

# SOME CONSIDERATIONS ON THE MATHEMATICAL FORMALISM OF A GENERIC PHYSICAL THEORY AND ON THE MATHEMATICAL FORMALISM OF QUANTUM MECHANICS

DANIEL V. TAUSK

## 1. THE MATHEMATICAL FORMALISM OF A NON RELATIVISTIC GENERIC PHYSICAL THEORY

Let me start by explaining how I think the mathematical formalism of a generic non relativistic<sup>1</sup> and deterministic physical theory should look like.

The first ingredient should be a *phase space*  $X$  which should be thought of as the set of all possible states of some (closed) system, which I shall call the *system under study*. Mathematically,  $X$  should be a set endowed with some additional structure. A point  $x \in X$  should be thought of as a description of the state of the system at a generic instant of time<sup>2</sup>. Also, we need a rule that assigns to each nonnegative<sup>3</sup> real number  $t$  a morphism  $U_t : X \rightarrow X$  of the phase space  $X$ ; we assume that  $U_t \circ U_s = U_{t+s}$ , for all  $t, s \geq 0$  and that  $U_0$  is the identity map of  $X$ . The interpretation of  $U_t$  is the following: if the system is at a state  $x \in X$  at some instant  $t_0 \in \mathbb{R}$  then the system will be at the state  $U_t(x)$  at the instant  $t_0 + t$ . We call the family of maps  $(U_t)_{t \geq 0}$  the *evolution law*. For example, in classical mechanics,  $X$  is a symplectic manifold and  $U_t$  is the flow of a Hamiltonian vector field (i.e., the symplectic gradient of a smooth real-valued Hamiltonian function on  $X$ ).

Of course, a theory with just a phase space  $X$  and an evolution law  $(U_t)_{t \geq 0}$  cannot be considered a physical theory, since it does not say anything about results of experiments. So, we need an additional ingredient: if  $\mathcal{C}$  is

---

*Date:* January 26th, 2008.

<sup>1</sup>By *non relativistic* I mean that it is assumed the existence of an absolute (observer-independent) time flow.

<sup>2</sup>It is not assumed that human beings should have an intuitive understanding of how  $x$  describes the system. For instance, we can imagine a planet populated by blind people that study classical mechanics and vector calculus, but they consider the space  $\mathbb{R}^3$  to be a weird abstraction coined by the local mathematicians. On the other hand, it should be clear that  $x$  is to be interpreted as some sort of description of the *system itself*, not of someone's knowledge about the system.

<sup>3</sup>I will assume that  $U_t$  is defined only for nonnegative  $t$ , although in most physical theories  $U_t$  is defined for every real number  $t$ . Notice that it is conceivable, in principle, that two different states  $x, x' \in X$  be lead to the same state  $U_t(x) = U_t(x')$  after a time  $t$ , i.e., that the map  $U_t$  be not injective (but this would make the physical theory *non reversible*). Also, in some cases, the domain of  $U_t$  may not be the entire phase space  $X$ .

an *experimental arrangement*, there should be an associated map  $f_{\mathfrak{C}}$  with domain<sup>4</sup>  $X$  such that, for each  $x \in X$ ,  $f_{\mathfrak{C}}(x)$  is the result obtained from the experiment with experimental arrangement  $\mathfrak{C}$ , when the system under study is at the state  $x$ . In classical mechanics,  $f_{\mathfrak{C}}$  is usually a smooth real-valued map on the symplectic manifold  $X$ .

So far, I have considered only deterministic physical theories, but a simple adaptation allows us to consider also non deterministic theories (at least, the kind of reasonable non deterministic physical theory that can be described in terms of probabilities). We just have to replace the (deterministic) evolution law  $(U_t)_{t \geq 0}$  by a *stochastic evolution law*:

$$\mathcal{U}_t : \mathcal{P}(X) \longrightarrow \mathcal{P}(X), \quad t \geq 0,$$

where  $\mathcal{P}(X)$  denotes the set of all probability measures in (some fixed  $\sigma$ -algebra of subsets of)  $X$ ; again, we assume that  $\mathcal{U}_t \circ \mathcal{U}_s = \mathcal{U}_{t+s}$ , for all  $t, s \geq 0$  and that  $\mathcal{U}_0$  is the identity map of  $\mathcal{P}(X)$ . The interpretation of  $\mathcal{U}_t$  is the following: if at an instant  $t_0 \in \mathbb{R}$  the state of the system is  $x \in X$  then the state of the system at the instant  $t_0 + t$  is unknowable at time  $t_0$ , but it is random with probability measure  $\mathcal{U}_t(\delta_x)$ , where  $\delta_x$  denotes the probability measure concentrated at  $x$  (i.e.,  $\delta_x(\{x\}) = 1$ )<sup>5</sup>.

