NOTES ON THE RIESZ REPRESENTATION THEOREM

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Resumo.

1. Introduction

Let $(X, \mathcal{A}, \mu)$ be a measure space, i.e., X is a set, $\mathcal{A}$ is a σ-algebra of subsets of X and $\mu : \mathcal{A} \to [0, +\infty]$ is a countably additive set function with $\mu(\emptyset) = 0$. Given a measurable function $f : X \to I\!R$ and $p \in [1, +\infty[$ we set:

$$\|f\|_p = \Big(\int_X |f|^p \, \mathrm{d}\mu \Big)^{\frac{1}{p}} \in [0, +\infty],$$

and:

$$\|f\|_\infty = \inf \big\{ c \in [0, +\infty] : |f| \le c, \ \mu\text{-almost everywhere} \big\} \in [0, +\infty].$$

We denote by $\mathcal{M}(X, \mathcal{A})$ the real vector space of measurable real valued maps on X and by $\overline{\mathcal{M}}(X, \mathcal{A}, \mu)$ the quotient of $\mathcal{M}(X, \mathcal{A})$ by the subspace of μ-almost everywhere vanishing maps. As usual, for $p \in [1, +\infty]$, $L^p(X, \mathcal{A}, \mu)$ will denote the subspace of $\overline{\mathcal{M}}(X, \mathcal{A}, \mu)$ consisting of classes of maps f with $\|f\|_p < +\infty$; under a (very standard) abuse of terminology, we will say that f is in $L^p(X, \mathcal{A}, \mu)$ meaning that the class of maps almost everywhere equal to f is in $L^p(X, \mathcal{A}, \mu)$. The vector space $L^p(X, \mathcal{A}, \mu)$ becomes a Banach space when endowed with the norm $\| \cdot \|_p$. Given $p, q \in \,]1, +\infty[$ with $\frac{1}{p} + \frac{1}{q} = 1$ then the well-known *Riesz Representation Theorem* states that the (q, p)-*Riesz map*:

$$L^q(X, \mathcal{A}, \mu) \ni g \longmapsto \alpha_g \in L^p(X, \mathcal{A}, \mu)^*, \tag{1.1}$$

where

$$\alpha_g(f) = \int_X f g \, \mathrm{d}\mu, \quad f \in L^p(X, \mathcal{A}, \mu) \tag{1.2}$$

is a linear isometry, where $L^p(X, \mathcal{A}, \mu)^*$ denotes the topological dual space of $L^p(X, \mathcal{A}, \mu)$. If $q = 1$ and $p = +\infty$ then the map (1.1) is an isometric immersion but it is not surjective even for fairly simple measure spaces $(X, \mathcal{A}, \mu)$. If, on the other hand, $q = +\infty$ and $p = 1$ then the map (1.1) is well-known to be a linear isometry *in the case where* $(X, \mathcal{A}, \mu)$ *is* σ*-finite*, i.e., if X can be covered by a countable number of sets of finite measure.

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In this article we are concerned about studying general conditions on the measure space $(X,\mathcal{A},\mu)$ under which the *Riesz map*:

$$L^\infty(X,\mathcal{A},\mu) \ni g \longmapsto \alpha_g \in L^1(X,\mathcal{A},\mu)^* \tag{1.3}$$

with α_g defined as in (1.2), is a linear isometry. Observe that if X is an arbitrary set, $\mathcal{A} = \wp(X)$ is the σ-algebra of all subsets of X and μ is the *counting measure* $\mu(A)$ = number of elements of A, then the Riesz map (1.3) is an isometry; however, if X is uncountable, $(X,\mathcal{A},\mu)$ is not σ-finite. Thus, σ-finiteness of the space is not a necessary condition for the Riesz map to be an isometry. In [1] a sufficient condition for the Riesz map to be an isometry which is weaker than σ-finiteness (and is satisfied by counting measures) is presented. The condition is that the space X should admit a decomposition; in [1] a *decomposition* for X is a partition $X = \bigcup_{i\in I} X_i$ of X into pairwise disjoint measurable sets X_i with $\mu(X_i) < +\infty$ for all $i \in I$ satisfying the following property: if A is a subset of X with $A \cap X_i \in \mathcal{A}$ and $\mu(A\cap X_i) = 0$ for all $i \in I$ then A is measurable and $\mu(A) = 0$. For instance, if $\mathcal{A} = \wp(X)$ and μ is the counting measure then $X = \bigcup_{x\in X}\{x\}$ is a decomposition for X.

2. Infinite Blocks

Definition 2.1. Let $(X,\mathcal{A},\mu)$ be a measure space. A measurable subset $B \in \mathcal{A}$ is called an *infinite block* for μ if $\mu(B) = +\infty$ and $\mu(A) \in \{0,+\infty\}$ for every $A \in \mathcal{A}$ contained in B. If there are no infinite blocks for μ then we call μ a *block-free* measure.

A subset of X will be called *σ-finite* for μ if it is equal to the union of a countable family of sets of finite measure μ. The intersection of a σ-finite set with an infinite block has measure zero. Note that if f is in $L^p(X,\mathcal{A},\mu)$ with $p < +\infty$ then the set $f^{-1}\big(I\!R\setminus\{0\}\big)$ is σ-finite because:

$$f^{-1}\big(I\!R\setminus\{0\}\big) = \bigcup_{n=1}^{\infty}\big\{x\in X : |f(x)| \ge \tfrac{1}{n}\big\};$$

thus $f|_B = 0$ almost everywhere, if B is an infinite block.

Lemma 2.2. *The Riesz map* (1.3) *is injective if and only if μ is block-free; in this case,* (1.3) *is an isometric immersion.*

Demonstração. If $B \subset X$ is an infinite block then $\alpha_g = 0$, where $g = \chi_B \ne 0$ is the characteristic function of B; thus (1.3) is not injective. Now assume that μ is block-free. Given $g \in L^\infty(X,\mathcal{A},\mu)$ then clearly $\|\alpha_g\| \le \|g\|_\infty$. Moreover, given $c > 0$ with $c < \|g\|_\infty$, we can find a measurable set A contained in the set $\big\{x\in X : |g(x)| \ge c\big\}$ with $0 < \mu(A) < +\infty$. Thus $f = \big(\mathrm{sign}(g)\big)\chi_A$ is a nonzero element of $L^1(X,\mathcal{A},\mu)$ with:

$$\alpha_g(f) \ge c\|f\|_1,$$

which proves that $\|\alpha_g\| \ge \|g\|_\infty$. Hence (1.3) is a (injective) isometric immersion. $\square$

In some spaces one can "factor out" the infinite blocks, i.e., write X as a disjoint union $X = X_0 \cup X_\infty$, where X_∞ is an infinite block and X_0 contains no infinite blocks. This is not always possible, as the following example shows.

Example 2.3. Let X be the rectangle $[0,1]^2$. We define $A \subset X$ to be measurable if the line $A^y = \big\{x \in [0,1] : (x,y) \in A\big\}$ is Lebesgue measurable for every $y \in [0,1]$. The measure $\mu(A)$ is defined as follows. If A^y is non empty for only a countable number of values of $y \in [0,1]$ then we set:

$$\mu(A) = \sum_{y\in[0,1]} \mathfrak{m}(A^y),$$

where $\mathfrak{m}$ denotes the Lebesgue measure on $I\!R$; otherwise, we set $\mu(A) = +\infty$. Observe that each column $\{x\} \times [0,1]$, $x \in [0,1]$, is an infinite block. We claim that X cannot be written as a disjoint union $X = X_0 \cup X_\infty$, where X_∞ is an infinite block and X_0 does not contain infinite blocks. Namely, if X_∞ is an infinite block then $\mathfrak{m}(X_\infty^y) = 0$ for all $y \in [0,1]$ and thus there exists a point $a(y) \in [0,1] \setminus X_\infty^y$; therefore $B = \big\{\big(a(y), y\big) : y \in [0,1]\big\}$ is an infinite block disjoint from X_∞.

Example 2.3 shows that "factoring out" is not the right way to get rid of the infinite blocks. The right way is to "fix" the measure μ by defining a "block-free" version of μ as follows; set:

$$\mu_{\rm bf}(A) = \sup\big\{\mu(E) : E \subset A,\ E \in \mathcal{A} \text{ and } \mu(E) < +\infty\big\}, \tag{2.1}$$

for all $A \in \mathcal{A}$. Obviously $\mu_{\rm bf}(A) \le \mu(A)$, for all $A \in \mathcal{A}$ and $\mu_{\rm bf}(A) = 0$ if and only if either $\mu(A) = 0$ or A is an infinite block. Moreover, we have the following:

Lemma 2.4. *The set function $\mu_{\rm bf} : \mathcal{A} \to [0,+\infty]$ defined in* (2.1) *has the following properties:*

(a) *given $A \in \mathcal{A}$, there exists a σ-finite subset E for μ contained in A with $\mu_{\rm bf}(A) = \mu(E)$;*
(b) *if $A \in \mathcal{A}$ does not contain infinite blocks for μ (in particular, if A is σ-finite for μ) then $\mu(A) = \mu_{\rm bf}(A)$;*
(c) *$\mu_{\rm bf}$ is a block-free measure;*
(d) *if $A \in \mathcal{A}$ is σ-finite for $\mu_{\rm bf}$ then A can be written as a disjoint union $A = A_0 \cup A_\infty$, $A_0, A_\infty \in \mathcal{A}$, with A_0 σ-finite for μ and $\mu_{\rm bf}(A_\infty) = 0$ (so that either $\mu(A_\infty) = 0$ or A_∞ is an infinite block for μ).*

Demonstração.

- *Proof of* (a).

  By the definition of $\mu_{\rm bf}$ there exists a sequence $(E_n)_{n\ge1}$ of subsets of A of finite measure with $\lim_{n\to\infty} \mu(E_n) = \mu_{\rm bf}(A)$. Set $E = \bigcup_{n=1}^\infty E_n$; then E is a σ-finite subset of A with $\mu(E) = \lim_{n\to\infty} \mu(E_1 \cup \ldots \cup E_n)$. The conclusion follows by observing that $\mu(E_n) \le \mu(E_1 \cup \ldots \cup E_n) \le \mu_{\rm bf}(A)$, for all $n \ge 1$.

- *Proof of* (b).
  If $\mu_{\rm bf}(A) = +\infty$ then also $\mu(A) = +\infty$. Assume that $\mu_{\rm bf}(A) < +\infty$. Let E be a measurable subset of A with $\mu(E) = \mu_{\rm bf}(A)$. If E' is a measurable subset of $A \setminus E$ with finite measure then:

$$\mu_{\rm bf}(A) = \mu(E) \leq \mu(E) + \mu(E') = \mu(E \cup E') \leq \mu_{\rm bf}(A),$$

  so that $\mu(E') = 0$. Since $A \setminus E$ is not an infinite block, it must be $\mu(A \setminus E) = 0$. Hence $\mu(A) = \mu(E) = \mu_{\rm bf}(A)$.

- *Proof of* (c).
  We prove that $\mu_{\rm bf}$ is countably additive. Let $(A_n)_{n\geq 1}$ be a sequence of pairwise disjoint measurable sets and let E be a σ-finite subset of $A = \bigcup_{n=1}^{\infty} A_n$ with $\mu(E) = \mu_{\rm bf}(A)$; for each $n \geq 1$, let E_n be a σ-finite subset of A_n with $\mu(E_n) = \mu_{\rm bf}(A_n)$. Then, keeping in mind the simple fact that the set function $\mu_{\rm bf}$ is monotonically increasing we have:

$$\sum_{n=1}^{\infty} \mu_{\rm bf}(A_n) = \sum_{n=1}^{\infty} \mu(E_n) = \mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \mu_{\rm bf}\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leq \mu_{\rm bf}(A) = \mu(E)$$
$$= \sum_{n=1}^{\infty} \mu(E \cap A_n) = \sum_{n=1}^{\infty} \mu_{\rm bf}(E \cap A_n) \leq \sum_{n=1}^{\infty} \mu_{\rm bf}(A_n).$$

  In order to prove that $\mu_{\rm bf}$ is block-free, observe that if $\mu_{\rm bf}(A) = +\infty$ for some $A \in \mathcal{A}$ then there exists a measurable subset E of A with $0 < \mu(E) < +\infty$; hence $\mu_{\rm bf}(E) = \mu(E) < +\infty$ and A is not an infinite block for $\mu_{\rm bf}$.

- *Proof of* (d).
  Write $A = \bigcup_{n=1}^{\infty} A_n$, with $\mu_{\rm bf}(A_n) < +\infty$ for all $n \geq 1$. For each n there exists a measurable subset E_n of A_n with $\mu(E_n) = \mu_{\rm bf}(A_n)$. Thus $\mu_{\rm bf}(E_n) = \mu_{\rm bf}(A_n)$ and $\mu_{\rm bf}(A_n \setminus E_n) = 0$. Set $A_0 = \bigcup_{n=1}^{\infty} E_n$ and $A_\infty = A \setminus A_0$. Then A_0 is σ-finite for μ and $A_\infty \subset \bigcup_{n=1}^{\infty}(A_n \setminus E_n)$, so that $\mu_{\rm bf}(A_\infty) = 0$. □

Definition 2.5. The measure $\mu_{\rm bf} : \mathcal{A} \to [0, +\infty]$ defined in (2.1) is called the *block-free version* of the measure $\mu : \mathcal{A} \to [0, +\infty]$.

Since the measure $\mu_{\rm bf}$ is absolutely continuous with respect to μ, the identity map of $\mathcal{M}(X, \mathcal{A})$ induces a linear map $\mathcal{M}(X, \mathcal{A}, \mu) \to \mathcal{M}(X, \mathcal{A}, \mu_{\rm bf})$; moreover, since $\mu_{\rm bf} \leq \mu$ such map takes $L^p(X, \mathcal{A}, \mu)$ to $L^p(X, \mathcal{A}, \mu_{\rm bf})$ and thus we obtain a canonical map:

$$L^p(X, \mathcal{A}, \mu) \longrightarrow L^p(X, \mathcal{A}, \mu_{\rm bf}), \tag{2.2}$$

for all $p \in [1, +\infty]$.

Lemma 2.6. *The map* (2.2) *is an isometry for* $p < +\infty$.

Demonstração. If f is in $L^p(X,\mathcal{A},\mu)$ then the set $f^{-1}\big(\mathbb{R}\setminus\{0\}\big)$ is σ-finite for μ and thus, by item (b) of Lemma 2.4, μ and μ_{bf} coincide on all measurable subsets of $f^{-1}\big(\mathbb{R}\setminus\{0\}\big)$. This proves that:

$$\int_X |f|^p \,\mathrm{d}\mu = \int_X |f|^p \,\mathrm{d}\mu_{\mathrm{bf}},$$

i.e., (2.2) is an isometric immersion. In order to prove that (2.2) is surjective we fix $f \in L^p(X,\mathcal{A},\mu_{\mathrm{bf}})$ and we exhibit an element of $L^p(X,\mathcal{A},\mu)$ that is equal μ_{bf}-almost everywhere to f. Since $f^{-1}\big(\mathbb{R}\setminus\{0\}\big)$ is σ-finite for μ_{bf}, item (d) of Lemma 2.4 allows us to write $f^{-1}\big(\mathbb{R}\setminus\{0\}\big) = A_0 \cup A_\infty$ with $A_0, A_\infty \in \mathcal{A}$, $A_0 \cap A_\infty = \emptyset$, A_0 σ-finite for μ and $\mu_{\mathrm{bf}}(A_\infty) = 0$. Thus $f\chi_{A_0} = f$ μ_{bf}-almost everywhere and, since μ and μ_{bf} coincide on all measurable subsets of A_0, we have $\int_X |f\chi_{A_0}|^p \,\mathrm{d}\mu = \int_X |f|^p \,\mathrm{d}\mu_{\mathrm{bf}} < +\infty$ and hence $f\chi_{A_0} \in L^p(X,\mathcal{A},\mu)$. □

Recall that if E and E' are Banach spaces then a bounded linear map $q : E \to E'$ is called a *quotient map* if it is surjective and:

$$\|q(x)\| = \inf\big\{\|y\| : y \in E,\ q(y) = q(x)\big\},$$

for all $x \in E$. Alternatively, q is a quotient map if it induces a linear isometry from the quotient Banach space $E/\mathrm{Ker}(q)$ onto E'.

