TENSOR PRODUCTS OF $\mathbf{L^2}$ SPACES

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Given (real or complex) inner product spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we can endow their (algebraic) tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ with a unique inner product satisyfing

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle,$$

for all $x_1, y_1 \in \mathcal{H}_1$ and all $x_2, y_2 \in \mathcal{H}_2$. When $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, the product $\mathcal{H}_1 \otimes \mathcal{H}_2$ is not in general complete and we define the *Hilbert space tensor product* $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ to be the completion of the algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. Now let $(X, \mathcal{A}, \mu)$ be a measure space and consider the Hilbert space $L^2(X, \mathcal{A}, \mu)$ of (real or complex valued) measurable quadratic integrable functions f on X, modulo the equivalence relation of μ-almost everywhere equality, endowed with the inner product defined by

$$\langle f, g \rangle = \int_X f \bar{g} \, \mathrm{d}\mu,$$

for all $f, g \in L^2(X, \mathcal{A}, \mu)$. It is well-known that if $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces, then the Hilbert space tensor product

$$L^2(X, \mathcal{A}, \mu) \hat{\otimes} L^2(Y, \mathcal{B}, \nu) \tag{1}$$

is naturally isometrically identified with the space $L^2(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$, where $\mathcal{A} \otimes \mathcal{B}$ denotes the product σ-algebra on $X \times Y$ and $\mu \otimes \nu$ the product measure on $\mathcal{A} \otimes \mathcal{B}$. The identification of the Hilbert space tensor product (1) with the space $L^2(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$ carries, for all $f \in L^2(X, \mathcal{A}, \mu)$ and $g \in L^2(Y, \mathcal{B}, \nu)$, the tensor product $f \otimes g$ to the mapping (also denoted by $f \otimes g$) defined by:

$$(f \otimes g)(x, y) = f(x)g(y), \quad \text{for all } x \in X,\ y \in Y. \tag{2}$$

The goal of this note is to investigate what happens with the tensor product (1) when the measure spaces involved are not σ-finite. In this case, the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$ is still well-defined (as the smallest σ-algebra which makes the projections measurable or, alternatively, the σ-algebra generated by the rectangles $A \times B$ with $A \in \mathcal{A}$, $B \in \mathcal{B}$) but the meaning of the product measure $\mu \otimes \nu$ is not so clear. In general, there always exists a measure $\rho : \mathcal{A} \otimes \mathcal{B} \to [0, +\infty]$ satisfying the condition

$$\rho(A \times B) = \mu(A)\nu(B), \quad \text{for all } A \in \mathcal{A},\ B \in \mathcal{B}, \tag{3}$$

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but such measure is not always unique and Fubini's Theorem might not hold. As usual, we adopt here the convention $0 \cdot (+\infty) = (+\infty) \cdot 0 = 0$ for the product in the extended real line. A measure ρ satisfying (3) can, for instance, be defined by setting, for each $C \in \mathcal{A} \otimes \mathcal{B}$, the value of $\rho(C)$ to be equal to the infimum of the sums $\sum_{n=1}^{\infty} \mu(A_n)\nu(B_n)$, where $(A_n)_{n\geq 1}$ is a sequence in $\mathcal{A}$, $(B_n)_{n\geq 1}$ is a sequence in $\mathcal{B}$ and C is contained in the union $\bigcup_{n=1}^{\infty}(A_n \times B_n)$. In what follows we assume that ρ satisfies the weaker condition

$$(4) \quad \rho(A \times B) = \mu(A)\nu(B), \quad \text{for all } A \in \mathcal{A},\ B \in \mathcal{B} \text{ such that } \mu(A) < +\infty \text{ and } \nu(B) < +\infty,$$

which is sufficient for the results we are going to prove. Obviously, condition (4) implies that the equality $\rho(A \times B) = \mu(A)\nu(B)$ also holds when A is a σ-finite subset of X and B is a σ-finite subset of Y; by a *σ-finite subset* we mean a countable union of measurable sets with finite measure. A measure space is called *semi-finite* if every measurable subset with infinite measure contains a measurable subset with finite positive measure. It is easy to show that this implies actually that every measurable subset with infinite measure contains measurable subsets with arbitrarily large finite measure[1]. Thus, if $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are semi-finite, we have that condition (4) implies that the equality $\rho(A \times B) = \mu(A)\nu(B)$ holds if both $\mu(A)$ and $\nu(B)$ are positive. However, under (4), it might happen for instance that $\mu(A) = 0$, $B \in \mathcal{B}$ is not σ-finite and $\rho(A \times B)$ is not zero, so that (3) does not hold (see Example 4).

