BANACH MANIFOLD STRUCTURE FOR GENERAL SETS OF MAPS

DANIEL VICTOR TAUSK

ABSTRACT. We present a method for introducing a Banach manifold structure on sets of maps $f : \Omega \to M$, where Ω is a set and M is a manifold. This new approach generalizes and simplifies the classical work developed on [4] and also the more recent work [5].

1. INTRODUCTION

The introduction of infinite-dimensional manifold structures on sets of maps is the foundation of modern calculus of variations and global analysis. It also plays the central role in concrete applications of infinite-dimensional Morse theory and general critical point theory.

In this paper we give conditions under which sets of maps can be endowed with Banach manifold structures. More specifically, we consider the following setup. Let Ω be an *arbitrary set* and assume that we are given a rule $\mathfrak{M}$ that assigns to each smooth manifold M a set $\mathfrak{M}(\Omega, M)$ of maps $f : \Omega \to M$ and a topology on the set $\mathfrak{M}(\Omega, M)$. We assume the validity of eight natural axioms for the rule $\mathfrak{M}$ and we show that for every manifold M the topological space $\mathfrak{M}(\Omega, M)$ can be endowed with the structure of a Banach manifold. Such structure will be explicitly described in terms of local charts. Moreover, the Banach manifold structure of $\mathfrak{M}(\Omega, M)$ is unique under some naturality conditions. The eight axioms and the detailed construction of the Banach manifold structure on $\mathfrak{M}(\Omega, M)$ is presented in Section 2. In Section 3 some concrete examples where the theory applies are discussed; first, we list a few rules $\mathfrak{M}$ for which the topological spaces $\mathfrak{M}(\Omega, M)$ can be easily described for arbitrary manifolds M. Then we give a general theorem showing that, given a Banach space $\mathcal{E}$ of maps $f : \Omega \to I\!R$ satisfying two simple conditions, then there exists a unique rule $\mathfrak{M}$ satisfying the eight axioms of Section 2 and with $\mathfrak{M}(\Omega, I\!R) = \mathcal{E}$. Using this theorem we are able to construct several other examples of rules $\mathfrak{M}$ for which the theory of Section 2 applies. We emphasize that the maps $f : \Omega \to M$ belonging to our Banach manifolds $\mathfrak{M}(\Omega, M)$ *need not be continuous* and in fact, in some examples, *one does not even need to fix a topology on* Ω. For example, for any set Ω, we obtain a Banach manifold structure on the set $\mathfrak{B}(\Omega, M)$ of all maps $f : \Omega \to M$ with relatively compact image. If Ω is an arbitrary measure space, we also define a Banach manifold based on the space of bounded maps $f : \Omega \to I\!R$ with $f \in L^p(\Omega, I\!R)$.

In what follows we will make a comparison between the constructions presented in this paper and others appearing in the literature. We also sketch the idea behind our construction of the manifold structure of $\mathfrak{M}(\Omega, M)$.

It should be pointed out that in this paper we only consider Banach and Hilbert manifold structures, which are in practice more applicable from the point of view of critical point

---

Date: April 2002.
2000 *Mathematics Subject Classification.* 46T10, 58D15, 46T05.
The author is partially supported by CNPq (Brazil).

theory. The theory of manifolds modelled on more general locally convex topological vector spaces have been recently developed in detail in [2].

The classical work [4] introduces a Banach manifold structure on a set of sections $\mathfrak{M}(E)$ of a smooth fiber bundle E over a smooth compact manifold with boundary Ω. The Banach manifold $\mathfrak{M}(E)$ is modelled on Banach spaces of the form $\mathfrak{M}(\xi)$, where ξ is a *vector bundle neighborhood* on E, i.e., ξ is an open subset of E endowed with the structure of a vector bundle over Ω. The rule $\mathfrak{M}$ that assigns to each vector bundle ξ a Banach space of sections $\mathfrak{M}(\xi)$ of ξ is supposed to satisfy some axioms. For example, $\mathfrak{M}(\xi)$ should have a continuous inclusion on the space of continuous sections of ξ (endowed with the compact-open topology) and the left composition maps $\mathfrak{M}(\xi) \to \mathfrak{M}(\eta)$ induced by smooth fiber bundle morphisms $\phi : \xi \to \eta$ should be continuous. Banach manifold structures for sets of maps $f : \Omega \to M$, where M is an arbitrary manifold, are obtained by identifying such maps with sections of the trivial fiber bundle $\Omega \times M \to \Omega$.

Following the approach of [4], one would not expect to obtain Banach manifold structures on sets of sections $\mathfrak{M}(E)$ of a fiber bundle E whose base manifold Ω is not compact. For the case of noncompact bases, the standard literature on the subject tends to present Frechét manifold structures; typically, the topology of $\mathfrak{M}(E)$ is induced by the restriction maps $\mathfrak{M}(E) \to \mathfrak{M}(E|_K)$, where $K \subset \Omega$ is a compact domain and $\mathfrak{M}(E|_K)$ is endowed with a Banach manifold structure.

In [5], Banach manifold structures for sets of maps with noncompact domains are studied. The basic example of the theory of [5] is the nonlinear version of the Banach space $C^0_{\mathrm{b}}(\Omega, I\!R)$ of bounded continuous maps $f : \Omega \to I\!R$ defined on a (not necessarily compact) topological space Ω, endowed with the sup norm. More precisely, it is introduced a Banach manifold structure on the set $C^0_{\mathrm{b}}(\Omega, M)$ of continuous maps $f : \Omega \to M$ with relatively compact image on M. The technique for introducing the Banach manifold structure on $C^0_{\mathrm{b}}(\Omega, M)$ is based on an infinite-dimensional version of the rank theorem, which is used to show that $C^0_{\mathrm{b}}(\Omega, M)$ is a smooth (embedded) submanifold of the Banach space $C^0_{\mathrm{b}}(\Omega, I\!R^n)$, where M is embedded in $I\!R^n$ using Whitney's theorem. Later, it is shown that the manifold structure of $C^0_{\mathrm{b}}(\Omega, M)$ does not depend on the particular embedding of M in the Euclidean space. We point out that there is no explicit description of a smooth atlas for the Banach manifold $C^0_{\mathrm{b}}(\Omega, M)$ in [5].