In the case of non deterministic physical theories, it is possible that the interaction between the system under study and the experimental apparatus of some experimental arrangement  $\mathfrak{C}$  be also non deterministic; thus, the value of the map  $f_{\mathfrak{C}}$  at a state  $x \in X$  will not be the result of the experiment, but a probability measure on the set of all possible results for the experiment. It is also conceivable that, although the evolution law  $(\mathcal{U}_t)_{t \geq 0}$  is non deterministic, the interaction of the system under study with a given experimental apparatus is so well-behaved that  $f_{\mathfrak{C}}(x)$  is really just the result of the experiment (not a probability measure). What *is not* reasonable is the other way around: a deterministic evolution law  $(U_t)_{t \geq 0}$  coupled with a stochastic (i.e., probability measure-valued) map  $f_{\mathfrak{C}}$ . Why is this not reasonable? Because we could in principle use the physical theory under

---

<sup>4</sup>The counter-domain of  $f_{\mathfrak{C}}$  is usually taken to be the set  $\mathbb{R}$  of real numbers, but it is not really important that the values of  $f_{\mathfrak{C}}$  be numbers at all;  $f_{\mathfrak{C}}(x)$  is just something that describes the result of the experiment. In fact, if we focuss on real-life experiments, it can always be assumed that the image of the map  $f_{\mathfrak{C}}$  is a finite set.

<sup>5</sup>If we start with a deterministic evolution law  $(U_t)_{t \geq 0}$  then, for each  $t \geq 0$ , we can define a map  $\mathcal{U}_t : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  by setting  $\mathcal{U}_t(P)(S) = P(U_t^{-1}(S))$ , for every  $P \in \mathcal{P}(X)$  and every (measurable) subset  $S$  of  $X$ . The study of  $(\mathcal{U}_t)_{t \geq 0}$  can be useful even in the deterministic case, since it is usually impossible for humans to have perfect knowledge about the state of a system. If the family of maps  $(\mathcal{U}_t)_{t \geq 0}$  has not arisen from a deterministic evolution law  $(U_t)_{t \geq 0}$  via such procedure then we are dealing with a truly indeterministic physical theory.

consideration to study the system consisting of both the original system under study and the experimental apparatus<sup>6</sup>; this would give us a “larger” phase space  $\bar{X}$  with (deterministic) evolution law  $(\bar{U}_t)_{t \geq 0}$ . The result of the experiment should then be completely determined by the state of the system under study  $x \in X$  and by the state of the experimental apparatus. In other words: where would the randomness in the result of the experiment come from, if the evolution law is assumed to be deterministic?<sup>7</sup>

## 2. THE STANDARD MATHEMATICAL FORMALISM OF QUANTUM MECHANICS

The standard mathematical formalism of Quantum Mechanics is precisely of that unreasonable type explained at the end of the previous section. The phase space  $X$  is a complex Hilbert space<sup>8</sup>  $\mathcal{H}$ . The evolution law  $(U_t)_{t \geq 0}$  is a one-parameter group of unitary operators, namely,  $U_t = \exp(-\frac{i}{\hbar} tH)$ , where  $H$  is a (usually unbounded) self-adjoint Hamiltonian operator on  $\mathcal{H}$ . So, the evolution law is deterministic. What about the results of experiments? Given an experimental arrangement  $\mathfrak{C}$  (of a certain type), one can associate to it a self-adjoint operator<sup>9</sup>  $A_{\mathfrak{C}}$  on  $\mathcal{H}$ . The possible results of the experiment are the eigenvalues of  $A_{\mathfrak{C}}$  and, assuming that the state of the system is a unit vector  $\psi \in \mathcal{H}$ , the probability that the result of the experiment be the eigenvalue  $\lambda \in \mathbb{R}$  is  $\|P_{\lambda}(\psi)\|^2$ , where  $P_{\lambda} : \mathcal{H} \rightarrow \mathcal{H}$  denotes the operator of orthogonal projection onto the eigenspace of  $A_{\mathfrak{C}}$  corresponding to  $\lambda$ .

There is nothing unreasonable about a non deterministic physical theory. But, as remarked at the end of the previous section, it is unreasonable to have a deterministic evolution law coupled with a stochastic map that tells the results of experiments. What is going on? It seems that the most reasonable answer is that the vector  $\psi \in \mathcal{H}$  is not a complete description of the state of the system under study.

---

<sup>6</sup>I am assuming here that the study of the experimental apparatus can be done without considering types of physical interactions that fall beyond the scope of the physical theory we are considering.

<sup>7</sup>One possible answer to this question is: the randomness comes from our lack of knowledge of the microscopic state of the experimental apparatus  $\mathfrak{C}$ . But, if this is the case, we should have a deterministic map  $f_{\mathfrak{C}}$  at least in the idealized situation where the experimental apparatus is described down to the microscopic level.

