CHARACTERIZATION OF TRACE CLASS OPERATORS

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Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a linear operator. For each Hilbert basis $(e_i)_{i\in I}$ of $\mathcal{H}$ we consider the sum:

$$\sum_{i\in I} \langle T(e_i), e_i\rangle. \tag{1}$$

If T is positive then it is well-known that (1) is independent of the Hilbert basis and it is called the *trace* of T. A bounded operator T is said to be *trace class* if $|T|$ (i.e., the square root of T^*T) has finite trace. It is also well-known that if T is trace class then the sum (1) is finite and independent of the Hilbert basis; in this case, the *trace* of T is also defined as the value of (1).

It is easy to find examples of bounded operators that are not trace class but for which the sum (1) is finite for some Hilbert basis of $\mathcal{H}$. For example, if $T : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ is the shift operator and $(e_i)_{i\in\mathbb{Z}}$ is the canonical Hilbert basis of $\ell_2(\mathbb{Z})$ then $\langle T(e_i), e_i\rangle = \langle e_{i+1}, e_i\rangle = 0$ for all $i \in \mathbb{Z}$ and yet T is unitary, so that $|T|$ is the identity and T is not trace class. If the Hilbert space is real it is also easy to find examples of unitary operators T which are antisymmetric, so that $\langle T(x), x\rangle = 0$ for all $x \in \mathcal{H}$ and the sum (1) is zero for every Hilbert basis. For instance, pick any real Hilbert space $\mathcal{H}$ and define $T : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ by $T(x, y) = (-y, x)$, for all $x, y \in \mathcal{H}$.

Here's a reasonable question: if $\mathcal{H}$ is a complex Hilbert space and T is a bounded operator on $\mathcal{H}$ such that the sum (1) is finite for every Hilbert basis, is it true that T is trace class? I think most people believe the answer to be "yes", though I wasn't able to find a published proof. So here we provide a simple proof of this fact.

The sum (1) is understood as the limit of the net of all sums of finite subfamilies of the family of its terms. Finiteness of such sum is equivalent to the finiteness of the sum of the absolute values of its terms. We note that if $\mathcal{H}$ is a separable infinite-dimensional Hilbert space then one can index Hilbert bases of $\mathcal{H}$ using the natural numbers and in that case one could also ask if the (not necessarily absolute) convergence of the series $\sum_{n=0}^{\infty}\langle T(e_n), e_n\rangle$ for every Hilbert basis implies that T is trace class. But this follows trivially from the result we prove below, as the reordering of a Hilbert basis is also a Hilbert basis, so that $\sum_{n=0}^{\infty}\langle T(e_n), e_n\rangle$ must converge absolutely.

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Definition 1. A linear operator T on a Hilbert space $\mathcal{H}$ is said to be a *finite trace operator* if the sum (1) is finite for every Hilbert basis of $\mathcal{H}$.

Note that T is finite trace if and only if

$$\sum_{i\in I} \big|\langle T(e_i), e_i\rangle\big| < +\infty,$$

for every Hilbert basis $(e_i)_{i\in I}$ of $\mathcal{H}$.

Theorem 2. *If $\mathcal{H}$ is a complex Hilbert space then every bounded operator T on $\mathcal{H}$ that is finite trace is also trace class.*

If T is a bounded operator on $\mathcal{H}$ we can write $T = T_1 + iT_2$, with T_1 and T_2 both self-adjoint by setting $T_1 = \frac{1}{2}(T + T^*)$ and $T_2 = \frac{1}{2i}(T - T^*)$. It then follows that $\langle T_1(x), x\rangle$ and $\langle T_2(x), x\rangle$ are respectively the real and imaginary part of the complex number $\langle T(x), x\rangle$, so that

$$\big|\langle T_1(x), x\rangle\big| \le \big|\langle T(x), x\rangle\big|, \quad \big|\langle T_2(x), x\rangle\big| \le \big|\langle T(x), x\rangle\big|,$$

for all $x \in \mathcal{H}$. Hence if T is finite trace then both T_1 and T_2 are finite trace. As linear combinations of trace class operators are trace class, it follows that it is sufficient to prove Theorem 2 in case T is self-adjoint.