Lemma 2.7. *The map* (2.2) *is a quotient map for* $p = +\infty$ *and the nonzero elements of its kernel are the maps* $f \in L^\infty(X,\mathcal{A},\mu)$ *such that* $f^{-1}\big(\mathbb{R}\setminus\{0\}\big)$ *is an infinite block.*

Demonstração. Let $f \in L^\infty(X,\mathcal{A},\mu_{\mathrm{bf}})$ be fixed and let c be the norm of f in $L^\infty(X,\mathcal{A},\mu_{\mathrm{bf}})$. We define $f_0 : X \to \mathbb{R}$ by setting $f_0(x) = f(x)$ if $|f(x)| \le c$ and $f_0(x) = 0$ otherwise. Then $f_0 \in L^\infty(X,\mathcal{A},\mu)$ and $f = f_0$ μ_{bf}-almost everywhere. Thus f_0 is mapped by (2.2) into f, which proves that (2.2) is surjective. We will now prove that the norm of f_0 in $L^\infty(X,\mathcal{A},\mu)$ is equal to c and that for every $f_1 \in L^\infty(X,\mathcal{A},\mu)$ with $f = f_1$ μ_{bf}-almost everywhere the norm of f_1 in $L^\infty(X,\mathcal{A},\mu)$ is greater than or equal to c. This will imply that (2.2) is a quotient map. For any $\varepsilon > 0$ we have:

$$\mu_{\mathrm{bf}}\Big(\big\{x \in X : |f_1(x)| > c-\varepsilon\big\}\Big) = \mu_{\mathrm{bf}}\Big(\big\{x \in X : |f(x)| > c-\varepsilon\big\}\Big) > 0$$

and thus:

$$\mu\Big(\big\{x \in X : |f_1(x)| > c-\varepsilon\big\}\Big) \ge \mu_{\mathrm{bf}}\Big(\big\{x \in X : |f_1(x)| > c-\varepsilon\big\}\Big) > 0.$$

This proves that the norm of f_1 (and the norm of f_0) in $L^\infty(X,\mathcal{A},\mu)$ is greater than or equal to c. Finally, since $|f_0(x)| \le c$ for all $x \in X$, the norm of f_0 in $L^\infty(X,\mathcal{A},\mu)$ is indeed equal to c. □

For $p, q \in [1, +\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have a commutative diagram:

$$\begin{tikzcd} L^p(X, \mathcal{A}, \mu)^* & L^p(X, \mathcal{A}, \mu_{\mathrm{bf}})^* \arrow[l] \\ L^q(X, \mathcal{A}, \mu) \arrow[u] \arrow[r] & L^q(X, \mathcal{A}, \mu_{\mathrm{bf}}) \arrow[u] \end{tikzcd} \tag{2.3}$$

where the vertical arrows are the (q, p)-Riesz maps (1.1) for $(X, \mathcal{A}, \mu)$ and $(X, \mathcal{A}, \mu_{\mathrm{bf}})$, the top horizontal arrow is the transpose of the map (2.2) and the bottom horizontal arrow is the version of the map (2.2) for L^q. If $p, q \in\,]1, +\infty[$ then (2.3) is just a commutative diagram of isometries. The most interesting case for us is $p = 1$ and $q = +\infty$; in this case we get a commutative diagram:

$$\begin{tikzcd} L^1(X, \mathcal{A}, \mu)^* & \\ L^\infty(X, \mathcal{A}, \mu) \arrow[u] \arrow[r] & L^\infty(X, \mathcal{A}, \mu_{\mathrm{bf}}) \arrow[ul] \end{tikzcd} \tag{2.4}$$

In diagram (2.4) the slanted arrow differs from the Riesz map of the space $(X, \mathcal{A}, \mu_{\mathrm{bf}})$ by an isometry; since μ_{bf} is block-free, the slanted arrow is an isometric immersion. Therefore diagram (2.4) shows us how to factor the Riesz map of the space $(X, \mathcal{A}, \mu)$ into a isometric immersion (the slanted arrow) and a quotient map (the horizontal arrow).

3. Full Measures

Recall that a measure space $(X, \mathcal{A}, \mu)$ is called *complete* if every subset of a measurable set of null measure is also measurable. If $(X, \mathcal{A}, \mu)$ is an arbitrary measure space then:

$$\overline{\mathcal{A}} = \big\{ A \cup N : A \in \mathcal{A} \text{ and } N \text{ contained in some } M \in \mathcal{A} \text{ with } \mu(M) = 0 \big\}$$

is a σ-algebra containing $\mathcal{A}$ and μ extends in a unique way to a measure $\bar{\mu} : \overline{\mathcal{A}} \to [0, +\infty]$ by setting $\bar{\mu}(A \cup N) = \mu(A)$ when $A \in \mathcal{A}$ and N is contained in some $M \in \mathcal{A}$ with $\mu(M) = 0$. The measure space $(X, \overline{\mathcal{A}}, \bar{\mu})$ is complete and it is called the *completion* of $(X, \mathcal{A}, \mu)$. We have the following:

Lemma 3.1. *If $(X, \overline{\mathcal{A}}, \bar{\mu})$ is the completion of $(X, \mathcal{A}, \mu)$ then for every $p \in [1, +\infty]$ the inclusion map of $\mathcal{M}(X, \mathcal{A})$ in $\mathcal{M}(X, \overline{\mathcal{A}})$ induces a linear isometry:*

$$L^p(X, \mathcal{A}, \mu) \longrightarrow L^p(X, \overline{\mathcal{A}}, \bar{\mu}). \tag{3.1}$$

Demonstração. Clearly (3.1) is an isometric immersion. Moreover, a standard argument using limits of simple functions shows that if $f : X \to \mathbb{R}$ is measurable with respect to $\overline{\mathcal{A}}$ then there exists a map $f_1 : X \to \mathbb{R}$ that is measurable with respect to $\mathcal{A}$ with $f = f_1$ $\bar{\mu}$-almost everywhere. Hence (3.1) is surjective. $\square$

For $p, q \in [1, +\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have a commutative diagram:

$$\begin{tikzcd} L^p(X, \mathcal{A}, \mu)^* & L^p(X, \overline{\mathcal{A}}, \bar{\mu})^* \arrow[l, "\cong"'] \\ L^q(X, \mathcal{A}, \mu) \arrow[u] \arrow[r, "\cong"] & L^q(X, \overline{\mathcal{A}}, \bar{\mu}) \arrow[u] \end{tikzcd} \tag{3.2}$$

where the vertical arrows are the (q, p)-Riesz maps (1.1) for $(X, \mathcal{A}, \mu)$ and $(X, \overline{\mathcal{A}}, \bar{\mu})$, the top horizontal arrow is the transpose of the map (3.1) and the bottom horizontal arrow is the version of the map (3.1) for L^q.

In this section we study another type of "completion" for measure spaces which is related to the Riesz Representation Theorem. We start with an example where the Riesz map (1.1) is not surjective.

Example 3.2. Let X be an uncountable set and let $\mathcal{A}$ be the σ-algebra consisting of all subsets of X that are either countable or have countable complement. Let $\mu : \mathcal{A} \to [0, +\infty]$ be the counting measure. Thus $L^1(X, \mathcal{A}, \mu)$ is the space of maps $f : X \to I\!R$ with $\sum_{x \in X} |f(x)| < +\infty$; note that this condition implies that $f^{-1}\big(I\!R \setminus \{0\}\big)$ is countable. Let S be a subset of X such that neither $X \setminus S$ nor S is countable. Then $\alpha(f) = \sum_{x \in S} f(x)$ is a bounded linear functional on $L^1(X, \mathcal{A}, \mu)$ with $\|\alpha\| = 1$. We claim that α is not on the image of the Riesz map (1.3). Namely, if $g \in L^\infty(X, \mathcal{A}, \mu)$ and $\alpha_g = \alpha$ then for every $x \in X$ we have $\chi_{\{x\}} \in L^1(X, \mathcal{A}, \mu)$ and:

$$g(x) = \alpha_g(\chi_{\{x\}}) = \alpha(\chi_{\{x\}}) = \chi_S(x),$$

so that $g = \chi_S$. But χ_S is not measurable.

Observe that the lack of surjectivity of the Riesz map in Example 3.2 is caused by the bad choice of σ-algebra for the domain of the measure. Indeed, note that the map $g = \chi_S$ has the property that $\alpha(f) = \int_X fg \, \mathrm{d}\mu$ for all $f \in L^1(X, \mathcal{A}, \mu)$, but the map g is not a valid representation for the functional α because it is not measurable. Observe however that the counting measure μ can be naturally extended to the σ-algebra $\wp(X)$ of all subsets of X. Such extension of μ does not change the space L^1 but it enlarges the space L^∞ in such a way that the Riesz map (1.3) is an isometry.

Definition 3.3. A measure $\mu : \mathcal{A} \to [0, +\infty]$ is called *full* if the following property holds; given $A \subset X$ such that $A \cap E \in \mathcal{A}$ for all $E \in \mathcal{A}$ with $\mu(E) < +\infty$ then $A \in \mathcal{A}$.

Set:

$$\mathcal{A}_{\mathrm{e}} = \big\{A \subset X : A \cap E \in \mathcal{A}, \text{ for all } E \in \mathcal{A} \text{ with } \mu(E) < +\infty\big\};$$

obviously $\mathcal{A} \subset \mathcal{A}_{\mathrm{e}}$. Now define $\mu_{\mathrm{e}} : \mathcal{A}_{\mathrm{e}} \to [0, +\infty]$ by setting:

$$\mu_{\mathrm{e}}(A) = \begin{cases} \mu(A), & \text{for } A \in \mathcal{A}, \\ +\infty, & \text{for } A \in \mathcal{A}_{\mathrm{e}} \text{ not in } \mathcal{A}. \end{cases} \tag{3.3}$$

We have the following:

Lemma 3.4. *The map $\mu_{\mathrm{e}} : \mathcal{A}_{\mathrm{e}} \to [0, +\infty]$ defined in* (3.3) *is a full measure.*

Demonstração. In order to prove that μ_{e} is a measure it suffices to show that if $(A_n)_{n\geq 1}$ is a sequence of pairwise disjoint sets in $\mathcal{A}_{\mathrm{e}}$ with $A_n \notin \mathcal{A}$ for some $n \geq 1$ then $\mu_{\mathrm{e}}\big(\bigcup_{n=1}^{\infty} A_n\big) = +\infty$. Assuming that $\mu_{\mathrm{e}}\big(\bigcup_{n=1}^{\infty} A_n\big) < +\infty$ we would have $E = \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ and $\mu(E) < +\infty$, which would imply $A_n \cap E = A_n \in \mathcal{A}$, contradicting our assumption. We have proven that μ_{e} is a measure. Now, if $A \subset X$ is such that $A \cap E \in \mathcal{A}_{\mathrm{e}}$ for all $E \in \mathcal{A}_{\mathrm{e}}$ with $\mu_{\mathrm{e}}(E) < +\infty$ then $A \cap E \in \mathcal{A}_{\mathrm{e}}$ for all $E \in \mathcal{A}$ with $\mu(E) < +\infty$; but $A \cap E \in \mathcal{A}_{\mathrm{e}}$ implies $A \cap E = (A \cap E) \cap E \in \mathcal{A}$. Hence $A \in \mathcal{A}_{\mathrm{e}}$ and μ_{e} is full. □

Definition 3.5. The measure $\mu_{\mathrm{e}} : \mathcal{A}_{\mathrm{e}} \to [0, +\infty]$ defined in (3.3) is called the *canonical full extension* of the measure $\mu : \mathcal{A} \to [0, +\infty]$.

Clearly if μ is full then $\mathcal{A}_{\mathrm{e}} = \mathcal{A}$ and $\mu_{\mathrm{e}} = \mu$.

Lemma 3.6. *A map $f : X \to I\!R$ is measurable with respect to $\mathcal{A}_{\mathrm{e}}$ if and only if $f\chi_E$ is measurable with respect to $\mathcal{A}$, for every $E \in \mathcal{A}$ with $\mu(E) < +\infty$.*

Demonstração. Simply observe that $f\chi_E$ is measurable with respect to $\mathcal{A}$ if and only if $f^{-1}(B) \cap E \in \mathcal{A}$ for every Borel subset B of $I\!R$. □

Corollary 3.7. *If $f : X \to I\!R$ is measurable with respect to $\mathcal{A}_{\mathrm{e}}$ and $E \subset X$ is σ-finite for μ then $f\chi_E$ is measurable with respect to $\mathcal{A}$.*

Demonstração. Let $(E_n)_{n\geq 1}$ be a sequence of sets of finite measure with $E_n \nearrow E$. Then $\lim_{n\to\infty} f\chi_{E_n} = f\chi_E$ and $f\chi_{E_n}$ is measurable with respect to $\mathcal{A}$ for all $n \geq 1$. □

Proposition 3.8. *For any $g \in L^\infty(X, \mathcal{A}_{\mathrm{e}}, \mu_{\mathrm{e}})$ we have $fg \in L^1(X, \mathcal{A}, \mu)$ for all $f \in L^1(X, \mathcal{A}, \mu)$ and formula* (1.2) *defines a bounded linear functional α_g on $L^1(X, \mathcal{A}, \mu)$. Conversely, if $g : X \to I\!R$ is a map such that fg is in $L^1(X, \mathcal{A}, \mu)$ for all $f \in L^1(X, \mathcal{A}, \mu)$ and such that* (1.2) *defines a bounded linear functional α_g on $L^1(X, \mathcal{A}, \mu)$ then g is in $L^\infty\big(X, \mathcal{A}_{\mathrm{e}}, (\mu_{\mathrm{e}})_{\mathrm{bf}}\big)$ and there exists $g_1 \in L^\infty(X, \mathcal{A}_{\mathrm{e}}, \mu_{\mathrm{e}})$ with $\alpha_g = \alpha_{g_1}$.*

Demonstração. Let $g \in L^\infty(X, \mathcal{A}_{\mathrm{e}}, \mu_{\mathrm{e}})$ be fixed. Given $f \in L^1(X, \mathcal{A}, \mu)$ then the set $E = f^{-1}\big(I\!R \setminus \{0\}\big)$ is σ-finite for μ and thus $fg = fg\chi_E$ is measurable with respect to $\mathcal{A}$, by Corollary 3.7. Moreover, there exists $c \geq 0$ with $|g| \leq c$ μ_{e}-almost everywhere; but "μ_{e}-almost everywhere" is the same as "μ-almost everywhere", so that $|fg| \leq c|f|$ μ-almost everywhere and thus fg is in $L^1(X, \mathcal{A}, \mu)$. Conversely, assume that $g : X \to I\!R$ is a map such that fg is in $L^1(X, \mathcal{A}, \mu)$ for all $f \in L^1(X, \mathcal{A}, \mu)$ and such that (1.2) defines a bounded linear functional α_g on $L^1(X, \mathcal{A}, \mu)$. Given $E \in \mathcal{A}$ with $\mu(E) < +\infty$ then χ_E is in $L^1(X, \mathcal{A}, \mu)$ and thus $g\chi_E$ is also in $L^1(X, \mathcal{A}, \mu)$; it follows from Lemma 3.6 that g is measurable with respect to $\mathcal{A}_{\mathrm{e}}$. Now set $c = \|\alpha_g\|$ and let us prove that $|g| \leq c$ $(\mu_{\mathrm{e}})_{\mathrm{bf}}$-almost everywhere, so that g is in $L^\infty\big(X, \mathcal{A}_{\mathrm{e}}, (\mu_{\mathrm{e}})_{\mathrm{bf}}\big)$. It suffices to show that if $E \in \mathcal{A}_{\mathrm{e}}$ is a subset of:

$$B = \big\{x \in X : |g(x)| > c\big\}$$

with $\mu_{\mathrm{e}}(E) < +\infty$ then $\mu_{\mathrm{e}}(E) = 0$. Let such a set E be fixed. Then $E \in \mathcal{A}$ and $\mu(E) < +\infty$, so that $\chi_E \in L^1(X, \mathcal{A}, \mu)$. The map $\mathrm{sign}(g)$ is in $L^\infty(X, \mathcal{A}_{\mathrm{e}}, \mu_{\mathrm{e}})$ and thus $\big(\mathrm{sign}(g)\big)\chi_E$ is in $L^1(X, \mathcal{A}, \mu)$. We have:

$$c\,\mu(E) \le \int_X |g|\chi_E \,\mathrm{d}\mu = \alpha_g\big[\big(\mathrm{sign}(g)\big)\chi_E\big] \le c\big\|\big(\mathrm{sign}(g)\big)\chi_E\big\|_1 = c\mu(E);$$

thus $\int_X \big(|g| - c\big)\chi_E \,\mathrm{d}\mu = 0$ and $\mu_{\mathrm{e}}(E) = \mu(E) = 0$.