<sup>8</sup>To be more precise,  $X$  should be the projectivization  $(\mathcal{H} \setminus \{0\})/\mathbb{C}^*$  of the Hilbert space  $\mathcal{H}$ . Namely, a state  $\psi \in \mathcal{H} \setminus \{0\}$  is identical to the state  $c\psi$  if  $c$  is any nonzero complex number.

<sup>9</sup>I will assume that  $A_{\mathfrak{C}}$  (is bounded and) has finite spectrum (see footnote 4). In this case, the spectrum of  $A_{\mathfrak{C}}$  consists only of eigenvalues and the Hilbert space  $\mathcal{H}$  is the orthogonal direct sum of the corresponding eigenspaces.

## 3. THE MATHEMATICAL FORMALISM OF BOHM'S THEORY

In Bohm's theory, the phase space corresponding to a system of  $n$  particles is the cartesian product:

$$X = \mathbb{R}^{3n} \times \mathcal{H},$$

where  $\mathcal{H}$  is the complex Hilbert space<sup>10</sup>  $L^2(\mathbb{R}^{3n})$  of complex-valued square integrable maps on  $\mathbb{R}^{3n}$ . An element of  $X$  is an ordered pair  $(q, \psi)$ , where  $q = (q_1, \dots, q_n)$  is an  $n$ -tuple of elements of  $\mathbb{R}^3$  (the positions of the particles) and  $\psi \in \mathcal{H}$  is the wave function of the standard quantum formalism. The evolution law  $U_t : X \rightarrow X$  is defined as follows: given  $(q_0, \psi_0) \in X$  and setting  $(q(t), \psi(t)) = U_t(q_0, \psi_0)$  then  $\psi(t) = \exp(-\frac{i}{\hbar} tH) \cdot \psi_0$  and  $q$  is the solution of the Cauchy problem (I am omitting the masses of the particles<sup>11</sup>):

$$(3.1) \quad q'(t) = \hbar \Im \left( \frac{\nabla(\psi(t))(q(t))}{\psi(t)(q(t))} \right), \quad q(0) = q_0,$$

where  $H$  denotes a self-adjoint Hamiltonian operator on  $\mathcal{H}$ ,  $\Im(z)$  denotes the imaginary part of a complex number  $z$  and  $\nabla\psi$  denotes the gradient of a map  $\psi$ .

Now, to an experimental arrangement  $\mathfrak{C}$ , one can associate a (deterministic!) map  $f_{\mathfrak{C}} : X \rightarrow \mathbb{R}$ . For some experimental arrangements, there exists also a self-adjoint operator  $A_{\mathfrak{C}}$  on  $\mathcal{H}$  with the following property: given a unit vector  $\psi \in \mathcal{H}$  and a real number  $\lambda$ , the integral of  $|\psi|^2$  over the set:

$$(3.2) \quad \{q \in \mathbb{R}^{3n} : f_{\mathfrak{C}}(q, \psi) = \lambda\}$$

is equal to  $\|P_{\lambda}(\psi)\|^2$ , where, as before,  $P_{\lambda} : \mathcal{H} \rightarrow \mathcal{H}$  denotes the operator of orthogonal projection onto the eigenspace of  $A_{\mathfrak{C}}$  corresponding to  $\lambda$  (set  $P_{\lambda} = 0$  if  $\lambda$  is not an eigenvalue of  $A_{\mathfrak{C}}$ ). Such integral is, of course, the probability of the set (3.2) with respect to the probability measure whose density is  $|\psi|^2$ . So, in this formalism, the randomness of the result of the experiment is only apparent: loosely speaking, it reflects ignorance on the value of  $q$ . The unreasonable aspect of the standard quantum formalism has disappeared.

3.1. *Remark.* It is possible that two different experimental arrangements  $\mathfrak{C}$ ,  $\mathfrak{C}'$  (with different associated functions  $f_{\mathfrak{C}}$ ,  $f_{\mathfrak{C}'}$ ) correspond to the same self-adjoint operator  $A_{\mathfrak{C}} = A_{\mathfrak{C}'}$ . Therefore it is not possible to talk about the "value of the observable associated to the self-adjoint operator  $A$  when the system is at the state  $(q, \psi) \in X$ " (such value would be  $f_{\mathfrak{C}}(q, \psi)$  for the

<sup>10</sup>Again, it would be more accurate to consider the projectivization of the Hilbert space (see footnote 8).

<sup>11</sup>Alternatively, we can consider the gradient in (3.1) to be relative to a Riemannian metric that is scaled according to the masses of the particles. Observe that in a mathematically precise formulation we must impose some suitable regularity conditions on  $\psi_0$  to guarantee that the Cauchy problem (3.1) satisfies the hypotheses of some theorem that establishes existence and uniqueness for solutions of first order ordinary differential equations.

experimental arrangement  $\mathfrak{C}$ , but  $f_{\mathfrak{C}'}(q, \psi)$  for the experimental arrangement  $\mathfrak{C}'$ ). This explains how Bohm's theory escapes from the so called *no hidden variables theorems*.

