrictions to $\mathbb{S}'$ should be considered, i.e., (22) becomes:

$$\frac{\partial z|_{\mathbb{S}'}}{\partial x_j|_{\mathbb{S}'}} = \sum_{i=1}^{m} \frac{\partial z}{\partial y_i}\Big|_{\mathbb{S}'} \frac{\partial y_i|_{\mathbb{S}'}}{\partial x_j|_{\mathbb{S}'}}.$$

This is probably looking too abstract and ugly, so let us see in a concrete example how this generalization of Proposition 5.10 works in practice. Imagine that we have a force field $\vec{F}$ over physical space, so that the force $\vec{F}$ is written as a function of the coordinates (x, y, z) of a point of physical space (with respect to some fixed coordinate system). Imagine also that we have a particle flying around, so that the coordinates (x, y, z) of the position of the particle are written as functions of time t. Now, for each t, we have a force (given by the force field $\vec{F}$) acting at the particle and we would like to talk about the time derivative of that force. Using our formalism of dynamical variables, we could for instance take the state space $\mathbb{S}$ to be $\mathbb{R}^4$ and the dynamical variables t, x, y and z to be the four projections of $\mathbb{R}^4$ onto $\mathbb{R}$. The force field $\vec{F}$ can now be thought as a dynamical variable over the state space $\mathbb{S}$ that is a function of x, y and z, so it makes sense to talk about the partial derivatives $\frac{\partial \vec{F}}{\partial x}$, $\frac{\partial \vec{F}}{\partial y}$ and $\frac{\partial \vec{F}}{\partial z}$. Note that the image of (x, y, z) is simply $\mathbb{R}^3$, which is an open subset of $\mathbb{R}^3$. Here the variables x, y and z are independent and we are not capturing the idea that x, y and z become a function of t when the trajectory of a particle is specified. Let now $\mathbb{S}'$ be the subset of $\mathbb{S} = \mathbb{R}^4$ which is what is usually called the *worldline* of the particle, i.e., it is the set of spacetime points in which the particle is present. The restriction of (x, y, z) to $\mathbb{S}'$ is now a function of the restriction of t to $\mathbb{S}'$ and this function is precisely the time parametrization of the trajectory of the particle. The restriction of $\vec{F}$ to $\mathbb{S}'$ is then a function of the restriction of t to $\mathbb{S}'$ and (with the appropriate differentiability assumptions) we can now calculate the derivative of $\vec{F}|_{\mathbb{S}'}$ with respect to $t|_{\mathbb{S}'}$ using the chain rule obtaining:

$$\frac{\mathrm{d}\vec{F}|_{\mathbb{S}'}}{\mathrm{d}t|_{\mathbb{S}'}} = \frac{\partial \vec{F}}{\partial x}\Big|_{\mathbb{S}'} \frac{\mathrm{d}x|_{\mathbb{S}'}}{\mathrm{d}t|_{\mathbb{S}'}} + \frac{\partial \vec{F}}{\partial y}\Big|_{\mathbb{S}'} \frac{\mathrm{d}y|_{\mathbb{S}'}}{\mathrm{d}t|_{\mathbb{S}'}} + \frac{\partial \vec{F}}{\partial z}\Big|_{\mathbb{S}'} \frac{\mathrm{d}z|_{\mathbb{S}'}}{\mathrm{d}t|_{\mathbb{S}'}}.$$

Allowing for some abuse of notation, we drop the annoying restrictions to $\mathbb{S}'$ and use the more familiar notation:

$$\frac{\mathrm{d}\vec{F}}{\mathrm{d}t} = \frac{\partial \vec{F}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \vec{F}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \vec{F}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}.$$

5.4. **The general definition of partial derivative.** The style of presentation of this subsection might be too abstract for the typical Calculus student. The discussion of partial derivatives involving dynamical variables presented so far is sufficient for most practical purposes. This subsection is for those that enjoy generality and elegance.