We will show first that a self-adjoint finite trace operator must be compact. Given a closed subspace V of $\mathcal{H}$, we denote by $P_V : \mathcal{H} \to V$ the orthogonal projection.

Lemma 3. *A bounded operator T on a Hilbert space $\mathcal{H}$ is compact if and only if for every $\varepsilon > 0$ there exists a finite-dimensional subspace V of $\mathcal{H}$ such that $\|P_{V^\perp} \circ T|_{V^\perp}\| < \varepsilon$.*

Proof. Recall that a bounded operator on a Hilbert space is compact if and only if it belongs to the operator-norm closure of the space of finite rank operators. Thus, if T is compact we can find a finite rank operator $S : \mathcal{H} \to \mathcal{H}$ with $\|T - S\| < \varepsilon$. If V denotes the image of S we obtain:

$$\|P_{V^\perp} \circ T|_{V^\perp}\| = \|P_{V^\perp} \circ (T - S)|_{V^\perp}\| \le \|T - S\| < \varepsilon.$$

Conversely, if for every $\varepsilon > 0$ we can find a finite-dimensional subspace V of $\mathcal{H}$ with $\|P_{V^\perp} \circ T|_{V^\perp}\| < \varepsilon$ then

$$T = (P_V + P_{V^\perp}) \circ T \circ (P_V + P_{V^\perp}) = S + P_{V^\perp} \circ T \circ P_{V^\perp},$$

where $S = P_V \circ T \circ (P_V + P_{V^\perp}) + P_{V^\perp} \circ T \circ P_V$ is a finite rank operator and:

$$\|P_{V^\perp} \circ T \circ P_{V^\perp}\| = \|P_{V^\perp} \circ T|_{V^\perp}\| < \varepsilon.$$

This proves that T is compact. □

Recall that if $T : \mathcal{H} \to \mathcal{H}$ is self-adjoint then

$$\|T\| = \sup\big\{\big|\langle T(x), x\rangle\big| : \|x\| \le 1,\ x \in \mathcal{H}\big\}$$

and in particular

$$\|P_V \circ T|_V\| = \sup\big\{\big|\langle T(x), x\rangle\big| : \|x\| \le 1,\ x \in V\big\}$$

for every closed subspace V of $\mathcal{H}$.

Lemma 4. *If T is a self-adjoint finite trace operator on $\mathcal{H}$ then T is compact.*

Proof. If T is not compact, Lemma 3 yields $\varepsilon > 0$ such that

$$\|P_{V^\perp} \circ T|_{V^\perp}\| = \sup\left\{\left|\langle T(x), x\rangle\right| : \|x\| \le 1,\ x \in V^\perp\right\} > \varepsilon,$$

for every finite-dimensional subspace V of $\mathcal{H}$. Setting $V = \{0\}$ we obtain a unit vector $e_1 \in \mathcal{H}$ with $\left|\langle T(e_1), e_1\rangle\right| > \varepsilon$. Given an orthonormal sequence $(e_i)_{i=1}^n$ we let V be the span of $\left\{e_i : i = 1, \dots, n\right\}$ and we obtain a unit vector $e_{n+1} \in V^\perp$ with $\left|\langle T(e_{n+1}), e_{n+1}\rangle\right| > \varepsilon$. Thus by recursion we obtain an infinite orthonormal sequence $(e_n)_{n\ge 1}$ with $\left|\langle T(e_n), e_n\rangle\right| > \varepsilon$ for all $n \ge 1$. This contradicts the assumption that T is finite trace as every orthonormal family is contained in a Hilbert basis. □