From [5] it becomes clear that compactness of the domain Ω is not a crucial property for the introduction of a Banach manifold structure on a set of maps $\mathfrak{M}(\Omega, M)$; indeed, one only needs compactness on the *image* (or in the closure of the image) of the maps in $\mathfrak{M}(\Omega, M)$. The problem is that this new view is in principle conflicting with the spirit of [4]; namely, if $s : \Omega \to E$ is a continuous section of a fiber bundle $E \to \Omega$ then the image of s is always closed in E and homeomorphic to Ω, so that compactness in the image is actually equivalent to compactness in the domain. One is therefore not expected to obtain examples of Banach manifolds of maps with noncompact domain if spaces of maps $f : \Omega \to M$ are considered as particular cases of spaces of sections of fiber bundles.

Any axiomatization for a rule $\mathfrak{M}$ that leads to the introduction of Banach manifold structures on sets of maps $\mathfrak{M}(\Omega, M)$ (or $\mathfrak{M}(E)$) should include some sort of axiom requiring continuity of left composition maps $f \mapsto \phi \circ f$ (at the very least, one needs continuity of $f \mapsto \phi \circ f$ when ϕ is a smooth diffeomorphism). If, as in [4], one models the Banach manifolds on Banach spaces $\mathfrak{M}(\xi)$ of sections of a vector bundle $\xi \to \Omega$ then the natural left composition maps $f \mapsto \phi \circ f$ to be considered are the ones induced by smooth fiber bundle morphisms $\phi : \xi \to \eta$ (see [4, Axiom §5]). In the case where $\xi = \Omega \times I\!R^m$, $\eta = \Omega \times I\!R^n$ are trivial vector bundles, such morphisms ϕ take the form

$\Omega \times I\!R^m \ni (a,x) \mapsto \big(a, \phi(a,x)\big) \in \Omega \times I\!R^n$ and the corresponding left composition map carries $f : \Omega \to I\!R^m$ to $\Omega \ni a \mapsto \phi\big(a, f(a)\big) \in I\!R^n$. If Ω is not compact then, in most examples, such left composition maps are not continuous or even well-defined, i.e., they do not take values in $\mathfrak{M}(\Omega, I\!R^n)$. Instead, one usually has only the continuity of the left composition map $f \mapsto \phi \circ f$ where ϕ *is applied only to* $f(a)$ *and not to* a. This gives another hint of the fact that it is more natural to work directly with sets of maps $f : \Omega \to M$ and not with sections of trivial fiber bundles.

It should be mentioned that the use of Banach differentiable structures on sets of maps with noncompact domains is indeed relevant. For instance, in the study of *Morse homology* in [6], one is lead to consider Sobolev spaces of maps of the form $H^1(I\!R, M)$ (see the introduction of [5] for more examples where noncompact domains are used).

We emphasize that, differently from [5], the constructions and the main[1] results of this paper *never use embeddings of the manifolds into Euclidean space*. This is possibly a matter of personal taste, but the author feels that this is the most elegant way of dealing with manifolds. Also, the explicit description of coordinate charts for $\mathfrak{M}(\Omega, M)$ is often useful when dealing with spaces of maps in practical applications.

Let us now give a sketch of the main ideas behind the construction of the differentiable structure of $\mathfrak{M}(\Omega, M)$. The most tempting way of defining a coordinate chart on $\mathfrak{M}(\Omega, M)$ is to consider the left composition map $f \mapsto \varphi \circ f$, where $\varphi : U \subset M \to I\!R^n$ is a coordinate chart on M. This kind of charts are indeed smoothly compatible with each other in the examples we consider, but obviously one cannot expect that they form an atlas for $\mathfrak{M}(\Omega, M)$; namely, there may be maps $f \in \mathfrak{M}(\Omega, M)$ whose image is not contained in the domain of a chart of M. Since we only consider maps f with relatively compact image, one can cover the image of f with a finite number of coordinate charts φ_i, $i = 1, \ldots, r$, but there is no visible way of combining the charts φ_i, $i = 1, \ldots, r$ into a chart for $\mathfrak{M}(\Omega, M)$ around f. Instead, let us take a look at the vector bundle neighborhoods of [4]. As explained before in this introduction, a vector bundle neighborhood ξ on a fiber bundle $E \to \Omega$ is an open set $\xi \subset E$ which has the structure of a vector bundle over Ω. Obviously, in our case, Ω may not even be a topological space, so we shouldn't talk about fiber bundles over Ω; but let us just for the moment assume that Ω is a manifold. For each $a \in \Omega$, the fiber ξ_a of ξ over a is an open subset of the fiber E_a of E over a; moreover, ξ_a is endowed with the structure of a real finite-dimensional vector space. Just for psychological reasons, it seems simpler to picture this situation in terms of a smooth diffeomorphism $\varphi_a : V_a \to \xi_a$, where $V_a \subset E_a$ is open and ξ_a is a real finite-dimensional vector space. Now φ_a is just a chart on E_a; in the case that $E = \Omega \times M$ is a trivial bundle, the vector bundle neighborhood ξ can be thought of as a family of charts $(\varphi_a)_{a \in \Omega}$ on M, where the domain of φ_a is an open subset V_a of M (depending on a) and the counter-domain of φ_a is a vector space ξ_a (also depending on a). So, rather than trying to cover the image of a map $f : \Omega \to M$ with a finite number of charts, we cover it with a *continuous family* of charts $(\varphi_a)_{a \in \Omega}$, where $f(a)$ belongs to the domain of φ_a for every $a \in \Omega$. This yields a chart on $\mathfrak{M}(\Omega, M)$ around f taking values in the space of sections of a vector bundle.

Now, the key observation here is that the *index set* for the family of charts $(\varphi_a)_{a \in \Omega}$ doesn't need to have anything to do with the domain Ω of the maps f. Instead, we consider an *arbitrary smooth manifold* X to parameterize the charts φ_a and we consider an arbitrary vector bundle ξ over X playing the role of the old vector bundle neighborhood. We then consider a smooth diffeomorphism $\varphi : V \to \widetilde{V}$ between open subsets $V \subset X \times M$ and

[1]We did use the existence of embeddings into Euclidean space occasionally in some less important remarks.

$\widetilde{V} \subset \xi$ such that for every $x \in X$, the map $\varphi_x = \varphi(x, \cdot)$ carries $V_x = V \cap \big(\{x\} \times M\big)$ to $\widetilde{V}_x = \widetilde{V} \cap \xi_x$. Thus $(\varphi_x)_{x\in X}$ is now a parameterized family of charts on M. In order to complete the construction of the chart on $\mathfrak{M}(\Omega, M)$ around $f : \Omega \to M$, we need a map $\sigma : \Omega \to X$ that tells us for each $a \in \Omega$, which of the charts φ_x should be used around $f(a)$; more explicitly, the chart on $\mathfrak{M}(\Omega, M)$ is of the form $f \mapsto \varphi \circ (\sigma, f)$ and it takes values on the space of sections $s : \Omega \to \xi$ of ξ along σ. It may sound a bit surprising that the construction of a chart around f does not use any sort of continuity of f and does not use approximations by smooth maps as in [4].