Let us recall the following simple fact that will be used freely throughout the remainder of the text: if $\mathcal{C} \subset \wp(X)$ is a collection of subsets of X, $\mathcal{A}$ is the σ-algebra of subsets of X generated by $\mathcal{C}$ and X' is a subset of X then, denoting by $\mathcal{C}|_{X'}$ the collection $\{C \cap X' : C \in \mathcal{C}\}$, we have that the σ-algebra of subsets of X' generated by $\mathcal{C}|_{X'}$ is equal to $\mathcal{A}|_{X'}$. It follows that if $\mathcal{A}$ is a σ-algebra of subsets of X, $\mathcal{B}$ is a σ-algebra of subsets of Y, X' is a subset of X and Y' is a subset of Y, then $(\mathcal{A} \otimes \mathcal{B})|_{X' \times Y'} = (\mathcal{A}|_{X'}) \otimes (\mathcal{B}|_{Y'})$. In particular, if $X' \in \mathcal{A}$ and $Y' \in \mathcal{B}$, then $(\mathcal{A} \otimes \mathcal{B}) \cap \wp(X' \times Y') = \big(\mathcal{A} \cap \wp(X')\big) \otimes \big(\mathcal{B} \cap \wp(Y')\big)$. Keeping this observation in mind, it follows that if X' is a σ-finite subset of X, Y' is a σ-finite subset of Y, and the measure ρ on $\mathcal{A} \otimes \mathcal{B}$ satisfies (4), then the restriction of ρ to $(\mathcal{A} \otimes \mathcal{B}) \cap \wp(X' \times Y')$ is equal to the unique product measure of the restriction of μ to $\mathcal{A} \cap \wp(X')$ with the restriction of ν to $\mathcal{B} \cap \wp(Y')$. For the product of σ-finite measure spaces Fubini's Theorem holds and hence, for ρ satisfying (4), Fubini's Theorem also holds for measurable maps f on $X \times Y$ whose support (i.e., the set of points in which f is not zero) is contained in a product $X' \times Y'$ of σ-finite subsets.

[1] If A is a measurable subset with infinite measure such that the supremum k of the measures of measurable subsets of A with finite measure is finite, then using countable unions we can obtain a measurable subset B of A whose measure is exactly k. But then $A \setminus B$ would be a measurable subset with infinite measure, all of whose measurable subsets of finite measure have null measure.

Noting that every element of an L^2 space has σ-finite support, we obtain that for $f \in L^2(X, \mathcal{A}, \mu)$ and $g \in L^2(Y, \mathcal{B}, \nu)$, the mapping $f \otimes g$ defined by (2) has support contained in the product of σ-finite subsets. Hence Fubini's Theorem can be applied to conclude that $f \otimes g$ is square integrable and that the equality

$$\begin{aligned}\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle &= \int_{X\times Y} (f_1 \otimes g_1)\overline{(f_2 \otimes g_2)} \,\mathrm{d}\rho \\ &= \Big(\int_X f_1 \bar{f}_2 \,\mathrm{d}\mu \Big)\Big(\int_Y g_1 \bar{g}_2 \,\mathrm{d}\nu \Big) = \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle\end{aligned}$$

holds for all $f_1, f_2 \in L^2(X, \mathcal{A}, \mu)$ and all $g_1, g_2 \in L^2(Y, \mathcal{B}, \nu)$.