Definition 5.11. Let $\mathcal{F}$ be an arbitrary set of dynamical variables defined on the same state space $\mathbb{S}$. We say that a dynamical variable y defined on $\mathbb{S}$ is a *function* of the elements of $\mathcal{F}$ if for all $\sigma_1, \sigma_2 \in \mathbb{S}$, we have that if $x(\sigma_1) = x(\sigma_2)$ for all $x \in \mathcal{F}$, then $y(\sigma_1) = y(\sigma_2)$.

Definition 5.11 is a generalization of Definition 5.9 to the case when the set $\mathcal{F} = \{x_1, \dots, x_n\}$ is not necessarily finite.

Definition 5.12. Let $\mathcal{F}$ be an arbitrary set of dynamical variables defined on the same state space $\mathbb{S}$. For each $\sigma \in \mathbb{S}$, we set:

$$\mathbb{S}(\sigma;\mathcal{F}) = \big\{\tau \in \mathbb{S} : x(\tau) = x(\sigma), \text{ for all } x \in \mathcal{F}\big\}.$$

In words, $\mathbb{S}(\sigma;\mathcal{F})$ is the set of states in which all the variables belonging to $\mathcal{F}$ take the same values as they take at σ. When we move inside of $\mathbb{S}(\sigma;\mathcal{F})$, we are varying the states while keeping the values of the variables x belonging to $\mathcal{F}$ fixed. It is easy to see that if a dynamical variable y is a function of the elements of the set of dynamical variables $\mathcal{F} \cup \{x\}$, then for all $\sigma \in \mathbb{S}$, the restriction $y|_{\mathbb{S}(\sigma;\mathcal{F})}$ of y to $\mathbb{S}(\sigma;\mathcal{F})$ is a function of the restriction $x|_{\mathbb{S}(\sigma;\mathcal{F})}$ of x to $\mathbb{S}(\sigma;\mathcal{F})$.

We are now ready to give our general definition of partial derivative.

Definition 5.13. Let x and y be real-valued dynamical variables on a state space $\mathbb{S}$ and $\mathcal{F}$ be an arbitrary set of dynamical variables on $\mathbb{S}$. Assume that:

(i) y is a function of the elements of $\mathcal{F} \cup \{x\}$;
(ii) for every $\sigma \in \mathbb{S}$, the image I_σ of the restriction of x to $\mathbb{S}(\sigma;\mathcal{F})$ has no isolated points;
(iii) for every $\sigma \in \mathbb{S}$, the unique function $f_\sigma : I_\sigma \to \mathbb{R}$ such that

$$y|_{\mathbb{S}(\sigma;\mathcal{F})} = f_\sigma(x|_{\mathbb{S}(\sigma;\mathcal{F})})$$

is differentiable.

Under these conditions, the *partial derivative of y with respect to x keeping $\mathcal{F}$ fixed* is the real-valued dynamical variable $\left[\frac{\partial y}{\partial x}\right]_{\mathcal{F}}$ on $\mathcal{S}$ defined by

$$\left[\frac{\partial y}{\partial x}\right]_{\mathcal{F}}(\sigma) = f'_\sigma\big(x(\sigma)\big),$$

for all $\sigma \in \mathbb{S}$.

This definition generalizes (21). Namely, if $x_1, \ldots, x_n$ are distinct dynamical variables on the state space $\mathbb{S}$ and if y is a function of the dynamical variables $x_1, \ldots, x_n$, then what we call $\frac{\partial y}{\partial x_i}$ in (21) is the same as $\left[\frac{\partial y}{\partial x_i}\right]_{\mathcal{F}}$ in the terminology of Definition 5.13, with $\mathcal{F} = \big\{x_j : j = 1, \ldots, n,\ j \neq i\big\}$. The advantage of Definition 5.13 is that it does not require a choice of variable ordering for $x_1, \ldots, x_n$ and that it works with infinite sets of fixed variables.

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