2. The General Axiomatization

Given sets A, B we will denote by B^A the set of all maps from A to B. Throughout this section we will consider fixed a set Ω and a rule $\mathfrak{M}$ that assigns to each manifold M a subset $\mathfrak{M}(\Omega, M)$ of M^Ω and a topology on the set $\mathfrak{M}(\Omega, M)$. By a *manifold* we will mean a smooth finite-dimensional real manifold whose topology is Hausdorff and second countable, where *smooth* means "of class C^∞".

Below we will list a few axioms concerning the rule $\mathfrak{M}$ that will allow us to construct a Banach manifold structure on the topological space $\mathfrak{M}(\Omega, M)$.

Axiom A0. There exists a manifold M_0 for which $\mathfrak{M}(\Omega, M_0)$ is nonempty.

Axiom A1. Given manifolds M, N and a smooth map $\phi : M \to N$ then $\phi \circ f \in \mathfrak{M}(\Omega, N)$ for all $f \in \mathfrak{M}(\Omega, M)$; moreover, the *left composition map*:

$$\mathrm{LC}(\phi) : \mathfrak{M}(\Omega, M) \longrightarrow \mathfrak{M}(\Omega, N)$$

defined by $\mathrm{LC}(\phi)(f) = \phi \circ f$ is continuous.

Obviously axiom (A1) implies that if $\phi : M \to N$ is a smooth diffeomorphism then $\mathrm{LC}(\phi)$ is a homeomorphism. Also, from axioms (A0) and (A1) we obtain that $\mathfrak{M}(\Omega, M)$ contains all constant maps; to see that, simply evaluate $\mathrm{LC}(\phi)$ in an arbitrary element of $\mathfrak{M}(\Omega, M_0)$, where $\phi : M_0 \to M$ is an arbitrary constant map.

Axiom A2. Let M_1, M_2 be manifolds and denote by pr_1, pr_2 the projections of the product $M_1 \times M_2$. The map:

$$\big(\mathrm{LC}(\mathrm{pr}_1), \mathrm{LC}(\mathrm{pr}_2)\big) : \mathfrak{M}(\Omega, M_1 \times M_2) \longrightarrow \mathfrak{M}(\Omega, M_1) \times \mathfrak{M}(\Omega, M_2) \tag{2.1}$$

is a homeomorphism, where the counter-domain of (2.1) is endowed with the standard product topology.

Obviously one can show by induction a version of axiom (A2) for arbitrary finite products of manifolds.

Axiom A3. For any manifold M the elements of $\mathfrak{M}(\Omega, M)$ have relatively compact image in M, i.e., the set $\overline{\mathrm{Im}(f)}$ is compact for all $f \in \mathfrak{M}(\Omega, M)$.

Axiom A4. Let M be a manifold and $U \subset M$ be an open subset. If $f \in \mathfrak{M}(\Omega, M)$ and $\overline{\mathrm{Im}(f)} \subset U$ then the map $f : \Omega \to U$ is in $\mathfrak{M}(\Omega, U)$.

If U is open in M then axiom (A1) implies that $\mathfrak{M}(\Omega, U)$ is a subset of $\mathfrak{M}(\Omega, M)$, provided that we identify U^Ω with a subset of M^Ω in the obvious way. Moreover, axioms (A3) and (A4) imply that:

$$\mathfrak{M}(\Omega, U) = \big\{ f \in \mathfrak{M}(\Omega, M) : \overline{\mathrm{Im}(f)} \subset U \big\}. \tag{2.2}$$

For later use we observe that (2.2) implies:

$$\mathfrak{M}(\Omega, U \cap V) = \mathfrak{M}(\Omega, U) \cap \mathfrak{M}(\Omega, V), \tag{2.3}$$

for any open subsets $U, V \subset M$. Also, axioms (A3) and (A4) imply that:

$$\mathfrak{M}\big(\Omega, \phi^{-1}(U)\big) = \mathrm{LC}(\phi)^{-1}\big(\mathfrak{M}(\Omega, U)\big), \tag{2.4}$$

for every smooth map $\phi : M \to N$ and every open subset $U \subset N$.

Axiom A5. By identifying U^Ω with a subset of M^Ω then $\mathfrak{M}(\Omega, U)$ is open in $\mathfrak{M}(\Omega, M)$ and has the topology induced from $\mathfrak{M}(\Omega, M)$.

Axiom A6. Given a manifold M and a point $a \in \Omega$ then the *evaluation map*:

$$\mathrm{eval}_a : \mathfrak{M}(\Omega, M) \longrightarrow M \tag{2.5}$$

given by $\mathrm{eval}_a(f) = f(a)$ is continuous.

Observe that axiom (A6) means that the topology of $\mathfrak{M}(\Omega, M)$ is finer than pointwise convergence topology, i.e., the topology induced from the product topology on M^Ω. In particular, all spaces $\mathfrak{M}(\Omega, M)$ are Hausdorff.

Before stating the last axiom, we prove the following:

2.1**. Lemma.** *If E is a finite-dimensional real vector space (regarded as a manifold in the canonical way) then the set $\mathfrak{M}(\Omega, E)$ is a subspace of the vector space E^Ω. For $t \in \mathbb{R}$, denote by $c_t : \Omega \to \mathbb{R}$ the constant map equal to t; assuming that the map:*

$$\mathbb{R} \ni t \longmapsto c_t \in \mathfrak{M}(\Omega, \mathbb{R}) \tag{2.6}$$

is continuous then $\mathfrak{M}(\Omega, E)$ is a topological vector space, i.e., the vector space operations of $\mathfrak{M}(\Omega, E)$ are continuous.

