A SCHEFFÉ-TYPE METHOD FOR MULTIPLE COMPARISONS

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Let V be a real finite dimensional vector space and assume that a parameter θ of some statistical model belongs to V. Moreover, assume that we have an unbiased normal estimator $\hat{\theta}$ for θ, i.e., $\hat{\theta}$ is a Gaussian V-valued random variable with expectation equal to θ. Let the covariance matrix of $\hat{\theta}$ be equal to $\sigma^2\Sigma$, where σ is a (possibly unknown) positive real number and $\Sigma \in V \otimes V$ is known and nondegenerate. In case σ is unknown, we assume that we have access to an estimator $\hat{\sigma} \geq 0$ for σ such that

$$m\,\frac{\hat{\sigma}^2}{\sigma^2} \sim \chi^2(m),$$

for some positive integer m. Moreover, we assume that $\hat{\sigma}$ is independent of $\hat{\theta}$.

We will regard Σ as an inner product on the dual space V^* and we will denote by Σ^{-1} the induced inner product on V. The norms induced by both inner products will be denoted by $\|\cdot\|$. If $\alpha \in V^*$ is a nonzero linear functional, then

$$\frac{\alpha(\hat{\theta}) - \alpha(\theta)}{\sigma\|\alpha\|} \sim N(0,1), \quad \frac{\alpha(\hat{\theta}) - \alpha(\theta)}{\hat{\sigma}\|\alpha\|} \sim t(m)$$

and these facts can be used to construct confidence intervals for $\alpha(\theta)$. Our goal is to construct simultaneous confidence intervals for all $\alpha(\theta)$ with α ranging over some subset Λ of V^*. We will consider only the simplest case in which Λ consists of all vectors of some subspace of V^*. Note that the general case can be reduced to this case by replacing Λ with its linear span, though in many cases such replacement will make our choice of simultaneous confidence intervals suboptimal.

Since every subspace of V^* is the annihilator W^{o} of some subspace W of V, we set $\Lambda = W^{\mathrm{o}}$, for some proper subspace W of V. We will make use of the following elementary results.

Lemma 1. *Let V be a real finite dimensional vector space endowed with an inner product $\langle\cdot,\cdot\rangle$ and let the dual space V^* be endowed with the induced inner product. Denote the corresponding norms both by $\|\cdot\|$. If W is a proper subspace of V then for every $v \in V$, the supremum of*

$$\big\{|\alpha(v)| : \alpha \in W^{\mathrm{o}},\ \|\alpha\| = 1\big\} \tag{1}$$

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is equal to the norm of the orthogonal projection of v in $W^\perp$.

Proof. Simply note that the set (1) is equal to:

$$\big\{\big|\langle z, P(v)\rangle\big| : z \in W^\perp,\ \|z\| = 1\big\},$$

where $P : V \to W^\perp$ denotes the orthogonal projection onto W. □

Lemma 2. *Let V be a real finite dimensional vector space and let X be a Gaussian V-valued random variable with mean zero and nondegenerate covariance matrix $\Sigma \in V \otimes V$. We regard Σ as an inner product on V^* and we endow V with the induced inner product Σ^{-1} and the corresponding norm $\|\cdot\|$. Let U be a subspace of V and let $P : V \to U$ denote the orthogonal projection onto U with respect to Σ^{-1}. If $r = \dim(U)$ then $\|P(X)\|^2 \sim \chi^2(r)$.*

Proof. Pick an orthonormal basis $(e_i)_{i=1}^n$ of V such that $(e_i)_{i=1}^r$ is a basis of U and denote by X_i, $i = 1, \ldots, n$ the coordinates of X, i.e., $X = \sum_{i=1}^n X_i e_i$. To conclude the proof, note that X_i, $i = 1, \ldots, n$ are independent standard normal and that:

$$\|P(X)\|^2 = \sum_{i=1}^r X_i^2. \qquad \square$$

In what follows we denote by $P : V \to W^\perp$ the orthogonal projection onto $W^\perp$ with respect to the inner product Σ^{-1} and we set:

$$r = \dim(V) - \dim(W) = \dim(W^\perp).$$

It follows from Lemma 1 that the supremum of

$$\frac{|\alpha(\hat\theta) - \alpha(\theta)|}{\sigma\|\alpha\|}$$

with α ranging over $W^{\mathrm{o}} \setminus \{0\}$ is equal to

$$\frac{\|P(\hat\theta - \theta)\|}{\sigma}.$$

Applying now Lemma 2 with $X = \frac{\hat\theta - \theta}{\sigma}$ we obtain that:

$$\frac{\|P(\hat\theta - \theta)\|^2}{\sigma^2} \sim \chi^2(r). \tag{2}$$

These observations yield the following method for constructing simultaneous confidence intervals in case σ is known.

Proposition 3. *Let $\gamma \in [0, 1[$ and pick $q \in [0, +\infty[$ such that the probability of $\chi^2(r) \le q$ is γ. We have that the interval with center $\alpha(\hat\theta)$ and radius $\sigma\sqrt{q}\,\|\alpha\|$ is a simultaneous γ-confidence interval for $\alpha(\theta)$ for all $\alpha \in W^{\mathrm{o}}$.* □

We deal with the case of unknown σ in a similar way, noting first that Lemma 1 yields that the supremum of

$$\frac{|\alpha(\hat\theta) - \alpha(\theta)|}{\hat\sigma\|\alpha\|}$$

with α ranging over $W^{\mathrm{o}} \setminus \{0\}$ is equal to

$$\frac{\|P(\hat{\theta} - \theta)\|}{\hat{\sigma}}$$

and again Lemma 2 gives us (2). Hence

$$\frac{\|P(\hat{\theta} - \theta)\|^2}{r\hat{\sigma}^2} \sim F(r, m)$$

and we obtain the following result.

Proposition 4. *Let $\gamma \in [0, 1[$ and pick $f \in [0, +\infty[$ such that the probability of $F(r, m) \leq f$ is γ. We have that the interval with center $\alpha(\hat{\theta})$ and radius $\hat{\sigma}\sqrt{rf}\,\|\alpha\|$ is a simultaneous γ-confidence interval for $\alpha(\theta)$ for all $\alpha \in W^{\mathrm{o}}$.*

□

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