In order to complete the proof, observe that by Lemma 2.7 the canonical map (analogous to (2.2), with $\mathcal{A}$ and μ replaced with $\mathcal{A}_{\mathrm{e}}$ and μ_{e}):

$$L^\infty(X, \mathcal{A}_{\mathrm{e}}, \mu_{\mathrm{e}}) \longrightarrow L^\infty\big(X, \mathcal{A}_{\mathrm{e}}, (\mu_{\mathrm{e}})_{\mathrm{bf}}\big) \tag{3.4}$$

is surjective and therefore there exists $g_1 \in L^\infty(X, \mathcal{A}_{\mathrm{e}}, \mu_{\mathrm{e}})$ that is mapped by (3.4) to g. The commutativity of diagram (2.4) (with $\mathcal{A}$ and μ replaced with $\mathcal{A}_{\mathrm{e}}$ and μ_{e}) implies that α_g is the same as α_{g_1}, when considered as linear functionals in $L^1(X, \mathcal{A}_{\mathrm{e}}, \mu_{\mathrm{e}})$; but since every $f \in L^1(X, \mathcal{A}, \mu)$ can be regarded as an element of $L^1(X, \mathcal{A}_{\mathrm{e}}, \mu_{\mathrm{e}})$ (see (3.5) below), it follows that α_g and α_{g_1} are also equal as linear functionals in $L^1(X, \mathcal{A}, \mu)$. $\square$

Since the measure μ_{e} extends μ, there is for every $p \in [1, +\infty]$ a canonical map:

$$L^p(X, \mathcal{A}, \mu) \longrightarrow L^p(X, \mathcal{A}_{\mathrm{e}}, \mu_{\mathrm{e}}) \tag{3.5}$$

induced by the inclusion of $\mathcal{M}(X, \mathcal{A})$ into $\mathcal{M}(X, \mathcal{A}_{\mathrm{e}})$.

Lemma 3.9. *The canonical map* (3.5) *is an isometry for $p < +\infty$ and an isometric immersion for $p = +\infty$.*

Demonstração. Since μ_{e} extends μ it is clear that if $f : X \to I\!R$ is measurable with respect to $\mathcal{A}$ then the p-norm of f computed using μ is the same as the one computed using μ_{e}; thus (3.5) is an isometric immersion for all p. For $p < +\infty$ we claim that (3.5) is surjective. Namely, if f is in $L^p(X, \mathcal{A}_{\mathrm{e}}, \mu_{\mathrm{e}})$ then the set $E = f^{-1}\big(I\!R \setminus \{0\}\big)$ is σ-finite for μ and hence, by Corollary 3.7, $f = f\chi_E$ is measurable with respect to μ. Thus f is in $L^p(X, \mathcal{A}, \mu)$. $\square$

For $p, q \in [1, +\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have a commutative diagram:

$$\begin{tikzcd}
L^p(X, \mathcal{A}, \mu)^* & L^p(X, \mathcal{A}_{\mathrm{e}}, \mu_{\mathrm{e}})^* \arrow[l] \\
L^q(X, \mathcal{A}, \mu) \arrow[u] \arrow[r] & L^q(X, \mathcal{A}_{\mathrm{e}}, \mu_{\mathrm{e}}) \arrow[u]
\end{tikzcd} \tag{3.6}$$

where the vertical arrows are the (q, p)-Riesz maps (1.1) for $(X, \mathcal{A}, \mu)$ and $(X, \mathcal{A}_{\mathrm{e}}, \mu_{\mathrm{e}})$, the top horizontal arrow is the transpose of the map (3.5) and the bottom horizontal arrow is the version of the map (3.5) for L^q.

Lemma 3.10. *Given a measure $\mu : \mathcal{A} \to [0, +\infty]$, then:*

(a) *if μ is complete then μ_{e} is also complete;*
(b) *if μ is full then μ_{bf} is also full;*

(c) *if μ is full and complete then μ_{bf} is also (full and) complete.*

Demonstração.

- *Proof of* (a).
  Given $A \in \mathcal{A}_{\mathrm{e}}$ with $\mu_{\mathrm{e}}(A) = 0$ then $A \in \mathcal{A}$ and $\mu(A) = 0$; thus, since μ is complete, every subset of A is in $\mathcal{A}$ and hence also in $\mathcal{A}_{\mathrm{e}}$.

- *Proof of* (b).
  Assume that $A \cap E$ is in $\mathcal{A}$ for every $E \in \mathcal{A}$ with $\mu_{\mathrm{bf}}(E) < +\infty$. Notice that $\mu(E) < +\infty$ implies $\mu_{\mathrm{bf}}(E) = \mu(E) < +\infty$, by item (b) of Lemma 2.4; thus $A \cap E$ is in $\mathcal{A}$ for every $E \in \mathcal{A}$ with $\mu(E) < +\infty$ and therefore, since μ is full, A is in $\mathcal{A}$.

- *Proof of* (c).
  Let $B \in \mathcal{A}$ be fixed with $\mu_{\mathrm{bf}}(B) = 0$. We claim that every subset A of B is in $\mathcal{A}$. If $\mu(B) = 0$ this follows from the completeness of μ. Otherwise, B is an infinite block for μ. Thus, for every $E \in \mathcal{A}$ with $\mu(E) < +\infty$, we have $\mu(B \cap E) = 0$; since $A \cap E$ is contained in $B \cap E$, it follows that $A \cap E$ is in $\mathcal{A}$, by the completeness of μ. Since μ is full, we get that A is also in $\mathcal{A}$. □

Definition 3.11. A measure $\mu : \mathcal{A} \to [0, +\infty]$ is called *perfect* if it is complete, full and block-free.

Given a measure $\mu : \mathcal{A} \to [0, +\infty]$, we denote by $\mu_{\mathfrak{p}} : \mathcal{A}_{\mathfrak{p}} \to [0, +\infty]$ the block-free version of the canonical full extension of the completion of μ; in symbols, $\mu_{\mathfrak{p}} = \big((\bar{\mu})_{\mathrm{e}}\big)_{\mathrm{bf}}$ and $\mathcal{A}_{\mathfrak{p}} = \big(\overline{\mathcal{A}}\big)_{\mathrm{e}}$. It follows from Lemma 3.10 that $\mu_{\mathfrak{p}}$ is a perfect measure.

Definition 3.12. The measure $\mu_{\mathfrak{p}} : \mathcal{A}_{\mathfrak{p}} \to [0, +\infty]$ defined above is called the *perfect version* of μ.

Remark 3.13. Given a measure space $(X, \mathcal{A}, \mu)$ and a measurable subset $Y \in \mathcal{A}$, then the σ-algebra $\mathcal{A}$ and the measure μ can be restricted to Y (see Remark 4.2 below), so that Y becomes itself a measure space. It is easy to check that the operation of taking the completion, the operation of taking the block-free version and the operation of taking the canonical full extension all commute with the operation of restricting to Y. It follows that also the operation of taking the perfect version commutes with the operation of restricting to Y. In particular, when $\mu(Y) < +\infty$, then the restriction to Y of the perfect version $\mu_{\mathfrak{p}}$ of μ is simply the completion of the restriction of μ to Y (since for finite measures, taking the perfect version is the same as taking the completion).

Obviously the σ-algebra $\mathcal{A}_{\mathfrak{p}}$ contains $\mathcal{A}$ and the inclusion map of $\mathcal{M}(X, \mathcal{A})$ in $\mathcal{M}(X, \mathcal{A}_{\mathfrak{p}})$ induces, for every $p \in [1, +\infty]$, a linear map:

$$L^p(X, \mathcal{A}, \mu) \longrightarrow L^p(X, \mathcal{A}_{\mathfrak{p}}, \mu_{\mathfrak{p}}) \tag{3.7}$$

which is just the composite of the maps (3.1), (3.5) and (2.2). Thus, we have the following:

Lemma 3.14. *The map* (3.7) *is an isometry for* $p < +\infty$*.*

Demonstração. It follows from Lemmas 3.1, 3.9 and 2.6. □

Combining the commutative diagrams (3.2), (3.6) and (2.3) we obtain a new commutative diagram:

$$\begin{tikzcd} L^p(X,\mathcal{A},\mu)^* & L^p(X,\mathcal{A}_{\mathfrak{p}},\mu_{\mathfrak{p}})^* \arrow[l] \\ L^q(X,\mathcal{A},\mu) \arrow[u] \arrow[r] & L^q(X,\mathcal{A}_{\mathfrak{p}},\mu_{\mathfrak{p}}) \arrow[u] \end{tikzcd} \tag{3.8}$$

where the vertical arrows are the (q,p)-Riesz maps (1.1) for $(X,\mathcal{A},\mu)$ and $(X,\mathcal{A}_{\mathfrak{p}},\mu_{\mathfrak{p}})$, the top horizontal arrow is the transpose of the map (3.7) and the bottom horizontal arrow is the version of the map (3.7) for L^q.

For perfect measures the Riesz map (1.3) is always an isometric immersion (recall Lemma 2.2) and it has the "best chances" of being surjective. In Section 4 we will present a perfect measure space for which the Riesz map is not surjective. When we switch from a measure μ to its perfect version $\mu_{\mathfrak{p}}$ we do not change (up to a natural isometric identification) the L^p spaces for $p < +\infty$ and we make the L^∞ space more suitable for the bijectivity of the Riesz map; we get rid of the kernel of the Riesz map by taking a quotient of L^∞ (recall Lemma 2.7) and we extend L^∞ to the largest possible space of maps g that correspond to functionals α_g on L^1 (recall Proposition 3.8).

4. A Strong Counter-Example to the Bijectivity of the Riesz Map

Given sets C_1, C_2 and a subset A of $X = C_1 \times C_2$ then for each $y \in C_2$ we denote by A^y the *line* $\big\{x \in C_1 : (x,y) \in A\big\}$ and for each $x \in C_1$ we denote by A_x the *column* $\big\{y \in C_2 : (x,y) \in A\big\}$. Assume that both C_1 and C_2 are uncountable. Let $\mathcal{A}$ denote the σ-algebra consisting of those subsets A of X such that:

- either A^y or $C_1 \setminus A^y$ is countable, for all $y \in C_2$;
- either A_x or $C_2 \setminus A_x$ is countable, for all $x \in C_1$.

Given $x \in C_1$, $y \in C_2$ we define $\mu_x : \mathcal{A} \to \{0,1\}$, $\mu^y : \mathcal{A} \to \{0,1\}$ by setting:

$$\mu_x(A) = \begin{cases} 0, & \text{if } A_x \text{ is countable,} \\ 1, & \text{if } C_2 \setminus A_x \text{ is countable,} \end{cases}$$

$$\mu^y(A) = \begin{cases} 0, & \text{if } A^y \text{ is countable,} \\ 1, & \text{if } C_1 \setminus A^y \text{ is countable,} \end{cases}$$

for all $A \in \mathcal{A}$. Finally, we consider the measure $\mu : \mathcal{A} \to [0, +\infty]$ defined by:

$$\mu(A) = \sum_{x \in C_1} \mu_x(A) + \sum_{y \in C_2} \mu^y(A),$$

for all $A \in \mathcal{A}$.

Lemma 4.1. *The measure μ defined above is perfect.*

Demonstração.

- *μ is complete.*
  Given $A \in \mathcal{A}$ with $\mu(A) = 0$ then A_x and A^y are countable for all $x \in C_1$, $y \in C_2$. Thus, if B is a subset of A, then B_x and B^y are also countable for all $x \in C_1$, $y \in C_2$. It follows that B is in $\mathcal{A}$.

- *μ is full.*
  Let $A \subset X$ be such that $A \cap E \in \mathcal{A}$ for all $E \in \mathcal{A}$ with $\mu(E) < +\infty$. Note that for all $y \in C_2$ we have $E = C_1 \times \{y\} \in \mathcal{A}$ and $\mu(E) = 1$. Thus $A \cap E$ is in $\mathcal{A}$ and $(A \cap E)^y = A^y$ is either countable or has countable complement in C_1. Similarly, by setting $E = \{x\} \times C_2$ we can show that A_x is either countable or has countable complement in C_2, for all $x \in C_1$. Hence $A \in \mathcal{A}$.

- *μ is block-free.*
  If $A \in \mathcal{A}$ is such that $\mu(A) = +\infty$ then either there exists $x \in C_1$ with $\mu_x(A) = 1$ or there exists $y \in C_2$ with $\mu^y(A) = 1$. If $\mu_x(A) = 1$ then $\{x\} \times A_x$ is a subset of A with $\mu\big(\{x\} \times A_x\big) = 1$; similarly, if $\mu^y(A) = 1$ then $A^y \times \{y\}$ is a subset of A with $\mu\big(A^y \times \{y\}\big) = 1$. Hence A is not an infinite block. □

Remark 4.2. In what follows we use the following simple fact. If $(X', \mathcal{A}', \mu')$ is an arbitrary measure space and A is a measurable subset of X' then we can regard A itself as a measure space endowed with the σ-algebra consisting of elements of $\mathcal{A}'$ contained in A and the measure obtained by restricting μ'. For any $p \in [1, +\infty]$ the space $L^p(A)$ corresponding to the measure space A can be (isometrically) identified with the subspace of $L^p(X', \mathcal{A}', \mu')$ consisting of maps that vanish outside A.

Lemma 4.3. *If the Riesz map* (1.3) *of the space $(X, \mathcal{A}, \mu)$ defined above is an isomorphism then there exists a subset R of $X = C_1 \times C_2$ such that R^y is countable for all $y \in C_2$ and $C_2 \setminus R_x$ is countable for all $x \in C_1$.*

Demonstração. Consider the measure $\nu : \mathcal{A} \to [0, +\infty]$ defined by:

$$\nu(A) = \sum_{x \in C_1} \mu_x(A),$$

for all $A \in \mathcal{A}$. Since $\nu(A) \le \mu(A)$, for all $A \in \mathcal{A}$, integration with respect to ν defines a bounded linear functional on $L^1(X, \mathcal{A}, \mu)$; more explicitly, the map $\alpha(f) = \int_X f \,\mathrm{d}\nu$ is a linear functional on $L^1(X, \mathcal{A}, \mu)$ with $\|\alpha\| \le 1$. If the Riesz map of the space $(X, \mathcal{A}, \mu)$ is an isomorphism then there must exist $g \in L^\infty(X, \mathcal{A}, \mu)$ with $\alpha_g = \alpha$. Let $y \in C_2$ be fixed and consider the line $X^y = C_1 \times \{y\}$. The Riesz map $L^\infty(X^y) \to L^1(X^y)^*$ of the space X^y is injective and it carries $g|_{X^y}$ to the restriction of α to the space $L^1(X^y)$ (see Remark 4.2). Since $\nu(X^y) = 0$, the functional α vanishes on $L^1(X^y)$ and thus the injectivity of the Riesz map of X^y implies that $g|_{X^y} = 0$ μ-almost everywhere. In particular, if $R = g^{-1}(1)$ then R^y is countable. Let now $x \in C_1$ be fixed and let us consider the restriction of α to $L^1(X_x)$. Note that ν and μ coincide on all measurable subsets of X_x, so that $\alpha(f) = \int_X f \,\mathrm{d}\mu$, for $f \in L^1(X_x)$. Again, the Riesz map of the space X_x is injective, which implies that $g|_{X_x} = 1$ μ-almost everywhere. Hence the set R_x has countable complement in C_2. This concludes the proof. □

Proposition 4.4. *If C_1 and C_2 are uncountable sets then the following conditions are equivalent:*

- $|C_1| = |C_2| = \aleph_1$;
- *there exists a subset R of $C_1 \times C_2$ such that R^y and $C_2 \setminus R_x$ are countable for all $x \in C_1$, $y \in C_2$.*

Demonstração. If $|C_1| = |C_2| = \aleph_1$ then we may assume that $C_1 = C_2 = \aleph_1$. The set R can thus be defined by $R = \big\{(x, y) \in \aleph_1 \times \aleph_1 : x \in y\big\}$. Conversely, assume that we are given a subset R of $C_1 \times C_2$ such that R^y and $C_2 \setminus R_x$ are countable for all $x \in C_1$, $y \in C_2$. Since C_1 is uncountable, there exists a subset A of C_1 with $|A| = \aleph_1$. We have $\bigcap_{x \in A} R_x = \emptyset$; namely, $y \in \bigcap_{x \in A} R_x$ would imply $A \subset R^y$, contradicting the assumption that R^y is countable. Thus:

$$C_2 = \bigcup_{x \in A} (C_2 \setminus R_x);$$

since $C_2 \setminus R_x$ is countable for all $x \in C_1$, we obtain $|C_2| \le \aleph_1 \cdot \aleph_0 = \aleph_1$, proving that $|C_2| = \aleph_1$. Now observe that $C_1 = \bigcup_{y \in C_2} R^y$; namely, for $x \in C_1$ the set $C_2 \setminus R_x$ is countable, so that there exists $y \in R_x$ and thus $x \in R^y$. Since R^y is countable for all $y \in C_2$, we have $|C_1| \le \aleph_1 \cdot \aleph_0 = \aleph_1$ and hence $|C_1| = \aleph_1$. □

Corollary 4.5. *If C_1 and C_2 are uncountable and either C_1 or C_2 has cardinality greater than $\aleph_1$ then the Riesz map* (1.3) *of the space $(X, \mathcal{A}, \mu)$ is not an isomorphism.*

Demonstração. It follows directly from Lemma 4.3 and Proposition 4.4. □

5. Essential Decompositions

Definition 5.1. Let $(X, \mathcal{A}, \mu)$ be a measure space. A family $(X_i)_{i \in I}$ of measurable subsets of X is called *essentially disjoint* if $\mu(X_i \cap X_j) = 0$, for all $i, j \in I$ with $i \ne j$. An *essential decomposition* for $(X, \mathcal{A}, \mu)$ is an essentially disjoint family $(X_i)_{i \in I}$ of measurable subsets of X satisfying the following properties:

- $0 < \mu(X_i) < +\infty$, for all $i \in I$;
- if $A \in \mathcal{A}$, $\mu(A) < +\infty$ and $\mu(A \cap X_i) = 0$ for all $i \in I$ then $\mu(A) = 0$.