Consider now the bilinear mapping

$$L^2(X, \mathcal{A}, \mu) \times L^2(Y, \mathcal{B}, \nu) \ni (f, g) \longmapsto f \otimes g \in \mathcal{H} \subset L^2(X \times Y, \mathcal{A} \otimes \mathcal{B}, \rho), \tag{5}$$

where $\mathcal{H}$ denotes the linear span of the image of such mapping. The standard argument showing that (5) satisfies the universal property of an algebraic tensor product works fine for non σ-finite spaces: namely, given a linearly independent finite sequence $(f_i)_{i=1}^n$ in $L^2(X, \mathcal{A}, \mu)$ and a linearly independent finite sequence $(g_j)_{j=1}^m$ in $L^2(Y, \mathcal{B}, \nu)$, we have to check that the finite family

$$(f_i \otimes g_j)_{1 \le i \le n,\ 1 \le j \le m}$$

is also linearly independent. Assuming that a linear combination

$$h = \sum_{i=1}^n \sum_{j=1}^m c_{ij} f_i \otimes g_j$$

is zero ρ-almost everywhere, we have to show that $c_{ij} = 0$ for all i and j. Let $C \in \mathcal{A} \otimes B$ be the set of points in which h is not zero, so that $\rho(C) = 0$. Since C is contained in a product of σ-finite subsets, we can apply Fubini's Theorem to the characteristic function of C and conclude that the fiber $C_x = \big\{ y \in Y : (x, y) \in C \big\}$ has zero ν-measure for μ-almost all $x \in X$. It then follows from the linear independence of $(g_j)_{j=1}^m$ that $\sum_{i=1}^n c_{ij} f_i(x) = 0$ for μ-almost all $x \in X$ and all j and hence the linear independence of $(f_i)_{i=1}^n$ implies that $c_{ij} = 0$ for all i and j.

We have thus proven the following result.

Proposition 1. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces and ρ be a measure on the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$ satisfying* (4). *If, for f in $L^2(X, \mathcal{A}, \mu)$ and g in $L^2(Y, \mathcal{B}, \nu)$, we identify the element $f \otimes g$ of the Hilbert space tensor product* (1) *with the element $f \otimes g$ of $L^2(X \times Y, \mathcal{A} \otimes \mathcal{B}, \rho)$ defined by* (2)*, then we obtain a linear isometric embedding*

$$L^2(X, \mathcal{A}, \mu) \hat{\otimes} L^2(Y, \mathcal{B}, \nu) \hookrightarrow L^2(X \times Y, \mathcal{A} \otimes \mathcal{B}, \rho) \tag{6}$$

of the Hilbert space tensor product $L^2(X, \mathcal{A}, \mu) \hat{\otimes} L^2(Y, \mathcal{B}, \nu)$ into the Hilbert space $L^2(X \times Y, \mathcal{A} \otimes \mathcal{B}, \rho)$. □

Let us now investigate conditions under which the embedding (6) is surjective. Given a measure ρ, we say that a measurable set C_1 is *ρ-almost contained* in a measurable set C_2 if $\rho(C_1 \setminus C_2) = 0$.

Theorem 2. *Under the assumptions of Proposition 1, the isometric embedding* (6) *is surjective if and only if every $C \in \mathcal{A} \otimes \mathcal{B}$ with $\rho(C) < +\infty$ is ρ-almost contained in a product of σ-finite subsets.*

Proof. Assume that every set with finite ρ-measure is ρ-almost contained in a product of σ-finite subsets. We check that the range of (6) contains all characteristic functions of sets with finite ρ-measure. Since such characteristic functions span a dense subspace, the conclusion will follow. Let $C \in \mathcal{A} \otimes \mathcal{B}$ be given with $\rho(C) < +\infty$. Removing a subset of C with ρ-measure zero does not change the equivalence class of the characteristic function of C, so we can assume that C is contained in a product $X' \times Y'$, where $X' \in \mathcal{A}$ and $Y' \in \mathcal{B}$ are σ-finite. Standard approximation results now imply that for every $\varepsilon > 0$ there exists a set R which equals a disjoint union $\bigcup_{i=1}^{n}(A_i \times B_i)$, where each $A_i \in \mathcal{A}$ has finite μ-measure, each $B_i \in \mathcal{B}$ has finite ν-measure and $\rho(C \triangle R) < \varepsilon$, where $C \triangle R$ denotes the symmetric difference of C and R. It follows that the characteristic function of R is in the range of (6) and the distance between the characteristic function of R and the characteristic function of C is less than $\sqrt{\varepsilon}$. To prove the converse, note that the set of functions in $L^2(X \times Y, \mathcal{A} \otimes \mathcal{B}, \rho)$ whose support is ρ-almost contained in a product of σ-finite subsets is a closed subspace of $L^2(X \times Y, \mathcal{A} \otimes \mathcal{B}, \rho)$ containing all products $f \otimes g$, with $f \in L^2(X, \mathcal{A}, \mu)$ and $g \in L^2(Y, \mathcal{B}, \nu)$. Assuming that (6) is surjective we then obtain that every function in $L^2(X \times Y, \mathcal{A} \otimes \mathcal{B}, \rho)$ has support ρ-almost contained in a product of σ-finite subsets; in particular, this holds for characteristic functions of sets of finite ρ-measure. □