Proof of Theorem 2. As discussed before we can assume that T is self-adjoint and thus Lemma 4 implies that T is compact. The Spectral Theorem for self-adjoint compact operators then yields a Hilbert basis $(e_i)_{i\in I}$ of $\mathcal{H}$ consisting of eigenvectors of T with corresponding eigenvalues $(\lambda_i)_{i\in I}$. Clearly e_i is an eigenvector of $|T|$ with eigenvalue $|\lambda_i|$ and hence the trace of $|T|$ is given by the sum

$$\sum_{i\in I}\langle |T|(e_i), e_i\rangle = \sum_{i\in I}|\lambda_i| = \sum_{i\in I}\left|\langle T(e_i), e_i\rangle\right|$$

which must be finite since T is finite trace. □

Remark 5. As discussed in the beginning, Theorem 2 does not hold if the Hilbert space $\mathcal{H}$ is real, since $\langle T(x), x\rangle = 0$ holds for all $x \in \mathcal{H}$ if T is antisymmetric. Nevertheless, our proof of Theorem 2 does work in case T is self-adjoint and thus we conclude that if T is a bounded finite trace operator on a real Hilbert space then $T = T_1 + T_2$, with T_1 self-adjoint trace class and T_2 antisymmetric. Namely, simply set $T_1 = \frac{1}{2}(T + T^*)$ and $T_2 = \frac{1}{2}(T - T^*)$.

Appendix A. Unbounded operators

The assumption in Theorem 2 that the operator T be bounded is not really necessary. Namely, we will prove below that a finite trace linear operator $T : \mathcal{H} \to \mathcal{H}$ must be bounded. We need a few simple lemmas.

Lemma 6. *Let X and Y be Banach spaces and let $T : X \to Y$ be a linear operator. If $\alpha \circ T$ is bounded for every bounded linear functional $\alpha \in Y^*$ then T is bounded.*

Proof. Follows easily from the Closed Graph Theorem (or from the fact that the image of the unit ball of X under T is weakly bounded and thus bounded). □

Lemma 7. *Let X and Y be Banach spaces and let $T : X \to Y$ be a linear operator. If X_0 is a closed subspace of X with finite codimension and if $T|_{X_0}$ is bounded then T is bounded.*

Proof. Write $X = X_0 \oplus X_1$ with X_1 finite-dimensional (and hence closed). If P_0 and P_1 are the projections corresponding to the direct sum decomposition $X = X_0 \oplus X_1$ then $T = T \circ P_0 + T \circ P_1$, with both $T \circ P_0$ and $T \circ P_1$ bounded. □

Lemma 8. *If $\mathcal{H}$ is a complex Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ is a bounded operator then:*

$$\sup\big\{\big|\langle T(x), x\rangle\big| : \|x\| \le 1,\ x \in \mathcal{H}\big\} \ge \frac{1}{2}\|T\|.$$

Proof. Set $T_1 = \frac{1}{2}(T + T^*)$ and $T_2 = \frac{1}{2i}(T - T^*)$, so that T_1 and T_2 are self-adjoint and $T = T_1 + iT_2$. To conclude the proof, note that

$$\sup\big\{\big|\langle T(x), x\rangle\big| : \|x\| \le 1,\ x \in \mathcal{H}\big\} \\ \ge \sup\big\{\big|\langle T_j(x), x\rangle\big| : \|x\| \le 1,\ x \in \mathcal{H}\big\} = \|T_j\|,$$

for $j = 1, 2$ and that $\|T\| \le \|T_1\| + \|T_2\|$. □

Corollary 9. *If $\mathcal{H}$ is a complex Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ is an unbounded linear operator then:*

$$\sup\big\{\big|\langle T(x), x\rangle\big| : \|x\| \le 1,\ x \in \mathcal{H}\big\} = +\infty.$$

Proof. Let $M > 0$ and pick $x \in \mathcal{H}$ with $\|x\| = 1$ and $\|T(x)\| \ge M$. Now choose $y \in \mathcal{H}$ with $\|y\| = 1$ and $\big|\langle T(x), y\rangle\big| = \|T(x)\|$. If V is the subspace spanned by x and y then $P_V \circ T|_V$ is a bounded operator such that:

$$\|P_V \circ T|_V\| \ge \big|\langle T(x), y\rangle\big| \ge M.$$

Lemma 8 now yields

$$\sup\big\{\big|\langle T(z), z\rangle\big| : \|z\| \le 1,\ z \in \mathcal{H}\big\} \\ \ge \sup\big\{\big|\langle T(z), z\rangle\big| : \|z\| \le 1,\ z \in V\big\} \ge \frac{M}{2}. \quad \square$$