An essential decomposition $(X_i)_{i \in I}$ for $(X, \mathcal{A}, \mu)$ in which the sets X_i are pairwise disjoint is called a *decomposition* for $(X, \mathcal{A}, \mu)$.

Remark 5.2. It is easy to check that if $(X_i)_{i \in I}$ is an essential decomposition (resp., a decomposition) for $(X, \mathcal{A}, \mu)$ then $(X_i)_{i \in I}$ is also an essential decomposition (resp., a decomposition) for the spaces $(X, \overline{\mathcal{A}}, \bar{\mu})$, $(X, \mathcal{A}, \mu_{\mathrm{bf}})$, $(X, \mathcal{A}_{\mathrm{e}}, \mu_{\mathrm{e}})$ and $(X, \mathcal{A}_{\mathfrak{p}}, \mu_{\mathfrak{p}})$ (in order to check that if $(X_i)_{i \in I}$ is an essential decomposition for $(X, \mathcal{A}, \mu)$ then it is also an essential decomposition for $(X, \mathcal{A}, \mu_{\mathrm{bf}})$, notice that if $A \in \mathcal{A}$ and if $\mu_{\mathrm{bf}}(A \cap X_i) = 0$ for all $i \in I$ then, for every $E \subset A$ with $E \in \mathcal{A}$ and $\mu(E) < +\infty$ we have:

$$\mu(E \cap X_i) = \mu_{\mathrm{bf}}(E \cap X_i) = 0,$$

so that $\mu(E) = 0$; this proves that $\mu_{\mathrm{bf}}(A) = 0$).

Clearly, if $(A_i)_{i \in I}$ is a *countable* essentially disjoint family of measurable subsets of X then $\mu\big(\bigcup_{i \in I} A_i\big) = \sum_{i \in I} \mu(A_i)$.

Lemma 5.3. *Let $(X_i)_{i \in I}$ be an essentially disjoint family of measurable subsets of X. If $A \in \mathcal{A}$ is σ-finite for μ then the set:*

$$\big\{ i \in I : \mu(A \cap X_i) > 0 \big\}$$

is countable.

Demonstração. Let us first consider the case in which $\mu(A) < +\infty$. Then, for any $\varepsilon > 0$, the set $\big\{ i \in I : \mu(A \cap X_i) \ge \varepsilon \big\}$ is finite; otherwise, it would contain an infinite countable set I' which would imply:

$$\mu(A) \ge \mu\Big(\bigcup_{i \in I'} (A \cap X_i) \Big) = \sum_{i \in I'} \mu(A \cap X_i) = +\infty.$$

Since:

$$\big\{ i \in I : \mu(A \cap X_i) > 0 \big\} = \bigcup_{k=1}^{\infty} \big\{ i \in I : \mu(A \cap X_i) \ge \tfrac{1}{k} \big\},$$

it follows that $\big\{ i \in I : \mu(A \cap X_i) > 0 \big\}$ is countable. Assume now that A is σ-finite, so that $A = \bigcup_{k=1}^{\infty} A_k$, with $\mu(A_k) < +\infty$, for all $k \ge 1$. We have:

$$\big\{ i \in I : \mu(A \cap X_i) > 0 \big\} = \bigcup_{k=1}^{\infty} \big\{ i \in I : \mu(A_k \cap X_i) > 0 \big\},$$

which proves that $\big\{ i \in I : \mu(A \cap X_i) > 0 \big\}$ is countable. □

Corollary 5.4. *Let $(X_i)_{i \in I}$ be an essential decomposition for $(X, \mathcal{A}, \mu)$. If $A \in \mathcal{A}$ is σ-finite for μ then:*

(a) *we can write $A = A_1 \cup A_0$, with $A_1, A_0 \in \mathcal{A}$ disjoint measurable sets, A_1 contained in the union of a countable subfamily of $(X_i)_{i \in I}$ and $\mu(A_0) = 0$;*

(b) $\mu(A) = \sum_{i\in I} \mu(A \cap X_i)$.

Demonstração. Set $I' = \big\{i \in I : \mu(A \cap X_i) > 0\big\}$, $A_1 = \bigcup_{i\in I'}(A \cap X_i)$ and $A_0 = A \setminus A_1$. We know that I' is countable and thus item (a) will be established once we show that $\mu(A_0) = 0$. We claim that $\mu(A_0 \cap X_i) = 0$ for all $i \in I$; namely, for $i \in I'$, the set $A_0 \cap X_i$ is empty and for $i \in I \setminus I'$ the set $A_0 \cap X_i$ is contained in $A \cap X_i$ and thus has null measure. This implies that every measurable subset of A_0 with finite measure has null measure; since A_0 is σ-finite, we have $\mu(A_0) = 0$. This concludes the proof of item (a). To prove item (b) observe that:

$$\mu(A) = \mu(A_1) = \sum_{i\in I'} \mu(A \cap X_i) = \sum_{i\in I} \mu(A \cap X_i). \qquad \square$$

Lemma 5.5. *Let $(X_i)_{i\in I}$ be an essential decomposition for $(X, \mathcal{A}, \mu)$. Then:*

(a) *for any $A \in \mathcal{A}$, we have $\mu_{\mathrm{bf}}(A) = \sum_{i\in I} \mu(A \cap X_i)$;*
(b) *if μ is complete and full then a set $A \subset X$ is measurable if and only if $A \cap X_i$ is measurable for all $i \in I$.*

Demonstração.

- *Proof of* (a).
  Let $E \subset A$ be a σ-finite set for μ with $\mu_{\mathrm{bf}}(A) = \mu(E)$ (recall item (a) of Lemma 2.4). By item (b) of Corollary 5.4 we have:

$$\mu_{\mathrm{bf}}(A) = \mu(E) = \sum_{i\in I} \mu(E \cap X_i). \tag{5.1}$$

  If $\mu_{\mathrm{bf}}(A) = +\infty$ then (5.1) implies:

$$+\infty = \sum_{i\in I} \mu(E \cap X_i) \le \sum_{i\in I} \mu(A \cap X_i),$$

  so that $\mu_{\mathrm{bf}}(A) = +\infty = \sum_{i\in I} \mu(A \cap X_i)$. Assume that $\mu_{\mathrm{bf}}(A) < +\infty$. Then $\mu_{\mathrm{bf}}(A \setminus E) = \mu_{\mathrm{bf}}(A) - \mu_{\mathrm{bf}}(E) = 0$. Thus $\mu\big((A \setminus E) \cap X_i\big) = 0$, which implies $\mu(A \cap X_i) = \mu(E \cap X_i)$, for all $i \in I$. The conclusion follows from (5.1).

- *Proof of* (b).
  If $A \subset X$ is measurable then obviously $A \cap X_i$ is measurable for all $i \in I$. Conversely, assume that $A \cap X_i$ is measurable for all $i \in I$. Since μ is full, in order to prove the measurability of A, it suffices to show that $A \cap E$ is measurable for every $E \in \mathcal{A}$ with $\mu(E) < +\infty$. By item (a) of Corollary 5.4 we can write $E = E_1 \cup E_0$, with $E_1, E_0 \in \mathcal{A}$ disjoint, $E_1 \subset \bigcup_{i\in I'} X_i$ for some countable subset I' of I and $\mu(E_0) = 0$. Now:

$$A \cap E_1 = \bigcup_{i\in I'} \big((A \cap X_i) \cap E_1\big),$$

which implies that $A \cap E_1$ is measurable. Moreover, since μ is complete, $A \cap E_0$ is also measurable and hence $A \cap E = (A \cap E_1) \cup (A \cap E_0)$ is measurable. □

Corollary 5.6. *Let $(X_i)_{i \in I}$ be an essential decomposition for $(X, \mathcal{A}, \mu)$. If μ is perfect then $\mathcal{A}$ consists of those subsets A of X such that $A \cap X_i$ is in $\mathcal{A}$ for all $i \in I$ and the measure μ is given by $\mu(A) = \sum_{i \in I} \mu(A \cap X_i)$.* □

Proposition 5.7. *Any measure space $(X, \mathcal{A}, \mu)$ admits an essential decomposition. Moreover, any essentially disjoint family $(X_i)_{i \in I}$ of measurable sets of positive finite measure can be extended to an essential decomposition for $(X, \mathcal{A}, \mu)$.*

Demonstração. Let Λ denote the collection of all subsets $\mathcal{C}$ of $\mathcal{A}$ such that:

- $\mu(A) \in \left]0, +\infty\right[$, for all $A \in \mathcal{C}$;
- $\mu(A_1 \cap A_2) = 0$, for all $A_1, A_2 \in \mathcal{C}$ with $A_1 \neq A_2$.

If Λ is partially ordered by inclusion then clearly every chain in Λ has an upper bound. Thus, Zorn's Lemma gives us a maximal element $\mathcal{C}$ of Λ. Note that if $A \in \mathcal{A}$, $\mu(A) < +\infty$ and $\mu(A \cap A') = 0$ for all $A' \in \mathcal{C}$ then $\mu(A) = 0$; otherwise, $\mathcal{C} \cup \{A\}$ would be an element of Λ containing $\mathcal{C}$ properly. It follows that the elements of $\mathcal{C}$ constitute an essential decomposition for $(X, \mathcal{A}, \mu)$. Finally, if $(X_i)_{i \in I}$ is an essentially disjoint family of measurable sets of positive finite measure then the set $\mathcal{C}_0 = \big\{X_i : i \in I\big\}$ is in Λ and one can again use Zorn's Lemma to obtain a maximal element $\mathcal{C}$ of Λ containing the set $\mathcal{C}_0$. □

Lemma 5.8. *A measure space $(X, \mathcal{A}, \mu)$ admits an essential decomposition $(X_i)_{i \in I}$ with I finite if and only if $\mu_{\mathrm{bf}}(X) < +\infty$.*

Demonstração. If $(X_i)_{i \in I}$ is an essential decomposition for $(X, \mathcal{A}, \mu)$ with I finite then, by item (a) of Lemma 5.5, we have $\mu_{\mathrm{bf}}(X) = \sum_{i \in I} \mu(X_i) < +\infty$. Conversely, if $\mu_{\mathrm{bf}}(X) < +\infty$ then by item (d) of Lemma 2.4 we can write $X = X_0 \cup X_\infty$, with $X_0, X_\infty \in \mathcal{A}$ disjoint, X_0 σ-finite for μ and $\mu_{\mathrm{bf}}(X_\infty) = 0$. Note that $\mu(X_0) = \mu_{\mathrm{bf}}(X_0) = \mu_{\mathrm{bf}}(X) < +\infty$. If $\mu_{\mathrm{bf}}(X) = 0$ then every measurable subset of X with finite measure has null measure; this implies that the empty family is an essential decomposition for $(X, \mathcal{A}, \mu)$. On the other hand, if $\mu_{\mathrm{bf}}(X) > 0$ then the unitary family consisting solely of the set X_0 is an essential decomposition for $(X, \mathcal{A}, \mu)$; namely, if $A \in \mathcal{A}$ has finite measure and $\mu(A \cap X_0) = 0$ then $\mu(A \cap X_\infty) = 0$ and hence $\mu(A) = 0$. □

Proposition 5.9. *If $\mu_{\mathrm{bf}}(X) = +\infty$ then, for any two essential decompositions $(X_i)_{i \in I}$, $(Y_j)_{j \in J}$ for $(X, \mathcal{A}, \mu)$, we have $|I| = |J|$, i.e., the sets I and J have the same cardinality.*

Demonstração. Note that by Lemma 5.8 the sets I and J are both infinite. For each $j \in J$ we set:

$$I_j = \big\{i \in I : \mu(Y_j \cap X_i) > 0\big\}.$$

By Lemma 5.3 the set I_j is countable. Now, for each $i \in I$ we have $\mu(X_i) > 0$ and $\mu(X_i) < +\infty$ so that there must exist $j \in J$ with $\mu(Y_j \cap X_i) > 0$, i.e., $i \in I_j$. We have shown that $I = \bigcup_{j\in J} I_j$, which implies that:

$$|I| \le |J| \cdot \aleph_0 = |J|.$$

Similarly, we have $|J| \le |I|$ and hence $|I| = |J|$. □

Definition 5.10. If $\mu_{\mathrm{bf}}(X) = +\infty$ then the *dimension* of the measure space $(X, \mathcal{A}, \mu)$ is defined by $\dim(X, \mathcal{A}, \mu) = |I|$, where $(X_i)_{i\in I}$ is an arbitrary essential decomposition for $(X, \mathcal{A}, \mu)$. If $\mu_{\mathrm{bf}}(X) < +\infty$ then we set $\dim(X, \mathcal{A}, \mu) = 1$ if $\mu_{\mathrm{bf}}(X) > 0$ and $\dim(X, \mathcal{A}, \mu) = 0$ if $\mu_{\mathrm{bf}}(X) = 0$.

Lemma 5.11. *We have* $\dim(X, \mathcal{A}, \mu) \le \aleph_0$ *if and only if* X *is* σ*-finite for the measure* μ_{bf}.

Demonstração. Assume that $\dim(X, \mathcal{A}, \mu) \le \aleph_0$. If $\dim(X, \mathcal{A}, \mu) < \aleph_0$ then $\mu_{\mathrm{bf}}(X)$ is finite. If $\dim(X, \mathcal{A}, \mu) = \aleph_0$ then there exists an essential decomposition $(X_i)_{i\in I}$ for $(X, \mathcal{A}, \mu)$ with I infinite and countable. For each $i \in I$ we have $\mu_{\mathrm{bf}}(X_i) = \mu(X_i) < +\infty$ and by item (a) of Lemma 5.5 we obtain $\mu_{\mathrm{bf}}\big(X \setminus \bigcup_{i\in I} X_i\big) = 0$. Thus X is σ-finite for μ_{bf}. Conversely, assume that X is σ-finite for μ_{bf}. If $\mu_{\mathrm{bf}}(X) < +\infty$ then the dimension of $(X, \mathcal{A}, \mu)$ is finite. Assume that $\mu_{\mathrm{bf}}(X) = +\infty$. By item (d) of Lemma 2.4 we can write $X = X_0 \cup X_\infty$, with $X_0, X_\infty \in \mathcal{A}$ disjoint, X_0 σ-finite for μ and $\mu_{\mathrm{bf}}(X_\infty) = 0$. Since X_0 is σ-finite for μ we can write $X_0 = \bigcup_{n=1}^{\infty} X_n$, where $(X_n)_{n\ge1}$ is a sequence of pairwise disjoint measurable sets of positive finite measure. We claim that $(X_n)_{n\ge1}$ is an essential decomposition for $(X, \mathcal{A}, \mu)$. Namely, if $A \in \mathcal{A}$ has finite measure and $\mu(A \cap X_n) = 0$ for all $n \ge 1$ then also $\mu(A \cap X_\infty) = 0$ and thus $\mu(A) = 0$. Hence $\dim(X, \mathcal{A}, \mu) = \aleph_0$. □

Definition 5.12. Let $(X_i)_{i\in I}$ be an essential decomposition for $(X, \mathcal{A}, \mu)$. A family $(X'_i)_{i\in I}$ of measurable subsets of X with $X'_i \subset X_i$ and $\mu(X_i \setminus X'_i) = 0$ for all $i \in I$ is called a *refinement* of the essential decomposition $(X_i)_{i\in I}$.

Clearly a refinement of an essential decomposition is again an essential decomposition. Moreover, a refinement $(X''_i)_{i\in I}$ of a refinement $(X'_i)_{i\in I}$ of an essential decomposition $(X_i)_{i\in I}$ is a refinement of $(X_i)_{i\in I}$.

Lemma 5.13. *If* $(X, \mathcal{A}, \mu)$ *admits a decomposition* $(X_i)_{i\in I}$ *then every essential decomposition* $(Y_j)_{j\in J}$ *for* $(X, \mathcal{A}, \mu)$ *can be refined to a decomposition for* $(X, \mathcal{A}, \mu)$.