Proof. We know that $\mathfrak{M}(\Omega, E)$ contains the constant maps, so it contains the identically zero map. Moreover, by axiom (A2), $\mathfrak{M}(\Omega, E \times E)$ can be identified with the product $\mathfrak{M}(\Omega, E) \times \mathfrak{M}(\Omega, E)$; since the sum map $E \times E \ni (v, w) \mapsto v + w \in E$ is smooth, axiom (A1) implies that $\mathfrak{M}(\Omega, E)$ is closed under addition and that the sum of $\mathfrak{M}(\Omega, E)$ is continuous. For fixed $t \in \mathbb{R}$, the homotety $E \ni v \mapsto tv \in E$ is smooth, and thus axiom (A1) implies that $\mathfrak{M}(\Omega, E)$ is closed under scalar multiplication. Finally, by axioms (A1) and (A2), the map $\mathfrak{M}(\Omega, \mathbb{R}) \times \mathfrak{M}(\Omega, E) \to \mathfrak{M}(\Omega, E)$ induced by scalar multiplication is continuous; thus, if (2.6) is continuous, then the scalar multiplication of $\mathfrak{M}(\Omega, E)$ is also continuous. $\square$

Axiom A7. The real vector space $\mathfrak{M}(\Omega, \mathbb{R})$ is *Banachble*, i.e., there exists a norm on $\mathfrak{M}(\Omega, \mathbb{R})$ that induces its topology and that makes it into a Banach space.

Obviously axiom (A2) implies that $\mathfrak{M}(\Omega, \mathbb{R}^n)$ is linearly homeomorphic to the topological direct sum $\bigoplus_n \mathfrak{M}(\Omega, \mathbb{R})$ and thus $\mathfrak{M}(\Omega, \mathbb{R}^n)$ is also a Banachble space, for all n. More generally, $\mathfrak{M}(\Omega, E)$ is a Banachble space for every real finite-dimensional vector space E, by axiom (A1).

2.2**. Lemma.** *Given manifolds M, N and a smooth embedding $\phi : N \to M$ then the map $\mathrm{LC}(\phi) : \mathfrak{M}(\Omega, N) \to \mathfrak{M}(\Omega, M)$ is a homeomorphism onto its image, which is given by:*

$$\mathrm{Im}\big(\mathrm{LC}(\phi)\big) = \big\{ f \in \mathfrak{M}(\Omega, M) : \overline{\mathrm{Im}(f)} \subset \mathrm{Im}(\phi) \big\}. \tag{2.7}$$

Moreover, if $\mathrm{Im}(\phi)$ is closed in M then $\mathrm{Im}\big(\mathrm{LC}(\phi)\big)$ is closed in $\mathfrak{M}(\Omega, M)$.

Proof. Since $\phi : N \to \mathrm{Im}(\phi)$ is a smooth diffeomorphism, by axiom (A1), we can assume without loss of generality that N is a submanifold of M and that ϕ is the inclusion map. We then identify $\mathfrak{M}(\Omega, N)$ with a subset of $\mathfrak{M}(\Omega, M)$, being $\mathrm{LC}(\phi)$ the inclusion map. Using a tubular neighborhood of N in M we can find an open set $U \subset M$ containing N and a smooth retraction $r : U \to N$, i.e., $r|_N = \mathrm{Id}_N$. Then $\mathrm{LC}(r)$ is a continuous left inverse of the inclusion of $\mathfrak{M}(\Omega, N)$ in $\mathfrak{M}(\Omega, U)$; observing that, by axiom (A5), $\mathfrak{M}(\Omega, U)$ has the topology induced from $\mathfrak{M}(\Omega, M)$, we conclude that $\mathfrak{M}(\Omega, N)$ has the topology induced from $\mathfrak{M}(\Omega, M)$, i.e., $\mathrm{LC}(\phi)$ is a homeomorphism onto its image. The inclusion of the lefthand side of (2.7) into the righthand side of (2.7) follows from axiom (A3). Moreover, if $f \in \mathfrak{M}(\Omega, M)$ and $\overline{\mathrm{Im}(f)} \subset N$ then $f \in \mathfrak{M}(\Omega, U)$, by axiom (A4) and thus $f = \mathrm{LC}(r)(f)$ is in $\mathfrak{M}(\Omega, N)$. This proves (2.7). Finally, the fact that $\mathfrak{M}(\Omega, N)$ is closed in $\mathfrak{M}(\Omega, M)$ if N is closed in M follows from axiom (A6). $\square$

Let $\pi : \xi \to X$ be a smooth vector bundle over a manifold X. Given $\sigma \in \mathfrak{M}(\Omega, X)$ we set:

$$\mathfrak{M}(\Omega, \xi; \sigma) = \big\{ s \in \mathfrak{M}(\Omega, \xi) : \pi \circ s = \sigma \big\},$$

and we endow $\mathfrak{M}(\Omega, \xi; \sigma)$ with the topology induced by $\mathfrak{M}(\Omega, \xi)$. If $U \subset \xi$ is open then we also write:

$$\mathfrak{M}(\Omega, U; \sigma) = \mathfrak{M}(\Omega, U) \cap \mathfrak{M}(\Omega, \xi; \sigma),$$

and we endow $\mathfrak{M}(\Omega, U; \sigma)$ with the topology induced by $\mathfrak{M}(\Omega, \xi)$, which, by axiom (A5), coincides with the topology induced by $\mathfrak{M}(\Omega, U)$. Moreover, $\mathfrak{M}(\Omega, U; \sigma)$ is open in $\mathfrak{M}(\Omega, \xi; \sigma)$.

2.3. **Lemma.** *If $\pi : \xi \to X$ is a smooth vector bundle over a manifold X and σ is in $\mathfrak{M}(\Omega, X)$ then $\mathfrak{M}(\Omega, \xi; \sigma)$ is a subspace of the vector space of all maps $s : \Omega \to \xi$ with $\pi \circ s = \sigma$. Moreover, $\mathfrak{M}(\Omega, \xi; \sigma)$ is a Banachble space.*