Corollary 3. *Under the assumptions of Proposition 1, we have that the embedding* (6) *is surjective if and only if the following condition holds: for every set C in $\mathcal{A} \otimes \mathcal{B}$ with $\rho(C) < +\infty$, if $\rho\big(C \cap (A \times B)\big) = 0$ for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$ with $\mu(A) < +\infty$ and $\nu(B) < +\infty$, then $\rho(C) = 0$.*

Proof. Consider the σ-ideal $\mathcal{J}$ in $\mathcal{A} \otimes \mathcal{B}$ consisting of sets that are contained in a product of σ-finite subsets. It is easy to see that the condition in the statement of the corollary is equivalent to the following: for every $C \in \mathcal{A} \otimes \mathcal{B}$ with $\rho(C) < +\infty$, if $\rho(C \cap I) = 0$ for all $I \in \mathcal{J}$, then $\rho(C) = 0$. Obviously, if every $C \in \mathcal{A} \otimes \mathcal{B}$ with $\rho(C) < +\infty$ is ρ-almost contained in an element of $\mathcal{J}$, then this condition holds. To prove the converse, let $C \in \mathcal{A} \otimes \mathcal{B}$ with $\rho(C) < +\infty$ be fixed and let us check that C is ρ-almost contained in an element of $\mathcal{J}$. Setting

$$k = \sup\big\{\rho(C \cap I) : I \in \mathcal{J}\big\} \le \rho(C) < +\infty,$$

the fact that $\mathcal{J}$ is closed under countable unions readily implies that the supremum is actually a maximum, i.e., that there exists $I \in \mathcal{J}$ such that

$\rho(C \cap I)$ equals k. Now it follows easily that $C' = C \setminus I$ is such that $\rho(C' \cap J) = 0$ for all $J \in \mathcal{J}$ and hence that $\rho(C') = 0$. □

Now we look for examples in which the embedding (6) is not surjective. It is easy to find such examples if only condition (4) is required to hold.

Example 4. Let $X = Y = [0,1]$, $\mathcal{A} = \mathcal{B}$ be the Borel σ-algebra of $[0,1]$ and μ, ν denote the measures defined by

$$\mu(A) = \big|A \setminus \{0\}\big| \quad \text{and} \quad \nu(B) = |B|,$$

for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, where $|C| \in [0,+\infty]$ denotes the number of elements of a set C. The product σ-algebra $\mathcal{A} \otimes \mathcal{B}$ is just the Borel σ-algebra of $[0,1]^2$. We define a measure ρ on $\mathcal{A} \otimes \mathcal{B}$ by setting

$$\rho(C) = \big|C \setminus \big(\{0\} \times [0,1]\big)\big| + \mathfrak{m}(C_0),$$

for all $C \in \mathcal{A} \otimes \mathcal{B}$, where $C_0 = \big\{y \in [0,1] : (0,y) \in C\big\}$ and $\mathfrak{m}$ denotes Lebesgue measure. We have that condition (4) holds, but (3) fails with $A = \{0\}$ and $B = [0,1]$. Using Corollary 3 we see that the embedding (6) is not surjective: namely, setting $C = \{0\} \times [0,1]$ we have $\rho(C) = 1$ and $\rho\big(C \cap (A \times B)\big) = 0$, for all finite subsets A and B of $[0,1]$.