Demonstração. For each $j \in J$, we set:

$$I_j = \big\{ i \in I : \mu(Y_j \cap X_i) > 0 \big\},$$

and for each $i \in I$ we set:

$$J_i = \big\{ j \in J : \mu(Y_j \cap X_i) > 0 \big\}.$$

By Lemma 5.3 the sets I_j and J_i are countable, for all $i \in I$, $j \in J$. For each $j \in J$ we consider the set:

$$Y_j^1 = \bigcup_{i \in I_j} (Y_j \cap X_i);$$

arguing as in the proof of Corollary 5.4 we obtain that $\mu(Y_j \setminus Y_j^1) = 0$, for all $j \in J$. Thus $(Y_j^1)_{j \in J}$ is a refinement of $(Y_j)_{j \in J}$. We claim that for all $j \in J$ we have:

$$\big\{k \in J : Y_k^1 \cap Y_j^1 \neq \emptyset\big\} \subset \bigcup_{i \in I_j} J_i; \tag{5.2}$$

namely, assume that $Y_k^1 \cap Y_j^1 \neq \emptyset$ for some $k, j \in J$. Since $Y_k^1 \subset \bigcup_{i \in I_k} X_i$, $Y_j^1 \subset \bigcup_{i \in I_j} X_i$ and since the sets X_i are pairwise disjoint, there must exist $i \in I_j \cap I_k$. Thus $i \in I_j$ and $k \in J_i$, proving the claim. Now (5.2) implies that the set $\big\{k \in J : Y_k^1 \cap Y_j^1 \neq \emptyset\big\}$ is countable and thus:

$$Z_j = \bigcup_{\substack{k \in J \\ k \neq j}} (Y_k^1 \cap Y_j^1)$$

has null measure, for all $j \in J$. Therefore, setting $Y_j' = Y_j^1 \setminus Z_j$ for all $j \in J$, we obtain a refinement $(Y_j')_{j \in J}$ of $(Y_j^1)_{j \in J}$; clearly, the sets Y_j' are pairwise disjoint. Hence $(Y_j')_{j \in J}$ is a decomposition that refines $(Y_j)_{j \in J}$. □

Proposition 5.14. *Let $(X, \mathcal{A}, \mu)$ be a measure space. If $\dim(X, \mathcal{A}, \mu) \leq \aleph_1$ then X admits a decomposition.*

Demonstração. Let $(X_i)_{i \in I}$ be an essential decomposition for X. Since $|I| \leq \aleph_1$, we can well-order the index set I so that for all $i \in I$ the initial segment $\big\{j \in I : j < i\big\}$ is countable. For all $i \in I$, set:

$$Y_i = X_i \setminus \bigcup_{j < i} (X_i \cap X_j).$$

Then $\mu(X_i \setminus Y_i) = 0$ and thus $(Y_i)_{i \in I}$ is a refinement of $(X_i)_{i \in I}$. Clearly, the sets Y_i, $i \in I$, are pairwise disjoint. □

6. Sums and Quotients

Given a family $(X_i)_{i \in I}$ of sets we denote by $\sum_{i \in I} X_i$ their *disjoint union* defined by:

$$\sum_{i \in I} X_i = \bigcup_{i \in I} \big(\{i\} \times X_i\big);$$

to simplify the notation, except in situations in which it may be confusing, we identify each $x \in X_i$ with (i, x), so that X_i is thought of as a subset of $\sum_{i \in I} X_i$.

Definition 6.1. Let $(X_i, \mathcal{A}_i, \mu_i)_{i\in I}$ be a family of measure spaces. The *external sum*:

$$\sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)$$

of such family is the measure space $(X, \mathcal{A}, \mu)$ defined as follows; we set $X = \sum_{i\in I} X_i$,

$$\mathcal{A} = \big\{A \subset X : A \cap X_i \in \mathcal{A}_i, \text{ for all } i \in I\big\},$$

and:

$$\mu(A) = \sum_{i\in I} \mu_i(A \cap X_i),$$

for all $A \in \mathcal{A}$.

Note that if $A \subset X_i$ then $A \in \mathcal{A}$ if and only if $A \in \mathcal{A}_i$ and in this case $\mu(A) = \mu_i(A)$. Moreover, if $\mu_i(X_i) \in \left]0, +\infty\right[$ for all $i \in I$ then the family $(X_i)_{i\in I}$ is a decomposition for the external sum $(X, \mathcal{A}, \mu) = \sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)$.

Remark 6.2. If $(X, \mathcal{A}, \mu) = \sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)$ then clearly a map f defined on X is measurable if and only if $f|_{X_i}$ is measurable for all $i \in I$.

Lemma 6.3. *Let $(X_i, \mathcal{A}_i, \mu_i)_{i\in I}$ be a family of measure spaces and let:*

$$(X, \mathcal{A}, \mu) = \sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)$$

denote its external sum. Then:

(a) *if each μ_i is complete then μ is complete;*
(b) *if each μ_i is full then μ is full;*
(c) *if each μ_i is block-free then μ is block-free;*
(d) *if each μ_i is perfect then μ is perfect.*

Demonstração.

- *Proof of* (a).
  Let $A \in \mathcal{A}$ be fixed with $\mu(A) = 0$. Given $B \subset A$ then $A \cap X_i \in \mathcal{A}_i$, $\mu_i(A \cap X_i) = 0$ and $B \cap X_i \subset A \cap X_i$, for all $i \in I$. Since μ_i is complete it follows that $B \cap X_i \in \mathcal{A}_i$ for all $i \in I$ and hence $B \in \mathcal{A}$.

- *Proof of* (b).
  Let $A \subset X$ be given and assume that $A \cap E \in \mathcal{A}$, for all $E \in \mathcal{A}$ with $\mu(E) < +\infty$. Let $i \in I$ be fixed. If $E \in \mathcal{A}_i$ and $\mu_i(E) < +\infty$ then $E \in \mathcal{A}$ and $\mu(E) = \mu_i(E) < +\infty$, so that $A \cap E \in \mathcal{A}$ and $(A \cap X_i) \cap E \in \mathcal{A}_i$. Since μ_i is full, it follows that $A \cap X_i$ is in $\mathcal{A}_i$ for all $i \in I$ and hence $A \in \mathcal{A}$.

- *Proof of* (c).
  Let $A \in \mathcal{A}$ be given with $\mu(A) = +\infty$. Thus $\mu_i(A \cap X_i) > 0$ for some $i \in I$. If $\mu_i(A \cap X_i) < +\infty$ then A is not an infinite block. If $\mu_i(A \cap X_i) = +\infty$ then, since μ_i is block-free, there exists $E \in \mathcal{A}_i$

contained in $A \cap X_i$ with $\mu_i(E) \in \,]0, +\infty[$, proving again that A is not an infinite block.

- *Proof of* (d).
  It follows from (a), (b) and (c). □

Let $(E_i)_{i\in I}$ be a family of Banach spaces. For $v = (v_i)_{i\in I} \in \prod_{i\in I} E_i$ and $p \in [1, +\infty]$ we define the *L^p-type norm* of v by:

$$\|v\|_p = \Big(\sum_{i\in I} \|v_i\|^p\Big)^{\frac{1}{p}}, \quad \text{if } p < +\infty, \qquad \text{and} \qquad \|v\|_\infty = \sup_{i\in I} \|v_i\|.$$

The *L^p-type (external) direct sum* of the family $(E_i)_{i\in I}$ is defined by:

$$\overline{\bigoplus_{i\in I}}^{\,p} E_i = \Big\{v \in \prod_{i\in I} E_i : \|v\|_p < +\infty\Big\}.$$

It is easy to check that $\overline{\bigoplus}^p_{i\in I} E_i$ is a subspace of $\prod_{i\in I} E_i$ and that it becomes a Banach space when endowed with the norm $\|\cdot\|_p$. Note that if $p < +\infty$ then for $v \in \overline{\bigoplus}^p_{i\in I} E_i$ we have $v_i = 0$ except for a countable number of indices $i \in I$.

Lemma 6.4. *Given $p, q \in [1, +\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ then for every bounded linear functional α on $\overline{\bigoplus}^p_{i\in I} E_i$ the family $(\alpha|_{E_i})_{i\in I}$ is in $\overline{\bigoplus}^q_{i\in I} E_i^*$; moreover, if $p < +\infty$ then $\alpha(v) = \sum_{i\in I} \alpha(v_i)$ for all $v = (v_i)_{i\in I} \in \overline{\bigoplus}^p_{i\in I} E_i$ and the map:*

$$\Big(\overline{\bigoplus_{i\in I}}^{\,p} E_i\Big)^* \ni \alpha \longmapsto (\alpha|_{E_i})_{i\in I} \in \overline{\bigoplus_{i\in I}}^{\,q} E_i^*; \tag{6.1}$$

is a linear isometry.

Demonstração. Use the same standard arguments that are used to prove the Riesz Representation Theorem for counting measures. □

Lemma 6.5. *If $(X_i, \mathcal{A}_i, \mu_i)_{i\in I}$ is a family of measure spaces then, for all $p \in [1, +\infty]$, the map:*

$$L^p\big(\textstyle\sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)\big) \ni f \longmapsto (f|_{X_i})_{i\in I} \in \displaystyle\overline{\bigoplus_{i\in I}}^{\,p} L^p(X_i, \mathcal{A}_i, \mu_i) \tag{6.2}$$

is a linear isometry.

Demonstração. If $f : \sum_{i\in I} X_i \to I\!R$ is a measurable map then it can be easily checked that:

$$\|f\|_p = \big\|\big(\|f|_{X_i}\|_p\big)_{i\in I}\big\|_p.$$

The conclusion follows from Remark 6.2. □

If $p \in [1, +\infty[$, $q \in]1, +\infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$ then the composition of the transpose of the linear isometry (6.2) with the linear isometry (6.1) gives us a linear isometry:

$$\overline{\bigoplus_{i\in I}}^{q} L^p(X_i, \mathcal{A}_i, \mu_i)^* \ni (\alpha_i)_{i\in I} \longmapsto \alpha \in L^p\big(\textstyle\sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)\big)^*, \tag{6.3}$$

where $(X_i, \mathcal{A}_i, \mu_i)_{i\in I}$ is an arbitrary family of measure spaces and α is defined by $\alpha(f) = \sum_{i\in I} \alpha_i(f|_{X_i})$, for all $f \in L^p\big(\sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)\big)$. We also have a commutative diagram:

$$\begin{tikzcd} L^p\big(\sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)\big)^* & \overline{\bigoplus}^{q}_{i\in I} L^p(X_i, \mathcal{A}_i, \mu_i)^* \arrow[l, "\cong"'] \\ L^q\big(\sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)\big) \arrow[u] \arrow[r, "\cong"'] & \overline{\bigoplus}^{q}_{i\in I} L^q(X_i, \mathcal{A}_i, \mu_i) \arrow[u] \end{tikzcd} \tag{6.4}$$

where the top horizontal arrow is the map (6.3), the bottom horizontal arrow is the version of the map (6.2) for L^q, the left vertical arrow is the (q,p)-Riesz map (1.1) for the space $\sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)$ and the right vertical arrow is given by the (q,p)-Riesz map (1.1) for the space $(X_i, \mathcal{A}_i, \mu_i)$ on each coordinate.

Remark 6.6. It follows from the commutativity of diagram (6.4) that if for each space $(X_i, \mathcal{A}_i, \mu_i)$ the (q,p)-Riesz map is an isometry then also for the external sum $\sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)$ the (q,p)-Riesz map is an isometry.

Definition 6.7. Given a measure space $(X, \mathcal{A}, \mu)$ and a map $\phi : X \to X'$ taking values in a set X' then the σ-algebra

$$\phi_*\mathcal{A} = \big\{A \subset X' : \phi^{-1}(A) \in \mathcal{A}\big\}$$

of subsets of X' is called *co-induced* by ϕ. If $\mathcal{A}'$ is a σ-algebra of subsets of X' contained in $\phi_*\mathcal{A}$ then the measure $(\phi_*\mu) : \mathcal{A}' \to [0, +\infty]$ defined by:

$$(\phi_*\mu)(A) = \mu\big(\phi^{-1}(A)\big), \quad A \in \mathcal{A}'$$

is called *co-induced* by ϕ on $\mathcal{A}'$. If $(X', \mathcal{A}', \mu')$ is a measure space then a map $\phi : X \to X'$ is called *measure preserving* if $\mathcal{A}' \subset \phi_*\mathcal{A}$ and $\mu' = \phi_*\mu$; if, in addition, we have $\mathcal{A}' = \phi_*\mathcal{A}$ then we call ϕ a *quotient map*. A bijective quotient map $\phi : X \to X'$ is called an *isomorphism*.

Note that an isomorphism from $(X, \mathcal{A}, \mu)$ to $(X', \mathcal{A}', \mu')$ is a bijective map $\phi : X \to X'$ such that, for all $A \subset X$, $\phi(A) \in \mathcal{A}'$ if and only if $A \in \mathcal{A}$, and, in this case, $\mu'\big(\phi(A)\big) = \mu(A)$.

Remark 6.8. If $\phi : X \to X'$ is a quotient map then clearly a map f defined on X' is measurable if and only if the composition $f \circ \phi$ is measurable.

Lemma 6.9. *Given measure spaces $(X, \mathcal{A}, \mu)$, $(X', \mathcal{A}', \mu')$ and a measure preserving map $\phi : X \to X'$ then for all $p \in [1, +\infty]$, the map* induced by ϕ

on L^p:

$$L^p(X', \mathcal{A}', \mu') \ni f \longmapsto f \circ \phi \in L^p(X, \mathcal{A}, \mu) \tag{6.5}$$

is an isometric immersion. Moreover, if ϕ is a quotient map and μ is complete then the image of (6.5) consists of those maps $g \in L^p(X, \mathcal{A}, \mu)$ for which there exists a map $f : X' \to I\!R$ such that $f \circ \phi = g$ μ-almost everywhere.

Demonstração. To prove the first part of the statement, simply observe that:

$$\|f \circ \phi\|_p = \|f\|_p,$$

for any measurable map $f : X' \to I\!R$. The second part of the statement follows from the equality above and from Remark 6.8, keeping in mind the following fact: if μ is complete, $g : X \to I\!R$ is measurable and $f \circ \phi = g$ μ-almost everywhere then $f \circ \phi$ is measurable. □

Given $p, q \in [1, +\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, measure spaces $(X, \mathcal{A}, \mu)$, $(X', \mathcal{A}', \mu')$ and a measure preserving map $\phi : X \to X'$ then we have a commutative diagram:

$$\begin{tikzcd} L^p(X', \mathcal{A}', \mu')^* & L^p(X, \mathcal{A}, \mu)^* \arrow[l] \\ L^q(X', \mathcal{A}', \mu') \arrow[u] \arrow[r] & L^q(X, \mathcal{A}, \mu) \arrow[u] \end{tikzcd} \tag{6.6}$$

where the vertical arrows are the (q,p)-Riesz maps (1.1) for $(X', \mathcal{A}', \mu')$ and $(X, \mathcal{A}, \mu)$, the top horizontal arrow is the transpose of the map induced by ϕ on L^p and the bottom horizontal arrow is the map induced by ϕ on L^q.

If $(X, \mathcal{A}, \mu)$ is a measure space and $\sim$ is an equivalence relation on X then we have a canonical map $\mathfrak{q} : X \to X/\!\sim$ and we may endow the quotient set $X/\!\sim$ with the σ-algebra $\mathfrak{q}_*\mathcal{A}$ and the measure $\mathfrak{q}_*\mu$ co-induced by $\mathfrak{q}$, so that the map $\mathfrak{q}$ becomes a quotient map in the sense of Definition 6.7.

Recall that a measurable subset of a measure space can be naturally regarded as a measure space (see Remark 4.2).

Lemma 6.10. *Let $\phi : X \to X'$ be a quotient map, where $(X, \mathcal{A}, \mu)$ and $(X', \mathcal{A}', \mu')$ are measure spaces. Then $\phi(X)$ is a measurable subset of X' whose complement has null measure. Moreover, if $\sim$ is the equivalence relation on X* induced *by ϕ, i.e.:*

$$x \sim y \Longleftrightarrow \phi(x) = \phi(y), \quad x, y \in X,$$

and the quotient set $X/\!\sim$ is endowed with the σ-algebra and measure co-induced by the canonical map $\mathfrak{q} : X \to X/\!\sim$ then ϕ induces an isomorphism

$\overline{\phi} : (X/{\sim}) \to \phi(X)$ *such that the diagram:*

$$\begin{tikzcd} X \arrow[d, "\mathfrak{q}"'] \arrow[dr, "\phi"] & \\ X/{\sim} \arrow[r, "\cong", "\overline{\phi}"'] & \phi(X) \end{tikzcd}$$

commutes.