Lemma 10. *Let $\mathcal{H}$ be a Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be an unbounded linear operator. If there exists a closed subspace V of $\mathcal{H}$ contained in the kernel of T such that $T[V^\perp] \subset V$ then T is not finite trace.*

Proof. Since T annihilates V we have $T = T \circ P_{V^\perp}$ and therefore the restriction $T|_{V^\perp} : V^\perp \to V$ must be unbounded. By Lemma 6 there exists $v \in V$ such that the linear functional $\alpha : V^\perp \ni x \mapsto \langle T(x), v\rangle$ is unbounded. We can assume $\|v\| = 1$. We will construct by recursion a linearly independent sequence $(x_n)_{n\ge1}$ in $V^\perp$ such that the vectors $e_n = x_n + v$, $n \ge 1$, are pairwise orthogonal and

$$\big|\langle T(e_n), e_n\rangle\big| = |\alpha(x_n)| \ge \|e_n\|^2 = \|x_n\|^2 + 1,$$

for all $n \ge 1$. One then obtains that T is not finite trace by considering a Hilbert basis of $\mathcal{H}$ containing the orthonormal sequence $\frac{e_n}{\|e_n\|}$, $n \ge 1$. To construct the sequence $(x_n)_{n\ge1}$, start with $x_1 \in V^\perp$ such that $\|x_1\| = 1$ and

$|\alpha(x_1)| \ge 2$. Given linearly independent vectors $x_1, \ldots, x_n \in V^\perp$, note that the linear map

$$y \longmapsto \big(\langle y, x_1\rangle, \ldots, \langle y, x_n\rangle\big)$$

defined on the linear span W of $x_1, \ldots, x_n$ is an isomorphism and obtain $y \in W$ with $\langle y, x_i\rangle = -1$, for $i = 1, \ldots, n$. We will now define $x_{n+1} = y + z$ with $z \in W^\perp \cap V^\perp$ an appropriately chosen unit vector. The fact that $y \in W$ and $z \in W^\perp$ is nonzero ensures that $x_{n+1} \notin W$ and thus that $(x_i)_{i=1}^{n+1}$ is linearly independent. Moreover, for $i = 1, \ldots, n$, we have:

$$\langle e_{n+1}, e_i\rangle = \langle x_{n+1}, x_i\rangle + 1 = \langle y, x_i\rangle + 1 = 0.$$

Finally, we need that $|\alpha(x_{n+1})| \ge \|x_{n+1}\|^2 + 1 = \|y\|^2 + 2$ and this will hold if z is chosen with:

$$|\alpha(z)| \ge |\alpha(y)| + \|y\|^2 + 2.$$

To see that this is possible simply note that Lemma 7 implies that the restriction of α to $W^\perp \cap V^\perp$ must be unbounded. □

Theorem 11. *If $\mathcal{H}$ is a complex Hilbert space then every finite trace linear operator T on $\mathcal{H}$ is bounded.*

Proof. Assume that T is unbounded. If for every finite-dimensional subspace V of $\mathcal{H}$ we have that $P_{V^\perp} \circ T|_{V^\perp}$ is unbounded then arguing as in the proof of Lemma 4 and using Corollary 9 we obtain an orthonormal sequence $(e_n)_{n\ge1}$ in $\mathcal{H}$ with $\big|\langle T(e_n), e_n\rangle\big| \ge 1$, for all $n \ge 1$, which contradicts the fact that T is finite trace. Now assume that there exists a finite dimensional subspace V of $\mathcal{H}$ such that $P_{V^\perp} \circ T|_{V^\perp}$ is bounded. Since every Hilbert basis of $V^\perp$ is contained in a Hilbert basis of $\mathcal{H}$, we easily see that $P_{V^\perp} \circ T|_{V^\perp} : V^\perp \to V^\perp$ is finite trace and thus, by Theorem 2, it is trace class. This implies that also $P_{V^\perp} \circ T \circ P_{V^\perp} : \mathcal{H} \to \mathcal{H}$ is trace class and in particular it is finite trace. We have

$$T = T \circ P_V + T \circ P_{V^\perp} = T \circ P_V + P_{V^\perp} \circ T \circ P_{V^\perp} + P_V \circ T \circ P_{V^\perp}.$$