Proof. By axiom (A3) we can cover $\overline{\mathrm{Im}(\sigma)}$ with a finite number of open sets $U_i \subset X$, $i = 1, \ldots, r$, such that ξ is trivial over U_i for each i. Denote by n the dimension of the fibers of ξ and for each $i = 1, \ldots, r$ let $\phi_i : \pi^{-1}(U_i) \to I\!R^n$ be a smooth map whose restriction to each fiber is an isomorphism. Let $(\lambda_i)_{i=1}^r$ be a smooth partition of unity on the open set $U = \bigcup_{i=1}^r U_i$ such that $\mathrm{supp}(\lambda_i) \subset U_i$ for all i. Consider the smooth map

$$\phi : \pi^{-1}(U) \longrightarrow \bigoplus_r I\!R^n \cong I\!R^{rn}$$

whose i-th coordinate equals $(\lambda_i \circ \pi)\phi_i$ on $\pi^{-1}(U_i)$ and equals zero on $\pi^{-1}(U \setminus U_i)$, for $i = 1, \ldots, r$. Then $(\pi, \phi) : \pi^{-1}(U) \to U \times I\!R^{rn}$ is a smooth vector bundle isomorphism from $\pi^{-1}(U) = \xi|_U$ onto a vector subbundle $\widetilde{\xi}$ of the trivial bundle $U \times I\!R^{rn}$. Observe that if $s \in \mathfrak{M}(\Omega, \xi; \sigma)$ then $\overline{\mathrm{Im}(s)} \subset \xi|_U$ and thus, by axioms (A4) and (A5), we have $\mathfrak{M}(\Omega, \xi; \sigma) = \mathfrak{M}(\Omega, \xi|_U; \sigma)$. Since $\widetilde{\xi}$ is a closed submanifold of $U \times I\!R^{rn}$, Lemma 2.2 implies that $\mathrm{LC}(\pi, \phi)$ is a linear homeomorphism between $\mathfrak{M}(\Omega, \xi|_U; \sigma)$ and the closed subspace $\mathfrak{M}(\Omega, \widetilde{\xi}; \sigma)$ of the Banachble space $\mathfrak{M}(\Omega, U \times I\!R^{rn}; \sigma) \cong \mathfrak{M}(\Omega, I\!R^{rn})$. $\square$

In order to construct a smooth atlas on the spaces $\mathfrak{M}(\Omega, M)$, we will have to prove the smoothness of certain left composition maps on Banachble spaces of the form $\mathfrak{M}(\Omega, \xi; \sigma)$. To that aim, we will employ a general lemma that allows one to establish differentiability of maps between Banach spaces.

2.4. **Definition.** Let E be a Banachble space. A *separating set of continuous linear maps* for E is a set Λ of continuous linear maps $\lambda : E \to F$, where F is a Banachble space that may depend on λ, such that for every nonzero $v \in E$ there exists $\lambda \in \Lambda$ with $\lambda(v) \neq 0$.

Given Banachble spaces E, F, we denote by $\mathrm{Lin}(E,F)$ the Banachble space of continuous linear maps from E to F.

2.5. **Lemma** (weak differentiation principle). *Let E, F be Banachble spaces, $U \subset E$ an open subset, $f : U \to F$ a map and Λ a separating set of continuous linear maps for F. If there exists a continuous map $g : U \to \mathrm{Lin}(E,F)$ such that:*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\lambda \circ f)(x+tv)\Big|_{t=0} = \lambda\big(g(x)v\big), \tag{2.8}$$

for all $x \in U$, $v \in E$, $\lambda \in \Lambda$ then f is of class C^1 and $\mathrm{d}f = g$.

Proof. See [5, Proposition 3.2]. □

Let X be a manifold and $\pi^1 : \xi^1 \to X$, $\pi^2 : \xi^2 \to X$ be smooth vector bundles over X. We denote by $\xi^1 \oplus \xi^2$ the *Whitney sum* of ξ^1 and ξ^2, i.e., the vector bundle over X whose fiber over $x \in X$ is $\xi^1_x \oplus \xi^2_x$. There exists an obvious diffeomorphism between $\xi^1 \oplus \xi^2$ and the closed submanifold of the product $\xi^1 \times \xi^2$ consisting of pairs $(v,w) \in \xi^1 \times \xi^2$ with $\pi^1(v) = \pi^2(w)$. Thus, using Lemma 2.2 and axiom (A2) we obtain a linear homeomorphism:

$$\mathfrak{M}(\Omega, \xi^1 \oplus \xi^2; \sigma) \cong \mathfrak{M}(\Omega, \xi^1; \sigma) \oplus \mathfrak{M}(\Omega, \xi^2; \sigma), \tag{2.9}$$

for any $\sigma \in \mathfrak{M}(\Omega, X)$.

Denote by $\mathrm{Lin}(\xi^1, \xi^2)$ the vector bundle over X whose fiber over $x \in X$ is the space $\mathrm{Lin}(\xi^1_x, \xi^2_x)$ of linear maps from ξ^1_x to ξ^2_x. Consider the smooth map:

$$\mathcal{C} : \mathrm{Lin}(\xi^1, \xi^2) \oplus \xi^1 \longrightarrow \xi^2$$

defined by $\mathcal{C}(T,v) = T(v)$, for all $T \in \mathrm{Lin}(\xi^1_x, \xi^2_x)$, $v \in \xi^1_x$, $x \in X$. By axiom (A1), the map $\mathrm{LC}(\mathcal{C})$ is continuous and using the identification given in (2.9) we obtain a continuous bilinear map:

$$\mathrm{LC}(\mathcal{C}) : \mathfrak{M}\big(\Omega, \mathrm{Lin}(\xi^1, \xi^2); \sigma\big) \times \mathfrak{M}(\Omega, \xi^1; \sigma) \longrightarrow \mathfrak{M}(\Omega, \xi^2; \sigma).$$

The continuous bilinear map $\mathrm{LC}(\mathcal{C})$ above then induces in a natural way a continuous linear map:

$$\mathcal{O} : \mathfrak{M}\big(\Omega, \mathrm{Lin}(\xi^1, \xi^2); \sigma\big) \longrightarrow \mathrm{Lin}\big(\mathfrak{M}(\Omega, \xi^1; \sigma), \mathfrak{M}(\Omega, \xi^2; \sigma)\big); \tag{2.10}$$

more explicitly, we have $\mathcal{O}(T)(s)(a) = T(a)s(a)$, for every $T \in \mathfrak{M}\big(\Omega, \mathrm{Lin}(\xi^1, \xi^2); \sigma\big)$, $s \in \mathfrak{M}(\Omega, \xi^1; \sigma)$ and $a \in \Omega$. The construction above will be used in the proof of Lemma 2.6 below.

Recall that, given smooth vector bundles $\pi^1 : \xi^1 \to X$, $\pi^2 : \xi^2 \to X$ over a manifold X then a map ϕ defined on a subset of ξ^1, taking values in ξ^2 is called *fiber preserving* if $\pi^2\big(\phi(v)\big) = \pi^1(v)$, for all v in the domain of ϕ.