Demonstração. We have $\phi(X) \in \mathcal{A}'$ because $\phi^{-1}\big(\phi(X)\big) = X$ is in $\mathcal{A}$. Moreover, since the inverse image of $X' \setminus \phi(X)$ by ϕ is empty we have $\mu'\big(X' \setminus \phi(X)\big) = 0$. The fact that $\overline{\phi}$ is an isomorphism follows straight forwardly from Definition 6.7. □

Proposition 6.11. *Let $(X_i, \mathcal{A}_i, \mu_i)$ be a family of complete measure spaces such that $0 < \mu_i(X_i) < +\infty$ for all $i \in I$. Let $\sim$ be an equivalence relation on $\sum_{i\in I} X_i$ satisfying the following property:*

- $(*)$ *for all $i, j \in I$ with $i \neq j$ the set $\big\{x \in X_i : x \sim y,$ for some $y \in X_j\big\}$ has null measure and for all $i \in I$ and all $x, y \in X_i$ we have $x \sim y$ if and only if $x = y$.*

Set $X = \big(\sum_{i\in I} X_i\big)/{\sim}$ and let $\mathcal{A}$ and μ be respectively the σ-algebra and the measure co-induced on X by the canonical map $\mathfrak{q} : \sum_{i\in I} X_i \to X$. Then $(X, \mathcal{A}, \mu)$ is a perfect measure space, $\big(\mathfrak{q}(X_i)\big)_{i\in I}$ is an essential decomposition for X and $\mathfrak{q}|_{X_i} : X_i \to \mathfrak{q}(X_i)$ is an isomorphism for all $i \in I$. Moreover, for $p < +\infty$, the map:

$$L^p(X, \mathcal{A}, \mu) \longmapsto L^p\big(\textstyle\sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)\big)$$

induced by $\mathfrak{q}$ on L^p is an isometry.

Demonstração. We divide the proof into several steps.

Step 1. *For $A \subset X$ we have $A \in \mathcal{A}$ if and only if $\mathfrak{q}^{-1}(A) \cap X_i \in \mathcal{A}_i$, for all $i \in I$; in this case, $\mu(A) = \sum_{i\in I} \mu_i\big(\mathfrak{q}^{-1}(A) \cap X_i\big)$.*

This follows directly from the definitions of quotient maps and external sums.

Step 2. *The measure μ is complete.*

Let $A \in \mathcal{A}$ be fixed with $\mu(A) = 0$ and let A' be a subset of A. Then $\mathfrak{q}^{-1}(A) \cap X_i$ is in $\mathcal{A}_i$ and $\mu_i\big(\mathfrak{q}^{-1}(A) \cap X_i\big) = 0$ for all $i \in I$. Since $\mathfrak{q}^{-1}(A') \cap X_i \subset \mathfrak{q}^{-1}(A) \cap X_i$ and μ_i is complete, it follows that $\mathfrak{q}^{-1}(A') \cap X_i \in \mathcal{A}_i$ for all $i \in I$ and hence $A' \in \mathcal{A}$.

Step 3. *For all $i \in I$, a subset E of $\mathfrak{q}(X_i)$ is in $\mathcal{A}$ if and only if $\mathfrak{q}^{-1}(E) \cap X_i$ is in $\mathcal{A}_i$; in this case, $\mu(E) = \mu_i\big(\mathfrak{q}^{-1}(E) \cap X_i\big)$.*

By step 1, it suffices to show that for $j \neq i$ we have $\mathfrak{q}^{-1}(E) \cap X_j \in \mathcal{A}_j$ and $\mu_j\big(\mathfrak{q}^{-1}(E) \cap X_j\big) = 0$. Note that:

$$\mathfrak{q}^{-1}(E) \cap X_j \subset \mathfrak{q}^{-1}\big(\mathfrak{q}(X_i)\big) \cap X_j = \big\{x \in X_j : x \sim y, \text{ for some } y \in X_i\big\}.$$

The conclusion follows from the completeness of the measure μ_j and from property $(*)$ of the equivalence relation $\sim$.

Step 4. *For all $i \in I$, $\mathfrak{q}(X_i)$ is in $\mathcal{A}$ and $\mathfrak{q}|_{X_i} : X_i \to \mathfrak{q}(X_i)$ is an isomorphism.*

Property $(*)$ implies that $\mathfrak{q}|_{X_i} : X_i \to \mathfrak{q}(X_i)$ is bijective. The fact that $\mathfrak{q}(X_i)$ is in $\mathcal{A}$ and $\mathfrak{q}|_{X_i} : X_i \to \mathfrak{q}(X_i)$ is an isomorphism follows then from step 3.

Step 5. *For $A \subset X$, we have $A \in \mathcal{A}$ if and only if $A \cap \mathfrak{q}(X_i) \in \mathcal{A}$, for all $i \in I$; in this case $\mu(A) = \sum_{i \in I} \mu\big(A \cap \mathfrak{q}(X_i)\big)$.*

For $i \in I$, note that:

$$\mathfrak{q}^{-1}\big(A \cap \mathfrak{q}(X_i)\big) \cap X_i = \mathfrak{q}^{-1}(A) \cap X_i;$$

thus, by step 3, $A \cap \mathfrak{q}(X_i) \in \mathcal{A}$ if and only if $\mathfrak{q}^{-1}(A) \cap X_i \in \mathcal{A}_i$ and, in this case, $\mu\big(A \cap \mathfrak{q}(X_i)\big) = \mu_i\big(\mathfrak{q}^{-1}(A) \cap X_i\big)$. The conclusion follows from step 1.

Step 6. *The family $\big(\mathfrak{q}(X_i)\big)_{i \in I}$ is an essential decomposition for X.*

By step 4, for all $i \in I$, $\mathfrak{q}(X_i)$ is in $\mathcal{A}$ and $\mu\big(\mathfrak{q}(X_i)\big) = \mu_i(X_i)$, so that $\mu\big(\mathfrak{q}(X_i)\big)$ is positive and finite. Moreover, if $A \in \mathcal{A}$ and $\mu\big(A \cap \mathfrak{q}(X_i)\big) = 0$ for all $i \in I$ then, by step 5, $\mu(A) = \sum_{i \in I} \mu\big(A \cap \mathfrak{q}(X_i)\big) = 0$.

Step 7. *The measure μ is full.*

Let $A \subset X$ be given and assume that $A \cap E \in \mathcal{A}$ for all $E \in \mathcal{A}$ with $\mu(E) < +\infty$. By Step 4, $\mathfrak{q}(X_i) \in \mathcal{A}$ and $\mu\big(\mathfrak{q}(X_i)\big) = \mu_i(X_i) < +\infty$, for all $i \in I$. Thus $A \cap \mathfrak{q}(X_i) \in \mathcal{A}$ for all $i \in I$ and hence $A \in \mathcal{A}$, by step 5.

Step 8. *The measure μ is block-free.*

Let $A \in \mathcal{A}$ be given with $\mu(A) = +\infty$. By step 5 we have:

$$\mu(A) = \sum_{i \in I} \mu\big(A \cap \mathfrak{q}(X_i)\big)$$

and thus $\mu\big(A \cap \mathfrak{q}(X_i)\big) > 0$ for some $i \in I$. But $\mu\big(A \cap \mathfrak{q}(X_i)\big) \le \mu\big(\mathfrak{q}(X_i)\big)$ and, by step 4, $\mu\big(\mathfrak{q}(X_i)\big) = \mu_i(X_i) < +\infty$, proving that A is not an infinite block for μ.

Step 9. *For $p < +\infty$, the map induced by $\mathfrak{q}$ on L^p is an isometry.*

By Lemma 6.3, the space $\sum_{i \in I}(X_i, \mathcal{A}_i, \mu_i)$ is complete and thus, by Lemma 6.9, it suffices to prove the following assertion: for every map g in $L^p\big(\sum_{i \in I}(X_i, \mathcal{A}_i, \mu_i)\big)$ there exists a map $f : X \to \mathbb{R}$ such that $f \circ \mathfrak{q} = g$ almost everywhere. By Lemma 6.5, the family $(g|_{X_i})_{i \in I}$ is in $\overline{\bigoplus}^p_{i \in I} L^p(X_i, \mathcal{A}_i, \mu_i)$ and, in particular, the set:

$$I' = I \setminus \big\{ i \in I : g|_{X_i} = 0 \text{ almost everywhere} \big\}$$

is countable. For $i \in I'$ we set:

$$Y_i = X_i \setminus \bigcup_{\substack{j \in I' \\ j \neq i}} \big\{x \in X_i : x \sim y, \text{ for some } y \in X_j\big\},$$

so that $\mu_i(X_i \setminus Y_i) = 0$ for all $i \in I'$ and the sets $\mathfrak{q}(Y_i)$, $i \in I'$, are pairwise disjoint. Consider the map $f : X \to I\!R$ such that $f \circ \mathfrak{q}|_{Y_i} = g|_{Y_i}$ for all $i \in I'$ and such that f vanishes outside $\bigcup_{i \in I'} \mathfrak{q}(Y_i)$. Let us show that $f \circ \mathfrak{q}|_{X_i} = g|_{X_i}$ almost everywhere for all $i \in I$; this will imply that $f \circ \mathfrak{q} = g$ almost everywhere and conclude the proof. If $i \in I'$ then $f \circ \mathfrak{q}|_{X_i}$ and $g|_{X_i}$ are equal on Y_i and $\mu(X_i \setminus Y_i) = 0$, so that $f \circ \mathfrak{q}|_{X_i} = g|_{X_i}$ almost everywhere. If $i \in I$ and $i \notin I'$ then $g|_{X_i} = 0$ almost everywhere and $f \circ \mathfrak{q}|_{X_i}$ vanishes outside the set:

$$\bigcup_{j \in I'} \big\{x \in X_i : x \sim y, \text{ for some } y \in X_j\big\}.$$

Hence $f \circ \mathfrak{q}|_{X_i} = g|_{X_i}$ almost everywhere. □

Proposition 6.12. *Let $(X, \mathcal{A}, \mu)$ be a perfect measure space and let $(X_i)_{i \in I}$ be an essential decomposition for X. For each $i \in I$, let $\mathcal{A}_i$ denote the σ-algebra of all elements of $\mathcal{A}$ contained in X_i and let μ_i denote the measure on $\mathcal{A}_i$ obtained by the restriction of μ, so that $(X_i, \mathcal{A}_i, \mu_i)$ is a complete measure space with $0 < \mu_i(X_i) < +\infty$; consider the external sum $\sum_{i \in I}(X_i, \mathcal{A}_i, \mu_i)$. Let $\phi : \sum_{i \in I} X_i \to X$ be the canonical map whose restriction to each X_i is equal to the inclusion map $X_i \to X$. Then ϕ is a quotient map and the equivalence relation $\sim$ induced on X by ϕ satisfies property $(*)$ in the statement of Proposition 6.11.*

Demonstração. To prove that ϕ is a quotient map it suffices to show that for all $A \subset X$ we have $A \in \mathcal{A}$ if and only if $A \cap X_i \in \mathcal{A}$, for all $i \in I$ and, in this case, $\mu(A) = \sum_{i \in I} \mu(A \cap X_i)$. But this it precisely the content of Lemma 5.5, since $\mu_{\mathrm{bf}} = \mu$. Finally, the fact that the equivalence relation $\sim$ satisfies property $(*)$ follows from the definition of essential decomposition. □

Corollary 6.13. *Under the conditions and notations of the statement of Proposition 6.12, assume that the quotient set $\big(\sum_{i \in I} X_i\big)/\!\sim$ is endowed with the σ-algebra and the measure co-induced by the canonical map*

$$\mathfrak{q} : \sum_{i \in I} X_i \longrightarrow \Big(\sum_{i \in I} X_i\Big)/\!\sim.$$

Then the map ϕ induces an isomorphism $\overline{\phi} : \big(\sum_{i \in I} X_i\big)/\!\sim \to \phi\big(\sum_{i \in I} X_i\big)$ such that the diagram:

$$\begin{tikzcd}
\sum_{i \in I} X_i \arrow[d, "\mathfrak{q}"'] \arrow[dr, "\phi"] & \\
\big(\sum_{i \in I} X_i\big)/\!\sim \arrow[r, "\cong", "\overline{\phi}"'] & \phi\big(\sum_{i \in I} X_i\big) \subset X
\end{tikzcd}$$

commutes and $\phi\big(\sum_{i\in I} X_i\big)$ *is a measurable subset of* X *whose complement has null measure.*

Demonstração. It follows from Lemma 6.10. □

Corollary 6.14. *Under the conditions and notations of the statement of Proposition 6.12, for all* $p \in [1, +\infty[$ *the map:*

$$L^p(X, \mathcal{A}, \mu) \longrightarrow L^p\big(\textstyle\sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)\big) \tag{6.7}$$

induced by ϕ *on* L^p *is an isometry.*

Demonstração. As in Corollary 6.13, we consider the maps $\mathfrak{q}$ and $\overline{\phi}$. By Proposition 6.11, the map induced by $\mathfrak{q}$ on L^p:

$$L^p\Big(\big(\textstyle\sum_{i\in I} X_i\big)/\!\sim\Big) \longrightarrow L^p\big(\textstyle\sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)\big)$$

is an isometry. Moreover, since $\overline{\phi}$ is an isomorphism, the map induced by $\overline{\phi}$ on L^p:

$$L^p\Big(\phi\big(\textstyle\sum_{i\in I} X_i\big)\Big) \longrightarrow L^p\Big(\big(\textstyle\sum_{i\in I} X_i\big)/\!\sim\Big)$$

is an isometry. Finally, since the complement of $\phi\big(\sum_{i\in I} X_i\big)$ in X has null measure we have:

$$L^p\Big(\phi\big(\textstyle\sum_{i\in I} X_i\big)\Big) = L^p(X, \mathcal{A}, \mu).$$

The conclusion follows. □

Remark 6.15. In Corollary 6.14 we actually don't need the hypothesis that the measure μ be perfect. Namely, if $(X_i)_{i\in I}$ is an essential decomposition for $(X, \mathcal{A}, \mu)$ and $\mu_{\mathfrak{p}}$ is the perfect version of μ then $(X_i)_{i\in I}$ is also an essential decomposition for $(X, \mathcal{A}_{\mathfrak{p}}, \mu_{\mathfrak{p}})$ (see Remark 5.2) and the restriction of $\mu_{\mathfrak{p}}$ to X_i is the completion $\overline{\mu_i}$ of μ_i (see Remark 3.13). Corollary 6.14 then says that, for $p \in [1, +\infty[$, the map:

$$L^p(X, \mathcal{A}_{\mathfrak{p}}, \mu_{\mathfrak{p}}) \longrightarrow L^p\big(\textstyle\sum_{i\in I}(X_i, \overline{\mathcal{A}_i}, \overline{\mu_i})\big) \tag{6.8}$$

induced by ϕ is an isometry. But, keeping in mind that $\sum_{i\in I}(X_i, \overline{\mathcal{A}_i}, \overline{\mu_i})$ is the completion of $\sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)$, Lemmas 3.1 and 3.14 give us isometries:

$$\begin{gathered} L^p\big(\textstyle\sum_{i\in I}(X_i, \mathcal{A}_i, \mu_i)\big) \cong L^p\big(\textstyle\sum_{i\in I}(X_i, \overline{\mathcal{A}_i}, \overline{\mu_i})\big), \\ L^p(X, \mathcal{A}, \mu) \cong L^p(X, \mathcal{A}_{\mathfrak{p}}, \mu_{\mathfrak{p}}), \end{gathered} \tag{6.9}$$

and the map (6.7) differs from the map (6.8) by the isometries (6.9), so that (6.7) is also an isometry.

The composition of the map (6.7) with the linear isometry (6.2) is given by:

$$L^p(X, \mathcal{A}, \mu) \ni f \longmapsto (f|_{X_i})_{i\in I} \in \overline{\bigoplus_{i\in I}}^{\,p} L^p(X_i, \mathcal{A}_i, \mu_i). \tag{6.10}$$

For $p < +\infty$, the map (6.10) is a linear isometry and for $p = +\infty$, (6.10) is, in general, just an isometric immersion. The commutativity of diagrams (6.4) and (6.6) gives us a new commutative diagram:

$$\begin{tikzcd} L^p(X,\mathcal{A},\mu)^* & \overline{\bigoplus}^q_{i\in I} L^p(X_i,\mathcal{A}_i,\mu_i)^* \arrow[l, "\cong"'] \\ L^q(X,\mathcal{A},\mu) \arrow[u] \arrow[r] & \overline{\bigoplus}^q_{i\in I} L^q(X_i,\mathcal{A}_i,\mu_i) \arrow[u, "\cong"] \end{tikzcd} \tag{6.11}$$

for all $p \in [1,+\infty[$, $q \in]1,+\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, where the top horizontal arrow is the composition of the map (6.3) with the transpose of the map (6.7), the bottom horizontal arrow is the version of the map (6.10) for L^q, the left vertical arrow is the (q,p)-Riesz map (1.1) for the space $(X,\mathcal{A},\mu)$ and the right vertical arrow is given by the (q,p)-Riesz map (1.1) for the space $(X_i,\mathcal{A}_i,\mu_i)$ on each coordinate. Note that, since $\mu_i(X_i) < +\infty$ for all $i \in I$, the right vertical arrow on diagram (6.11) is an isometry, by the Riesz Representation Theorem for spaces of finite measure.