An example where the stronger condition (3) is satisfied is a little harder to find. Given a set X, a σ-algebra $\mathcal{A}$ of subsets of X and a subset H of X, we denote by $\mathcal{A}[H]$ the σ-algebra of subsets of X generated by $\mathcal{A} \cup \{H\}$. It is easy to check that:

$$\mathcal{A}[H] = \big\{(A_1 \cap H) \cup (A_2 \setminus H) : A_1, A_2 \in \mathcal{A}\big\}.$$

Example 5. Let $X = Y = [0,1]$ and $\mathcal{A}$ be the Borel σ-algebra of $[0,1]$. Let H be a subset of $[0,1]$ whose inner Lebesgue measure is zero and whose outer Lebesgue measure is 1; equivalently, every Borel subset contained in H and every Borel subset contained in $[0,1] \setminus H$ have null Lebesgue measure (for instance, using transfinite recursion one can construct a *Bernstein subset* H of $[0,1]$, i.e., a set H such that both H and $[0,1] \setminus H$ intersect every uncountable Borel subset of $[0,1]$). Set $\mathcal{B} = \mathcal{A}[H]$ and consider the measures μ and ν defined by

$$\mu(A) = |A \setminus H| \quad \text{and} \quad \nu(B) = |B|,$$

for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$. If $\mathfrak{m}$ denotes Lebesgue measure, then we obtain a measure θ on $\mathfrak{B}$ by setting

$$\theta\big((A_1 \cap H) \cup (A_2 \setminus H)\big) = \mathfrak{m}(A_1),$$

for all $A_1, A_2 \in \mathcal{A}$. Namely, if $(A_1 \cap H) \cup (A_2 \setminus H) = (A_1' \cap H) \cup (A_2' \setminus H)$ and $A_1, A_1', A_2, A_2' \in \mathcal{A}$, then the symmetric difference $A_1 \mathbin{\triangle} A_1'$ is disjoint from H and hence $\mathfrak{m}(A_1 \mathbin{\triangle} A_1') = 0$. Note that θ is an extension of Lebesgue measure on $\mathcal{A}$. Denote by $d : [0,1] \to [0,1]^2$ the diagonal inclusion defined by $d(x) = (x,x)$, for all $x \in [0,1]$. We define a measure ρ on $\mathcal{A} \otimes \mathcal{B}$ by setting

$$\rho(C) = \big|C \setminus \big(H \times [0,1]\big)\big| + \theta\big(d^{-1}[C]\big),$$

for all $C \in \mathcal{A} \otimes \mathcal{B}$. Note that, since $d : \big([0,1], \mathcal{B}\big) \to \big([0,1]^2, \mathcal{A} \otimes \mathcal{B}\big)$ is measurable, we have that $d^{-1}[C]$ is in $\mathcal{B}$, for all $C \in \mathcal{A}\otimes\mathcal{B}$, so that $\theta\big(d^{-1}[C]\big)$ is well-defined. We claim that condition (3) holds. To prove the claim, let $A \in \mathcal{A}$, $B \in \mathcal{B}$ and set $C = A \times B$. Note that $\big|C \setminus \big(H \times [0,1]\big)\big| = \mu(A)\nu(B)$ and that if $\mu(A)\nu(B) < +\infty$, then either B is finite or A is contained in H, in which case $\theta\big(d^{-1}[C]\big) = \theta(A \cap B) = 0$. Now we use Corollary 3 to show that the embedding (6) is not surjective. Set $C = d[H] = d\big[[0,1]\big] \cap \big([0,1] \times H\big)$. Since the image of d is a Borel subset of $[0,1]^2$ and thus an element of $\mathcal{A} \otimes \mathcal{A}$, it follows that C is in $\mathcal{A} \otimes \mathcal{B}$. Moreover, $\rho(C) = \theta(H) = 1$. To see that $\rho\big(C \cap (A \times B)\big) = 0$, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $\mu(A) < +\infty$ and $\nu(B) < +\infty$, simply note that $\rho\big(C \cap (A \times B)\big) = \theta(H \cap A \cap B) = 0$, because B is finite.

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