2.6. **Lemma.** *Let $\pi^1 : \xi^1 \to X$, $\pi^2 : \xi^2 \to X$ be smooth vector bundles over a manifold X and let $\phi : U \to \xi^2$ be a smooth fiber preserving map defined on an open subset $U \subset \xi^1$. Given $\sigma \in \mathfrak{M}(\Omega, X)$ then:*

$$\mathrm{LC}(\phi) : \mathfrak{M}(\Omega, U; \sigma) \longrightarrow \mathfrak{M}(\Omega, \xi^2; \sigma)$$

is a smooth map on the open subset $\mathfrak{M}(\Omega, U; \sigma)$ of the Banachble space $\mathfrak{M}(\Omega, \xi^1; \sigma)$.

Proof. Denote by $\mathbb{F}\phi$ the *fiber derivative* of ϕ which is the smooth fiber preserving map:

$$\mathbb{F}\phi : U \longrightarrow \mathrm{Lin}(\xi^1, \xi^2)$$

defined by $\mathbb{F}\phi(v) = \mathrm{d}(\phi|_{\xi^1_x \cap U})(v)$, for all $v \in \xi^1_x \cap U$, $x \in X$. Our strategy is to use Lemma 2.5 to show that $\mathrm{LC}(\phi)$ is of class C^1 and that:

$$\mathrm{d}\big(\mathrm{LC}(\phi)\big) = \mathcal{O} \circ \mathrm{LC}(\mathbb{F}\phi).$$

The smoothness of $\mathrm{LC}(\phi)$ will then follow by induction. We already know that $\mathcal{O}$ is continuous and that $\mathrm{LC}(\mathbb{F}\phi)$ is continuous, by axiom (A1). For each $a \in \Omega$, denote by:

$$\mathrm{eval}_a : \mathfrak{M}(\Omega, \xi^2; \sigma) \longrightarrow \xi^2_{\sigma(a)}$$

the map of evaluation at a. We know from axiom (A6) that eval_a is a continuous linear map and then obviously $\Lambda = \big\{\mathrm{eval}_a : a \in \Omega\big\}$ is a separating set of continuous linear maps for $\mathfrak{M}(\Omega, \xi^2; \sigma)$. The verification of the hypothesis (2.8) on Lemma 2.5, with $f = \mathrm{LC}(\phi)$, $g = \mathcal{O} \circ \mathrm{LC}(\mathbb{F}\phi)$, is now straightforward. □

We will now consider a fixed manifold M and we will construct a smooth atlas for $\mathfrak{M}(\Omega, M)$. Let $\pi : \xi \to X$ be a smooth vector bundle over a manifold X and let:

$$\varphi : X \times M \supset V \longrightarrow \widetilde{V} \subset \xi$$

be a smooth diffeomorphism, where V is open in $X \times M$ and $\widetilde{V}$ is open in ξ; we assume in addition that φ is *fiber preserving* in the sense that $\pi \circ \varphi = \mathrm{pr}_1|_V$, where pr_1 denotes the first projection of the product $X \times M$. Given $\sigma \in \mathfrak{M}(\Omega, X)$ we write:

$$\mathfrak{M}(\Omega, M; \sigma, V) = \big\{ f \in \mathfrak{M}(\Omega, M) : (\sigma, f) \in \mathfrak{M}(\Omega, V) \big\} \subset \mathfrak{M}(\Omega, M), \tag{2.11}$$

and we consider the map:

$$\mathrm{LC}(\varphi; \sigma) : \mathfrak{M}(\Omega, M; \sigma, V) \longrightarrow \mathfrak{M}(\Omega, \widetilde{V}; \sigma) \subset \mathfrak{M}(\Omega, \xi; \sigma), \tag{2.12}$$

defined by $\mathrm{LC}(\varphi; \sigma)(f) = \varphi \circ (\sigma, f)$. The set $\mathfrak{M}(\Omega, M; \sigma, V)$ is open in $\mathfrak{M}(\Omega, V)$, by axioms (A2) and (A5); moreover, $\mathfrak{M}(\Omega, \widetilde{V}; \sigma)$ is open in the Banachble space $\mathfrak{M}(\Omega, \xi; \sigma)$ and $\mathrm{LC}(\varphi; \sigma)$ is a homeomorphism, by axiom (A1). Thus $\mathrm{LC}(\varphi; \sigma)$ is a (topological) local chart in the topological space $\mathfrak{M}(\Omega, M)$. We will call X the *auxiliary manifold* and σ the *auxiliary map* corresponding to the chart $\mathrm{LC}(\varphi; \sigma)$.

Our goal now is to show that the local charts (2.12) form a smooth atlas on $\mathfrak{M}(\Omega, M)$.

2.7**. Lemma.** *Let $\pi : \xi \to X$, $\pi' : \eta \to Y$ be smooth vector bundles over manifolds X, Y and let $\sigma \in \mathfrak{M}(\Omega, X)$, $\tau \in \mathfrak{M}(\Omega, Y)$ be fixed. Given smooth fiber preserving diffeomorphisms:*

$$\varphi : X \times M \supset V \longrightarrow \widetilde{V} \subset \xi, \quad \psi : Y \times M \supset W \longrightarrow \widetilde{W} \subset \eta$$

between open sets $V, \widetilde{V}, W, \widetilde{W}$ then the local charts $\mathrm{LC}(\varphi; \sigma)$ and $\mathrm{LC}(\psi; \tau)$ on $\mathfrak{M}(\Omega, M)$ are smoothly compatible, i.e., the transition map $\mathrm{LC}(\psi; \tau) \circ \mathrm{LC}(\varphi; \sigma)^{-1}$ is a smooth diffeomorphism between open sets.

Proof. The strategy of the proof is two modify the charts $\mathrm{LC}(\varphi; \sigma)$ and $\mathrm{LC}(\psi; \tau)$ so that they both correspond to the same auxiliary manifold $X \times Y$ and the same auxiliary map $(\sigma, \tau) \in \mathfrak{M}(\Omega, X \times Y)$. To this aim, consider the smooth vector bundles:

$$\pi \times \mathrm{Id}_Y : \xi \times Y \longrightarrow X \times Y, \quad \mathrm{Id}_X \times \pi' : X \times \eta \longrightarrow X \times Y$$

over the manifold $X \times Y$. By axiom (A2), we have obvious linear homeomorphisms:

$$\mathfrak{M}(\Omega, \xi; \sigma) \cong \mathfrak{M}\big(\Omega, \xi \times Y; (\sigma, \tau)\big), \quad \mathfrak{M}(\Omega, \eta; \tau) \cong \mathfrak{M}\big(\Omega, X \times \eta; (\sigma, \tau)\big),$$

where $(\sigma,\tau)\in\mathfrak{M}(\Omega,X\times Y)$. By taking restrictions of the linear homeomorphisms above we obtain the following homeomorphisms:

$$\mathfrak{M}(\Omega,\widetilde{V};\sigma)\cong\mathfrak{M}\big(\Omega,\widetilde{V}\times Y;(\sigma,\tau)\big),\quad \mathfrak{M}(\Omega,\widetilde{W};\tau)\cong\mathfrak{M}\big(\Omega,X\times\widetilde{W};(\sigma,\tau)\big). \tag{2.13}$$