Proposition 6.16. *If $p,q \in]1,+\infty[$ and $\frac{1}{p} + \frac{1}{q} = 1$ then, for* any *measure space $(X,\mathcal{A},\mu)$, the (q,p)-Riesz map (1.1) is a linear isometry.*

Demonstração. By Lemma 3.14 and by the commutativity of diagram (3.8) we may replace μ by its perfect version $\mu_{\mathfrak{p}}$. Thus, we may assume that μ is perfect. Since $p,q < +\infty$, Corollary 6.14 implies that both horizontal arrows on diagram (6.11) are linear isometries. The conclusion follows. □

Proposition 6.17. *If a perfect measure space $(X,\mathcal{A},\mu)$ admits a decomposition $(X_i)_{i\in I}$ then its Riesz map (1.3) is a linear isometry.*

Demonstração. If $(X_i)_{i\in I}$ is a decomposition for $(X,\mathcal{A},\mu)$ then the map ϕ appearing on the statement of Proposition 6.12 is injective and hence the map q on the statement of Corollary 6.13 is simply the identity map of $\sum_{i\in I} X_i$. Thus $\overline{\phi} = \phi$ is an isomorphism from $\sum_{i\in I} X_i$ onto a measurable subset of X whose complement has null measure. This proves that the map induced by ϕ on L^∞ is a linear isometry and thus both horizontal arrows of diagram (6.11) are linear isometries for $p = 1$, $q = +\infty$. The conclusion follows. □

Corollary 6.18. *Let $(X,\mathcal{A},\mu)$ be a perfect measure space. If*

$$\dim(X,\mathcal{A},\mu) \leq \aleph_1$$

then the Riesz map (1.3) of X is a linear isometry.

Demonstração. It follows from Proposition 5.14. □

Commutative diagram (6.11) allows us to see exactly what's the obstacle for the bijectivity of the Riesz map (1.3) of a perfect measure space $(X,\mathcal{A},\mu)$. Namely, the Riesz map (1.3) of $(X,\mathcal{A},\mu)$ is a linear isometry if and only if the bottom horizontal arrow of diagram (6.11) is surjective. Thus, we have the following:

Lemma 6.19. *Let $(X, \mathcal{A}, \mu)$ be a perfect measure space and let $(X_i)_{i \in I}$ be an essential decomposition for X. For $q = +\infty$, the image of the bottom horizontal arrow of diagram (6.11) consists of the families $(f_i)_{i \in I}$ in $\overline{\bigoplus}_{i \in I}^{\infty} L^\infty(X_i, \mathcal{A}_i, \mu_i)$ for which there exists a map $f : X \to I\!R$ such that $f|_{X_i} = f_i$ almost everywhere, for all $i \in I$. In particular, the Riesz map (1.3) of $(X, \mathcal{A}, \mu)$ is a linear isometry if and only if the following condition holds: given a family $(f_i)_{i \in I}$ in $\prod_{i \in I} L^\infty(X_i, \mathcal{A}_i, \mu_i)$ with $\sup_{i \in I} \|f_i\|_\infty < +\infty$ then there exists a map $f : X \to I\!R$ with $f|_{X_i} = f_i$ almost everywhere, for all $i \in I$.* □

We have now the following:

Proposition 6.20. *Let $(X, \mathcal{A}, \mu)$ be a perfect measure space with*

$$\dim(X, \mathcal{A}, \mu) \leq 2^{\aleph_0}.$$

Then the Riesz map (1.3) of X is a linear isometry if and only if X admits a decomposition.

Demonstração. If X admits a decomposition then its Riesz map is a linear isometry, by Proposition 6.17. Now assume that the Riesz map of X is a linear isometry and let $(X_i)_{i \in I}$ be an essential decomposition for X. Since $|I|$ is less than or equal to the continuum, we can find an injective map $c : I \to [0, 1]$. For all $i \in I$, let $f_i \in L^\infty(X_i, \mathcal{A}_i, \mu_i)$ be the constant map equal to $c(i)$. By Lemma 6.19, there exists a map $f : X \to I\!R$ such that $f|_{X_i} = c(i)$ almost everywhere, for all $i \in I$. Set $Y_i = X_i \cap f^{-1}\big(c(i)\big)$, so that $\mu(X_i \setminus Y_i) = 0$. Thus $(Y_i)_{i \in I}$ is a refinement of $(X_i)_{i \in I}$. For $i \neq j$, since $c(i) \neq c(j)$, we have $Y_i \cap Y_j = \emptyset$ and hence $(Y_i)_{i \in I}$ is a decomposition for X. □

7. Some Examples

In this section we give a few examples that illustrate the relations between several of the concepts introduced in the earlier sections.

Example 7.1. Let X be an uncountable set and let $\mathcal{A}$ be the σ-algebra consisting of those subsets of X which are either countable or have countable complement in X. Let S be a subset of X that is not in $\mathcal{A}$. Define a measure $\mu : \mathcal{A} \to [0, +\infty]$ by taking, for each $A \in \mathcal{A}$, $\mu(A)$ to be the number of elements of $A \cap S$. Notice that if $A \in \mathcal{A}$ has countable complement in X then $A \cap S = S \setminus (X \setminus A)$ is uncountable and therefore $\mu(A) = +\infty$. The measure μ is clearly complete, since if $A \in \mathcal{A}$ has null measure then A is countable and therefore every subset of A is in $\mathcal{A}$. The measure μ is also block-free, since if $A \in \mathcal{A}$ is such that $\mu(A) = +\infty$ then $A \cap S$ is an infinite set and, for every $x \in A \cap S$, the unitary set $\{x\}$ is a measurable subset of A with $\mu\big(\{x\}\big) = 1$. The measure μ is not full; moreover, we claim that the

σ-algebra $\mathcal{A}_e$ coincides with the set $\wp(X)$ of all subsets of X. Namely, if $A \in \wp(X)$ is an arbitrary subset of X and if $E \in \mathcal{A}$ has finite measure then E is countable and therefore $A \cap E$ is also countable, so that $A \cap E \in \mathcal{A}$. Now consider the canonical full extension $\mu_e : \wp(X) \to [0, +\infty]$ of μ. We claim that the set $X \setminus S$ is an infinite block for μ_e. Namely, since $X \setminus S$ is not in $\mathcal{A}$, we have $\mu_e(X \setminus S) = +\infty$. Now, given a subset A of $X \setminus S$, if A is not countable then A cannot be in $\mathcal{A}$ (since the complement of A in X contains S and therefore cannot be countable), so that $\mu_e(A) = +\infty$. If A is countable then $A \in \mathcal{A}$ and since $A \cap S$ is empty, we have $\mu_e(A) = \mu(A) = 0$.

Example 7.1 illustrates the fact that the canonical full extension of a (complete) block-free measure is not in general block-free. It also illustrates the fact that the block-free version of the canonical full extension of a measure is not in general the same as the canonical full extension of the block-free version of that measure, i.e., $(\mu_{\mathrm{bf}})_e \neq (\mu_e)_{\mathrm{bf}}$ (even when both $(\mu_{\mathrm{bf}})_e$ and $(\mu_e)_{\mathrm{bf}}$ are defined in the same σ-algebra).

Example 7.2. Let Λ be an arbitrary uncountable set and set $X = [0, 1] \times \Lambda$. Consider the σ-algebra $\mathcal{A}$ consisting of all subsets A of X such that the λ-th line:

$$A^\lambda = \big\{x \in [0, 1] : (x, \lambda) \in A\big\}$$

is a Borel subset of $[0, 1]$, for all $\lambda \in \Lambda$. Define a measure $\mu : \mathcal{A} \to [0, +\infty]$ as follows: if $A \in \mathcal{A}$ has an uncountable number of nonempty lines (i.e., if the set of those $\lambda \in \Lambda$ with $A^\lambda \neq \emptyset$ is uncountable) set $\mu(A) = +\infty$; otherwise, set:

$$\mu(A) = \sum_{\lambda \in \Lambda} \mathfrak{m}(A^\lambda),$$

where $\mathfrak{m}$ denotes Lebesgue measure. The measure μ is not block-free; namely, if $A \in \mathcal{A}$ has an uncountable number of nonempty lines and if all the lines of A have zero Lebesgue measure then A is an infinite block for μ. The measure μ is full; namely, given $A \in \mathcal{A}_e$ then for every $\lambda \in \Lambda$ the set $[0, 1] \times \{\lambda\} \in \mathcal{A}$ has finite measure and therefore:

$$A \cap \big([0, 1] \times \{\lambda\}\big) \in \mathcal{A}.$$

But the λ-th line of $A \cap \big([0, 1] \times \{\lambda\}\big)$ is equal to the λ-th line A^λ of A and therefore A^λ is a Borel subset of $[0, 1]$ for all $\lambda \in \Lambda$, i.e., $A \in \mathcal{A}$. The measure μ is not complete and the completion $\bar{\mu} : \overline{\mathcal{A}} \to [0, +\infty]$ of μ is defined in the σ-algebra $\overline{\mathcal{A}}$ consisting of those subsets $A \subset X$ such that A^λ is a Lebesgue measurable subset of $[0, 1]$ for all $\lambda \in \Lambda$ and such that the set:

$$\big\{\lambda \in \Lambda : A^\lambda \text{ is not Borel}\big\} \tag{7.1}$$

is countable. Namely, if $A \in \overline{\mathcal{A}}$ then $A = B \cup N$, with $N \subset M$, $B, M \in \mathcal{A}$ and $\mu(M) = 0$. For all $\lambda \in \Lambda$, $A^\lambda = B^\lambda \cup N^\lambda$, with $N^\lambda \subset M^\lambda$, B^λ, M^λ are Borel sets and M^λ has null Lebesgue measure, so that A^λ is Lebesgue measurable. Moreover, the set (7.1) is contained in the set:

$$\big\{\lambda \in \Lambda : M^\lambda \neq \emptyset\big\}, \tag{7.2}$$

which is countable, since $\mu(M)$ is finite. Conversely, if all the lines of A are Lebesgue measurable and if the set (7.1) is countable then one can assemble subsets B, N, M of X such that, for all $\lambda \in \Lambda$, $A^\lambda = B^\lambda \cup N^\lambda$, $N^\lambda \subset M^\lambda$, B^λ, M^λ are Borel subsets of $[0,1]$, M^λ has null Lebesgue measure and it is empty whenever A^λ is Borel. Then $A = B \cup N$, $N \subset M$, and B, M are in $\mathcal{A}$. The set (7.2) is equal to the set (7.1), so that the set (7.2) is countable, $\mu(M) = 0$ and A is in $\overline{\mathcal{A}}$. We claim that the measure $\bar{\mu}$ is not full. First, observe that if $E \in \overline{\mathcal{A}}$ is such that $\bar{\mu}(E) < +\infty$ then E has a countable number of nonempty lines; namely, if E has an uncountable number of nonempty lines, let E_0 contain exactly one point from each line of E. Then $E_0 \in \mathcal{A}$, $E_0 \subset E$ and $\bar{\mu}(E_0) = \mu(E_0) = +\infty$, so that $\bar{\mu}(E) = +\infty$. Now let S be a Lebesgue measurable subset of $[0,1]$ that is not Borel. The set $A = S \times \Lambda$ is not in $\overline{\mathcal{A}}$. We claim that A is in $\big(\overline{\mathcal{A}}\big)_{\mathrm{e}}$. Namely, if $E \in \overline{\mathcal{A}}$ is such that $\bar{\mu}(E) < +\infty$ then E has only a countable number of nonempty lines and therefore $A \cap E$ has only a countable number of lines that are not Borel (since it has only a countable number of nonempty lines). Moreover, all the lines of $A \cap E$ are Lebesgue measurable and hence $A \cap E$ is in $\overline{\mathcal{A}}$.

Example 7.2 illustrates the fact that the completion of a full measure may not be full and that the canonical full extension of the completion of a measure may not coincide with the completion of the canonical full extension of that measure, i.e., it can happen that $(\bar{\mu})_{\mathrm{e}} \ne \overline{(\mu_{\mathrm{e}})}$.

For the examples that follow we consider the following setup. Let $(Y, \mathcal{B}, \nu)$ be a measure space with ν a non zero finite measure and let $\mathcal{B}'$ be a σ-algebra of subsets of Y contained in $\mathcal{B}$. Let Λ be an arbitrary uncountable set and set $X = Y \times \Lambda$. Let $\mathcal{A}$ be the σ-algebra of subsets of X consisting of those $A \subset X$ such that the λ-th line:

$$A^\lambda = \big\{ y \in Y : (y, \lambda) \in A \big\}$$

is in $\mathcal{B}$, for all $\lambda \in \Lambda$. In $\mathcal{A}$ we define a measure $\mu : \mathcal{A} \to [0, +\infty]$ by setting:

$$\mu(A) = \sum_{\lambda \in \Lambda} \nu(A^\lambda),$$

for all $A \in \mathcal{A}$. Notice that $(X, \mathcal{A}, \mu)$ is isomorphic to the external sum:

$$\sum_{\lambda \in \Lambda} (Y, \mathcal{B}, \nu);$$

since ν (being finite) is full and block-free, it follows from Lemma 6.3 that μ is also full and block-free and that it is perfect if ν is complete. Moreover, it follows from Remark 6.6 that the $(\infty, 1)$-Riesz map for the space $(X, \mathcal{A}, \mu)$ is an isometry. Let $\mathcal{A}'$ be the σ-algebra of subsets of X consisting of those $A \in \mathcal{A}$ such that the set:

$$\big\{ \lambda \in \Lambda : A^\lambda \notin \mathcal{B}' \big\} \tag{7.3}$$

is countable. Denote by μ' the restriction of μ to $\mathcal{A}'$. The measure μ' is block-free, since given $A \in \mathcal{A}'$ with $\mu'(A) = +\infty$, there exists $\lambda \in \Lambda$ with

$\nu(A^\lambda) > 0$, so that $A^\lambda \times \{\lambda\}$ is a subset of A with $A^\lambda \times \{\lambda\} \in \mathcal{A}'$ and $\mu'\big(A^\lambda \times \{\lambda\}\big) = \nu(A^\lambda) \in \,]0, +\infty[$. We claim that:

$$(\mathcal{A}')_{\mathrm{e}} \subset \mathcal{A}, \tag{7.4}$$

i.e., the σ-algebra $(\mathcal{A}')_{\mathrm{e}}$ in which the canonical full extension of μ' is defined is contained in $\mathcal{A}$. Namely, given $A \in (\mathcal{A}')_{\mathrm{e}}$, then for each $\lambda \in \Lambda$ the set $Y \times \{\lambda\}$ is in $\mathcal{A}'$ and $\mu'\big(Y \times \{\lambda\}\big) = \nu(Y) < +\infty$, so that $A \cap \big(Y \times \{\lambda\}\big)$ is in $\mathcal{A}'$ and in particular the λ-th line of $A \cap \big(Y \times \{\lambda\}\big)$ is in $\mathcal{B}$. But the λ-th line of $A \cap \big(Y \times \{\lambda\}\big)$ is A^λ and this proves that A is in $\mathcal{A}$.

We also observe that both for μ and μ' the family $\big(Y \times \{\lambda\}\big)_{\lambda \in \Lambda}$ is a decomposition.

In each of the following two examples we consider the set up just described and we make a few extra assumptions about the space $(Y, \mathcal{B}, \nu)$ and the σ-algebra $\mathcal{B}'$.