Since $T \circ P_V$ is a finite rank bounded operator it is trace class and thus $T' = P_V \circ T \circ P_{V^\perp}$ must be finite trace. But T' must be unbounded and this contradicts Lemma 10. □

Though Theorem 2 really requires a complex Hilbert space, it turns out that Theorem 11 does hold in the real case. Namely, there are two points in the proof of Theorem 11 that use the fact that the Hilbert space is complex. First when we use Corollary 9 and second when we use Theorem 2 two conclude that $P_{V^\perp} \circ T|_{V^\perp}$ is trace class. If $\mathcal{H}$ is real, we cannot apply Theorem 2 but as mentioned in Remark 5 we obtain that $P_{V^\perp} \circ T|_{V^\perp}$ is the sum of trace class self-adjoint operator with an antisymmetric operator. It then follows that also $P_{V^\perp} \circ T \circ P_{V^\perp}$ is the sum of a trace class self-adjoint operator with an antisymmetric operator and thus we can conclude that $P_{V^\perp} \circ T \circ P_{V^\perp}$ is finite trace, as in the proof of Theorem 11. As for Corollary 9, though Lemma 8 does not hold in the real case it is possible to prove Corollary 9 directly in the real case.

Lemma 12. *Let T be a linear operator on a real Hilbert space $\mathcal{H}$. If*

$$\sup\big\{\big|\langle T(x),x\rangle\big| : \|x\|\le 1,\ x\in\mathcal{H}\big\}$$

is finite then T is bounded.

Proof. Let $c=\sup\big\{\big|\langle T(x),x\rangle\big| : \|x\|\le 1,\ x\in\mathcal{H}\big\}$. Consider the bilinear form B on $\mathcal{H}$ defined by $B(x,y)=\langle T(x),y\rangle$, for all $x,y\in\mathcal{H}$. Set

$$B_1(x,y)=\frac{1}{2}\big(B(x,y)+B(y,x)\big),\quad B_2(x,y)=\frac{1}{2}\big(B(x,y)-B(y,x)\big),$$

for all $x,y\in\mathcal{H}$, so that $B=B_1+B_2$, B_1 is a symmetric bilinear form and B_2 is an antisymmetric bilinear form. If V is a finite-dimensional subspace of $\mathcal{H}$ then there exists a self-adjoint operator $S:V\to V$ with

$$B_1(x,y)=\langle S(x),y\rangle,$$

for all $x,y\in V$. Thus

$$\begin{aligned}\|S\|&=\sup\big\{|B_1(x,x)| : \|x\|\le 1,\ x\in V\big\}\\&=\sup\big\{|B(x,x)| : \|x\|\le 1,\ x\in V\big\}\le c\end{aligned}$$

and therefore $|B_1(x,y)|\le c$, for all $x,y\in V$ with $\|x\|\le 1$, $\|y\|\le 1$. Since V is arbitrary, we conclude that $|B_1(x,y)|\le c$, for all $x,y\in\mathcal{H}$ with $\|x\|\le 1$, $\|y\|\le 1$. This implies that there exists a bounded self-adjoint operator $T_1:\mathcal{H}\to\mathcal{H}$ with $B_1(x,y)=\langle T_1(x),y\rangle$, for all $x,y\in\mathcal{H}$. Setting $T_2=T-T_1$, we obtain that T_2 must be antisymmetric and therefore bounded by the Closed Graph Theorem. Hence T is bounded. □

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