Also, by axiom (A2), we have (recall (2.11)):

$$\mathfrak{M}(\Omega,M;\sigma,V)=\mathfrak{M}\big(\Omega,M;(\sigma,\tau),\mathfrak{s}(Y\times V)\big), \tag{2.14}$$

$$\mathfrak{M}(\Omega,M;\tau,W)=\mathfrak{M}\big(\Omega,M;(\sigma,\tau),X\times W\big), \tag{2.15}$$

where $\mathfrak{s}:Y\times X\times M\to X\times Y\times M$ is the map that swaps the first two coordinates. Consider the smooth fiber preserving diffeomorphisms:

$$\overline{\varphi}:X\times Y\times M\supset\mathfrak{s}(Y\times V)\longrightarrow\widetilde{V}\times Y\subset\xi\times Y,$$
$$\overline{\psi}:X\times Y\times M\supset X\times W\longrightarrow X\times\widetilde{W}\subset X\times\eta,$$

defined by $\overline{\varphi}(x,y,m)=\big(\varphi(x,m),y\big)$, $\overline{\psi}(x,y,m)=\big(x,\psi(y,m)\big)$. Using the identities (2.14), (2.15) and the identifications (2.13), the charts $\mathrm{LC}(\varphi;\sigma)$ and $\mathrm{LC}(\psi;\tau)$ are identified respectively with the charts:

$$\mathrm{LC}\big(\overline{\varphi};(\sigma,\tau)\big):\mathfrak{M}\big(\Omega,M;(\sigma,\tau),\mathfrak{s}(Y\times V)\big)\longrightarrow\mathfrak{M}\big(\Omega,\widetilde{V}\times Y;(\sigma,\tau)\big), \tag{2.16}$$

$$\mathrm{LC}\big(\overline{\psi};(\sigma,\tau)\big):\mathfrak{M}\big(\Omega,M;(\sigma,\tau),X\times W)\big)\longrightarrow\mathfrak{M}\big(\Omega,X\times\widetilde{W};(\sigma,\tau)\big). \tag{2.17}$$

By (2.3), the intersection of the domains of the charts above is $\mathfrak{M}\big(\Omega,M;(\sigma,\tau),Z\big)$, where:

$$Z=\mathfrak{s}(Y\times V)\cap(X\times W)\subset X\times Y\times M.$$

Hence, the transition map between the charts $\mathrm{LC}\big(\overline{\varphi};(\sigma,\tau)\big)$ and $\mathrm{LC}\big(\overline{\psi};(\sigma,\tau)\big)$, which, up to the identifications (2.13), is equal to the transition map between $\mathrm{LC}(\varphi;\sigma)$ and $\mathrm{LC}(\psi;\tau)$, is given by:

$$\mathrm{LC}\big(\overline{\psi}\circ\overline{\varphi}^{-1}\big):\mathfrak{M}\big(\Omega,\overline{\varphi}(Z);(\sigma,\tau)\big)\longrightarrow\mathfrak{M}\big(\Omega,\overline{\psi}(Z);(\sigma,\tau)\big);$$

since $\overline{\psi}\circ\overline{\varphi}^{-1}:\overline{\varphi}(Z)\to\overline{\psi}(Z)$ is a smooth fiber preserving diffeomorphism, Lemma 2.6 implies that the map above is a smooth diffeomorphism between open sets, which proves that $\mathrm{LC}(\varphi;\sigma)$ and $\mathrm{LC}(\psi;\tau)$ are smoothly compatible. □

A topological space $\mathcal{X}$ is called *hereditarily paracompact* if every subspace of $\mathcal{X}$ is paracompact (or, equivalently, if every open subspace of $\mathcal{X}$ is paracompact). Under our conventions, all manifolds are hereditarily paracompact. We have the following:

2.8**. Lemma.** *Let $f:\mathcal{X}\to\mathcal{Y}$ be a local homeomorphism, where $\mathcal{X}$, $\mathcal{Y}$ are topological spaces, with $\mathcal{Y}$ Hausdorff and hereditarily paracompact. If $S\subset\mathcal{X}$ is a subset*[2] *such that $f|_S:S\to f(S)$ is a homeomorphism then there exists an open set $Z\subset\mathcal{X}$ containing S such that $f|_Z:Z\to f(Z)$ is a homeomorphism.*

Proof. The proof follows a standard argument that is used in some proofs on the existence of tubular neighborhoods (see for instance [3, §5, Chapter IV]). □

2.9**. Proposition.** *The local charts of the form* (2.12) *form a smooth atlas on the topological space $\mathfrak{M}(\Omega,M)$.*

[2]Actually, if one assumes that $f(S)$ be closed in $\mathcal{Y}$ then it would be sufficient to assume that $\mathcal{Y}$ be Hausdorff and paracompact, rather than hereditarily paracompact.

Proof. By Lemma 2.7, it suffices to find for every $f \in \mathfrak{M}(\Omega, M)$ a chart of the form (2.12) on $\mathfrak{M}(\Omega, M)$ whose domain contains f. Choose an arbitrary Riemannian metric on M (or an arbitrary connection on TM) and denote by $\exp$ the corresponding exponential map, which is a smooth M-valued map on an open subset D of TM. If $\pi : TM \to M$ denotes the projection then, by the inverse function theorem, the map $(\pi, \exp) : D \to M \times M$ is a smooth local diffeomorphism on an open subset of D containing the zero section of TM; thus, by Lemma 2.8, there exists an open set $\widetilde{V} \subset TM$ containing the zero section which is mapped diffeomorphically by $(\pi, \exp)$ onto an open subset $V \subset M \times M$ containing the diagonal. The desired chart $\mathrm{LC}(\varphi; \sigma)$ is now obtained by taking $\xi = TM$, $X = M$, $\varphi = (\pi, \exp)^{-1} : V \to \widetilde{V}$ and $\sigma = f$. □

2.10. *Remark.* As we have already observed, axiom (A6) implies that the space $\mathfrak{M}(\Omega, M)$ is Hausdorff. Actually, since M can be embedded in $I\!R^n$ for some n by Whitney's theorem, Lemma 2.2 implies that $\mathfrak{M}(\Omega, M)$ is homeomorphic to a subspace of the Banachble space $\mathfrak{M}(\Omega, I\!R^n)$. Thus $\mathfrak{M}(\Omega, M)$ is T4, metrizable and hereditarily paracompact. Moreover, $\mathfrak{M}(\Omega, M)$ is second countable if the Banachble space $\mathfrak{M}(\Omega, I\!R)$ is separable.