Example 7.3. Assume that the restriction of ν to $\mathcal{B}'$ is complete. We claim that:

$$(\mathcal{A}')_{\mathrm{e}} = \mathcal{A}.$$

Namely, by (7.4) it suffices to check that $\mathcal{A} \subset (\mathcal{A}')_{\mathrm{e}}$. Let $A \in \mathcal{A}$ and let $E \in \mathcal{A}'$ be such that:

$$\mu'(E) = \mu(E) = \sum_{\lambda \in \Lambda} \nu(E^\lambda) < +\infty.$$

Obviously, $A \cap E \in \mathcal{A}$. If $\lambda \in \Lambda$ does not belong to the countable set:

$$\big\{\lambda \in \Lambda : \nu(E^\lambda) > 0\big\} \cup \big\{\lambda \in \Lambda : E^\lambda \not\in \mathcal{B}'\big\},$$

then $(A \cap E)^\lambda = A^\lambda \cap E^\lambda$ is a subset of E^λ, where $E^\lambda \in \mathcal{B}'$ and $\nu(E^\lambda) = 0$. The completeness of $\nu|_{\mathcal{B}'}$ thus implies that $(A \cap E)^\lambda$ is in $\mathcal{B}'$, so that $A \cap E$ is in $\mathcal{A}'$ and A is in $(\mathcal{A}')_{\mathrm{e}}$. Notice that, if $\mathcal{B}'$ is not $\mathcal{B}$ itself, then also $\mathcal{A}'$ is not $\mathcal{A}$ itself and we have thus proven that the measure μ' is not full.

If the measure ν is complete then the measure μ' is also complete; namely, given $A \in \mathcal{A}'$ with $\mu'(A) = \mu(A) = 0$ and given B contained in A then B is in $\mathcal{A}$ (as remarked earlier, the completeness of ν implies the completeness of μ) and, by the completeness of $\nu|_{\mathcal{B}'}$ and from the fact that $\nu(A^\lambda) = 0$ for all $\lambda \in \Lambda$, the set:

$$\big\{\lambda \in \Lambda : B^\lambda \not\in \mathcal{B}'\big\}$$

is contained in the countable set:

$$\big\{\lambda \in \Lambda : A^\lambda \not\in \mathcal{B}'\big\},$$

so that B is in $\mathcal{A}'$. Now assume that the following condition holds:

($\bullet$) for every $U \in \mathcal{B}$, there exists $U_1 \in \mathcal{B}'$ such that $\nu(U \mathbin{\triangle} U_1) = 0$.

Here $U \mathbin{\triangle} U_1$ denotes the *symmetric difference* $(U \setminus U_1) \cup (U_1 \setminus U)$. It follows easily from condition ($\bullet$) that for all $A \in \mathcal{A}$ there exists $A_1 \in \mathcal{A}'$ such that $\mu(A \mathbin{\triangle} A_1) = 0$; namely, simply assemble A_1 in such a way that, for each $\lambda \in \Lambda$, the λ-th line A_1^λ of A_1 is in $\mathcal{B}'$ and such that $\nu(A^\lambda \mathbin{\triangle} A_1^\lambda) = 0$. Now, a standard argument using limits of simple functions shows that given a map $f : X \to I\!R$ that is measurable with respect to $\mathcal{A}$ there exists a map $f_1 : X \to I\!R$ that is measurable with respect to $\mathcal{A}'$ and such that $f = f_1$ μ-almost everywhere. Thus, just like in the proof of Lemma 3.1, it follows that the inclusion map of $\mathcal{M}(X, \mathcal{A}')$ in $\mathcal{M}(X, \mathcal{A})$ induces a linear isometry:

$$L^p(X, \mathcal{A}', \mu') \longrightarrow L^p(X, \mathcal{A}, \mu),$$

for all $p \in [1, +\infty]$. As remarked earlier, the $(\infty, 1)$-Riesz map of the space $(X, \mathcal{A}, \mu)$ is an isometry and, using a commutative diagram:

$$\begin{tikzcd}
L^1(X, \mathcal{A}', \mu')^* & L^1(X, \mathcal{A}, \mu)^* \arrow[l, "\cong"'] \\
L^\infty(X, \mathcal{A}', \mu') \arrow[u] \arrow[r, "\cong"] & L^\infty(X, \mathcal{A}, \mu) \arrow[u]
\end{tikzcd}$$

we conclude that the $(\infty, 1)$-Riesz map of the space $(X, \mathcal{A}', \mu')$ is also an isometry.

Concrete examples in which both ν and $\nu|_{\mathcal{B}'}$ are complete, $\mathcal{B}' \neq \mathcal{B}$, and in which condition $(\bullet)$ is satisfied are not hard to find. One possibility is the following: assume Y is uncountable, let $\mathcal{B}$ consist of sets that are either countable or have countable complement in Y, let ν give measure zero to countable sets and measure 1 to sets having countable complement and let $\mathcal{B}' = \{\emptyset, Y\}$. Another possibility is the following: let κ_1, κ_2 be infinite cardinals with $\kappa_1 < \kappa_2$, assume that the cardinality $|Y|$ of Y is greater than κ_2, let $\mathcal{B}$ consist of sets B with $|B| \le \kappa_2$ or $|Y \setminus B| \le \kappa_2$, ν be such that $\nu(B) = 0$ if $|B| \le \kappa_2$, $\nu(B) = 1$ if $|Y \setminus B| \le \kappa_2$, and $\mathcal{B}'$ consist of sets B with $|B| \le \kappa_1$ or $|Y \setminus B| \le \kappa_1$.

Example 7.3 illustrates the fact that a measure need not be full in order for its $(\infty, 1)$-Riesz map to be an isometry. This might look a little surprising, since the $(\infty, 1)$-Riesz map corresponding to the canonical full extension of a measure is an extension of the $(\infty, 1)$-Riesz map corresponding to the original measure. So, if the $(\infty, 1)$-Riesz map of the original measure is an isometry, what happens to its extension? It turns out that such extension is not injective (and the canonical full extension of the original measure is not block-free).

Example 7.4. Let us now assume that the measure space $(Y, \mathcal{B}, \nu)$ and the σ-algebra $\mathcal{B}'$ satisfy the following condition:

- $(\bullet\bullet)$ for every $U \in \mathcal{B} \setminus \mathcal{B}'$, there exists $U_0 \in \mathcal{B}'$ with $\nu(U_0) = 0$ and $U \cap U_0 \notin \mathcal{B}'$.

Let us show that the measure μ' is full. Let $A \in (\mathcal{A}')_{\mathrm{e}}$ be given. We know (see (7.4)) that A is in $\mathcal{A}$. Let us show that the set (7.3) is countable. Assemble a set $E \subset X$ satisfying the following property: for each $\lambda \in \Lambda$, if A^λ is in $\mathcal{B}'$, set $E^\lambda = \emptyset$; if A^λ is not in $\mathcal{B}'$, use $(\bullet\bullet)$ in order to find $E^\lambda \in \mathcal{B}'$

with $\nu(E^\lambda)=0$ and $A^\lambda\cap E^\lambda\not\in\mathcal{B}'$. Then E is in $\mathcal{A}'$ and $\mu'(E)=0$; since A is in $(\mathcal{A}')_{\mathrm{e}}$, we must have $A\cap E\in\mathcal{A}'$ and therefore the set:

$$\big\{\lambda\in\Lambda:(A\cap E)^\lambda=A^\lambda\cap E^\lambda\not\in\mathcal{B}'\big\} \tag{7.5}$$

must be countable. But, from our construction, the set (7.5) coincides with the set (7.3).

Denote by $\overline{\mathcal{B}'}$ the σ-algebra in which the completion of the measure $\nu|_{\mathcal{B}'}$ is defined. Assume that:

(i) $\mathcal{B}$ is not contained in $\overline{\mathcal{B}'}$;
(ii) every $U\in\mathcal{B}$ with $\nu(U)=0$ is in $\overline{\mathcal{B}'}$.

We will show that the $(\infty,1)$-Riesz map of the measure space $(X,\mathcal{A}',\mu')$ is not surjective. Let $S\in\mathcal{B}$ be such that $S\not\in\overline{\mathcal{B}'}$. Define a measure ρ in the σ-algebra $\mathcal{A}'$ by setting:

$$\rho(A)=\sum_{\lambda\in\Lambda}\nu(A^\lambda\cap S),$$

for all $A\in\mathcal{A}'$. Since $\rho(A)\le\mu'(A)$ for all $A\in\mathcal{A}'$, the map:

$$\alpha:L^1(X,\mathcal{A}',\mu')\ni f\longmapsto\int_X f\,\mathrm{d}\rho\in I\!R$$

is a bounded linear functional with $\|\alpha\|\le1$. Assume by contradiction that there exists $g\in L^\infty(X,\mathcal{A}',\mu')$ such that:

$$\alpha(f)=\alpha_g(f)\stackrel{\text{def}}{=}\int_X fg\,\mathrm{d}\mu',$$

for all $f\in L^1(X,\mathcal{A}',\mu')$. For each $\lambda\in\Lambda$, we define maps[1]:

$$\mathfrak{i}_\lambda:L^1(Y,\mathcal{B},\nu)\longrightarrow L^1(X,\mathcal{A}',\mu'),$$
$$\mathfrak{r}_\lambda:L^\infty(X,\mathcal{A}',\mu')\longrightarrow L^\infty(Y,\mathcal{B},\nu),$$

as follows; $\mathfrak{r}_\lambda$ is just given by composition on the right with the measurable map:

$$(Y,\mathcal{B},\nu)\ni y\longmapsto(y,\lambda)\in(X,\mathcal{A}',\mu').$$

The map $\mathfrak{i}_\lambda$ carries $f\in L^1(Y,\mathcal{B},\nu)$ to the map $\mathfrak{i}_\lambda(f)\in L^1(X,\mathcal{A}',\mu')$ that vanishes outside the line $Y\times\{\lambda\}$ and such that $\mathfrak{i}_\lambda(f)(y,\lambda)=f(y)$, for all $y\in Y$. It is easily seen that the diagram:

$$\begin{tikzcd} L^1(Y,\mathcal{B},\nu)^* & L^1(X,\mathcal{A}',\mu')^* \arrow[l, "\mathfrak{i}_\lambda^*"'] \\ L^\infty(Y,\mathcal{B},\nu) \arrow[u] & L^\infty(X,\mathcal{A}',\mu') \arrow[l, "\mathfrak{r}_\lambda"] \arrow[u] \end{tikzcd} \tag{7.6}$$

[1]A more informal (perhaps easier to understand) description of what we are about to do is the following: identify $(Y,\mathcal{B},\nu)$ with the λ-th line $Y\times\{\lambda\}$ of $(X,\mathcal{A}',\mu')$ and identify $L^1(Y,\mathcal{B},\nu)$ with a subspace of $L^1(X,\mathcal{A}',\mu')$ as explained in Remark 4.2. Under such identifications, the map $\mathfrak{i}_\lambda$ is just inclusion and the map $\mathfrak{r}_\lambda$ takes L^∞ maps defined in X to their restriction to the λ-th line.

in which the vertical arrows are $(\infty, 1)$-Riesz maps is commutative. The map $\mathfrak{r}_\lambda$ carries $g \in L^\infty(X, \mathcal{A}', \mu')$ to the map:

$$\mathfrak{r}_\lambda(g) : Y \ni y \longmapsto g(y, \lambda) \in I\!R \tag{7.7}$$

which is in $L^\infty(Y, \mathcal{B}, \nu)$. We claim that the map (7.7) is ν-almost everywhere equal to the characteristic function χ_S. Since $\alpha = \alpha_g$, the commutativity of diagram (7.6) implies that the linear functional:

$$\alpha \circ \mathfrak{i}_\lambda = \alpha_g \circ \mathfrak{i}_\lambda = \mathfrak{i}_\lambda^*(\alpha_g) \in L^1(Y, \mathcal{B}, \nu)^*$$

corresponds by the $(\infty, 1)$-Riesz map of the space $(Y, \mathcal{B}, \nu)$ to the map (7.7). But it is easy to see that:

$$(\alpha \circ \mathfrak{i}_\lambda)(f) = \int_S f \, \mathrm{d}\nu = \int_Y f \chi_S \, \mathrm{d}\nu,$$

for all $f \in L^1(Y, \mathcal{B}, \nu)$, so that the $(\infty, 1)$-Riesz map of the space $(Y, \mathcal{B}, \nu)$ also carries χ_S to $\alpha \circ \mathfrak{i}_\lambda$. The claim that (7.7) is ν-almost everywhere equal to χ_S then follows from the injectivity of such Riesz map. Now, denoting by $g^{-1}(1)^\lambda$ the λ-th line of the set $g^{-1}(1)$ (which is the same as the set in which the map (7.7) takes the value 1), we conclude that the sets:

$$S \setminus g^{-1}(1)^\lambda, \quad g^{-1}(1)^\lambda \setminus S, \tag{7.8}$$

(are in $\mathcal{B}$ and) have measure zero with respect to ν, for all $\lambda \in \Lambda$. It follows from assumption (ii) that the sets (7.8) are in $\overline{\mathcal{B}'}$. Now, since $g^{-1}(1)$ is in $\mathcal{A}'$, there exists $\lambda \in \Lambda$ such that the line $g^{-1}(1)^\lambda$ is in $\mathcal{B}'$ and therefore:

$$S = \big[g^{-1}(1)^\lambda \setminus \big(g^{-1}(1)^\lambda \setminus S\big)\big] \cup \big(S \setminus g^{-1}(1)^\lambda\big)$$

is in $\overline{\mathcal{B}'}$, contradicting our assumptions. Thus the bounded linear functional $\alpha \in L^1(X, \mathcal{A}', \mu')^*$ is not in the image of the $(\infty, 1)$-Riesz map of the space $(X, \mathcal{A}', \mu')$.

Let us now present a concrete example of a measure space $(Y, \mathcal{B}, \nu)$ and of a σ-algebra $\mathcal{B}'$ satisfying ($\bullet\bullet$), (i) and (ii). Let:

$$Y = [0, 1] \times \{0, 1\},$$

$\mathcal{B}$ be the σ-algebra consisting of all sets of the form:

$$\big(B_0 \times \{0\}\big) \cup \big(B_1 \times \{1\}\big), \tag{7.9}$$

with $B_0, B_1 \subset [0, 1]$ Lebesgue measurable. Let the measure ν of the set (7.9) be equal to the sum of the Lebesgue measures of B_0 and B_1. Let $\mathcal{B}'$ consist of the sets of the form (7.9) with B_0, B_1 Lebesgue measurable and $B_0 = B_1$. In order to prove property ($\bullet\bullet$), let $U \in \mathcal{B} \setminus \mathcal{B}'$ be of the form (7.9); then $B_0 \neq B_1$ and there exists $x \in B_0 \bigtriangleup B_1$. The set $U_0 = \{x\} \times \{0, 1\}$ is in $\mathcal{B}'$, $\nu(U_0) = 0$ and $U \cap U_0$ is not in $\mathcal{B}'$. It is easy to see that the σ-algebra $\overline{\mathcal{B}'}$ consists of the sets of the form (7.9) with B_0, B_1 Lebesgue measurable and such that $B_0 \bigtriangleup B_1$ has Lebesgue measure equal to zero. Thus, $\overline{\mathcal{B}'}$ is properly contained in $\mathcal{B}$ and every element of $\mathcal{B}$ having measure zero is in $\overline{\mathcal{B}'}$.

Example 7.4 shows that in Proposition 6.17 the hypothesis that the measure be complete is important. Namely, the measure μ' is full, block-free, admits a decomposition, but its $(\infty,1)$-Riesz map is not an isometry. This is somewhat surprising, since by completing a measure we do not change its Riesz map. What happens here is that the completion of μ' is not full and the $(\infty,1)$-Riesz map of the perfect version $(\mu')_{\mathfrak{p}}$ of μ' is a proper extension[2] of the $(\infty,1)$-Riesz map of μ'. Clearly, the $(\infty,1)$-Riesz map of $(\mu')_{\mathfrak{p}}$ is an isometry, by Proposition 6.17, since $(\mu')_{\mathfrak{p}}$ admits a decomposition (see Remark 5.2).

Referências

[1] Royden, ...

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[2]It is easy to determine explicitly what the perfect version $(\mu')_{\mathfrak{p}}$ of μ' is. First, it follows from Remark 3.13 that the restriction of $(\mu')_{\mathfrak{p}}$ to each line $Y\times\{\lambda\}$ is (up to the obvious identification of Y with $Y\times\{\lambda\}$) the completion $\bar\nu$ of ν, which is defined in the σ-algebra $\overline{\mathcal{B}}$. Moreover, since the lines $Y\times\{\lambda\}$, $\lambda\in\Lambda$, constitute a decomposition for μ', it follows from Remark 5.2 that they also constitute a decomposition for $(\mu')_{\mathfrak{p}}$; therefore it follows from Corollary 5.6 that $(\mathcal{A}')_{\mathfrak{p}}$ consists of those sets $A\subset X$ such that A^λ is in $\overline{\mathcal{B}}$, for all $\lambda\in\Lambda$ and that $(\mu')_{\mathfrak{p}}(A)$ is equal to $\sum_{\lambda\in\Lambda}\bar\nu(A^\lambda)$. Notice that if ν is complete then $(\mu')_{\mathfrak{p}}$ coincides with μ (and, in general, it coincides with the completion of μ).