3.2. *Remark.* Another attractive aspect of the mathematical formalism of Bohm's theory is the possibility of defining a *forgetful map* that receives the state of a system and returns the state of a subsystem (forgetting about the rest of the system). In classical mechanics, the phase space of a system of  $n$  particles (with no constraints) is  $\mathbb{R}^{3n} \times \mathbb{R}^{3n}$ ; given  $k \leq n$ , we have a projection map:

$$\mathbb{R}^{3n} \times \mathbb{R}^{3n} \longrightarrow \mathbb{R}^{3k} \times \mathbb{R}^{3k}$$

that forgets about the positions and momentums of  $n-k$  of those  $n$  particles. In the standard quantum formalism, we can't define a forgetful map:

$$L^2(\mathbb{R}^{3n}) \longrightarrow L^2(\mathbb{R}^{3k})$$

on the entire phase space  $L^2(\mathbb{R}^{3n})$ . Such a forgetful map can be defined only in the set:

$$(3.3) \quad \{\psi \otimes \tilde{\psi} : \psi \in L^2(\mathbb{R}^{3k}), \tilde{\psi} \in L^2(\mathbb{R}^{3(n-k)})\} \subset L^2(\mathbb{R}^{3n}),$$

where:

$$\psi \otimes \tilde{\psi} : \mathbb{R}^{3n} \cong \mathbb{R}^{3k} \times \mathbb{R}^{3(n-k)} \ni (q, \tilde{q}) \longmapsto \psi(q)\tilde{\psi}(\tilde{q}) \in \mathbb{C};$$

the forgetful map takes<sup>12</sup>  $\psi \otimes \tilde{\psi}$  to  $\psi$ . On the other hand, in Bohm's theory, we have a forgetful map defined by<sup>13</sup>:

$$\mathbb{R}^{3n} \times L^2(\mathbb{R}^{3n}) \ni ((q, \tilde{q}), \Psi) \longmapsto (q, \Psi(\cdot, \tilde{q})) \in \mathbb{R}^{3k} \times L^2(\mathbb{R}^{3k}),$$

where  $(q, \tilde{q}) \in \mathbb{R}^{3k} \times \mathbb{R}^{3(n-k)} \cong \mathbb{R}^{3n}$ . What is so great about a forgetful map? It allows us to study the effect of the experimental apparatus on the system under study: if  $X$  denotes the phase space corresponding to the system under study and  $\bar{X}$  denotes the ("larger") phase space that describes both the system under study and some experimental apparatus then we can use the evolution law  $(\bar{U}_t)_{t \geq 0}$  in  $\bar{X}$  to find out the final state  $\bar{x} \in \bar{X}$  of the pair consisting of the system under study and the experimental apparatus; then we can apply the forgetful map  $\bar{X} \rightarrow X$  to  $\bar{x}$  to obtain the state  $x \in X$  of the system under study after the experiment. In the standard mathematical formalism of Quantum Mechanics, the state  $\bar{x}$  is usually outside the domain (3.3) of the forgetful map (i.e., the system under study and the experimental apparatus are *entangled*), so we can't go back<sup>14</sup> to  $x \in X$ .

<sup>12</sup>Careful readers will notice that the map  $\psi \otimes \tilde{\psi} \mapsto \psi$  is not well-defined, because if  $c$  is a nonzero complex number then  $(c\psi) \otimes (\frac{1}{c}\tilde{\psi}) = \psi \otimes \tilde{\psi}$ . But this is not really a problem, because we should actually be working with the projectivization of the Hilbert spaces (see footnote 8).

<sup>13</sup>To be precise, if  $\Psi \in L^2(\mathbb{R}^{3n})$ , it may not be the case that  $\Psi(\cdot, \tilde{q})$  is in  $L^2(\mathbb{R}^{3k})$  for all  $\tilde{q} \in \mathbb{R}^{3(n-k)}$ , but by Fubini's theorem, we have  $\Psi(\cdot, \tilde{q}) \in L^2(\mathbb{R}^{3k})$  for *almost every*  $\tilde{q} \in \mathbb{R}^{3(n-k)}$ .

<sup>14</sup>It is for this reason that the standard mathematical formalism of Quantum Mechanics needs the so called *projection postulate* that tells us what happens to the system under

---

study after the experimental procedure is completed. The projection postulate makes the standard mathematical formalism of Quantum Mechanics weird, because no one can explain precisely what types of physical processes make the system abandon the standard evolution law  $(U_t)_{t \geq 0}$  and evolve according to the projection postulate.