2.11. *Remark.* Obviously if the Banachble space $\mathfrak{M}(\Omega, I\!R)$ is *Hilbertable* then the proof of Lemma 2.3 shows that the spaces $\mathfrak{M}(\Omega, \xi; \sigma)$ are also Hilbertable and therefore $\mathfrak{M}(\Omega, M)$ is a Hilbert manifold for any manifold M.

From now on, on this section, we will assume that the spaces $\mathfrak{M}(\Omega, M)$ are endowed with the Banach manifold structure defined by the charts (2.12) and we will prove a few basic results about such manifold structure.

The next two propositions are rather trivial though important for the completeness of the theory.

2.12**. Proposition.** *If E is a finite-dimensional real vector space (regarded as a manifold in the canonical way) then the Banachble space $\mathfrak{M}(\Omega, E)$ has its canonical manifold structure, i.e., the manifold structure induced by the atlas containing the identity map of $\mathfrak{M}(\Omega, E)$.*

Proof. Let X be a one point (zero-dimensional) manifold, $\xi = E \to X$ be the trivial bundle over X whose unique fiber is E and let $\sigma \in \mathfrak{M}(\Omega, X)$ be the unique constant map. The chart $\mathrm{LC}(\varphi; \sigma) : \mathfrak{M}(\Omega, E; \sigma, X \times E) = \mathfrak{M}(\Omega, E) \to \mathfrak{M}(\Omega, \xi; \sigma) = \mathfrak{M}(\Omega, E)$ induced by the obvious diffeomorphism $\varphi : X \times E \to \xi$ is equal to the identity map. □

2.13**. Proposition.** *If M is a manifold and $U \subset M$ is an open subset then $\mathfrak{M}(\Omega, U)$ is an open submanifold of $\mathfrak{M}(\Omega, M)$.*

Proof. By axiom (A5), $\mathfrak{M}(\Omega, U)$ is open in $\mathfrak{M}(\Omega, M)$. Moreover, the charts we have defined for $\mathfrak{M}(\Omega, U)$ are also charts for $\mathfrak{M}(\Omega, M)$. □

2.14**. Proposition.** *Given manifolds M, N and a smooth map $\phi : M \to N$ then the left composition map $\mathrm{LC}(\phi) : \mathfrak{M}(\Omega, M) \to \mathfrak{M}(\Omega, N)$ is smooth.*

Proof. Let $f \in \mathfrak{M}(\Omega, M)$ be fixed and choose local charts:

$$\mathrm{LC}(\varphi; \sigma) : \mathfrak{M}(\Omega, M; \sigma, V) \longrightarrow \mathfrak{M}(\Omega, \widetilde{V}; \sigma),$$
$$\mathrm{LC}(\psi; \tau) : \mathfrak{M}(\Omega, N; \tau, W) \longrightarrow \mathfrak{M}(\Omega, \widetilde{W}; \tau),$$

whose domains contain respectively f and $\phi \circ f$. As usual, the definition of the charts above involve manifolds X, Y, maps $\sigma \in \mathfrak{M}(\Omega, X)$, $\tau \in \mathfrak{M}(\Omega, Y)$, smooth vector bundles $\xi \to X$, $\eta \to Y$, open sets $V \subset X \times M$, $W \subset Y \times N$, $\widetilde{V} \subset \xi$, $\widetilde{W} \subset \eta$ and smooth fiber preserving diffeomorphisms $\varphi : V \to \widetilde{V}$, $\psi : W \to \widetilde{W}$. Now we proceed as in the proof of Lemma 2.7 in order to modify the charts $\mathrm{LC}(\varphi;\sigma)$ and $\mathrm{LC}(\psi;\tau)$ so that they correspond to the same auxiliary manifold $X \times Y$ and the same auxiliary map (σ,τ). We will then obtain charts $\mathrm{LC}\big(\overline{\varphi};(\sigma,\tau)\big)$ and $\mathrm{LC}\big(\overline{\psi};(\sigma,\tau)\big)$ similar to (2.16) and (2.17); here the domain of $\overline{\varphi}$ is open in $X \times Y \times M$ while the domain of $\overline{\psi}$ is open in $X \times Y \times N$. The coordinate representation of the map $\mathrm{LC}(\phi)$ with respect to the charts $\mathrm{LC}\big(\overline{\varphi};(\sigma,\tau)\big)$ and $\mathrm{LC}\big(\overline{\psi};(\sigma,\tau)\big)$ is given by (see also (2.4)):

$$\mathrm{LC}\big(\overline{\psi};(\sigma,\tau)\big) \circ \mathrm{LC}(\phi) \circ \mathrm{LC}\big(\overline{\varphi};(\sigma,\tau)\big)^{-1} = \mathrm{LC}\big(\overline{\psi} \circ \overline{\phi} \circ \overline{\varphi}^{-1}\big),$$

where $\overline{\phi} = \mathrm{Id} \times \mathrm{Id} \times \phi : X \times Y \times M \to X \times Y \times N$. Since $\overline{\psi} \circ \overline{\phi} \circ \overline{\varphi}^{-1}$ is a smooth fiber preserving map between open subsets of the vector bundles $\xi \times Y$ and $X \times \eta$, the conclusion follows from Lemma 2.6. □

2.15**. Corollary.** *Given manifolds M, N and a smooth diffeomorphism $\phi : M \to N$ then also* $\mathrm{LC}(\phi) : \mathfrak{M}(\Omega,M) \to \mathfrak{M}(\Omega,N)$ *is a smooth diffeomorphism.* □

2.16**. Corollary.** *Given manifolds M, N and a smooth embedding $\phi : N \to M$ then the left composition map* $\mathrm{LC}(\phi) : \mathfrak{M}(\Omega,N) \to \mathfrak{M}(\Omega,M)$ *is a smooth embedding.*

Proof. As in the proof of Lemma 2.2, we can find an open subset $U \subset M$ containing $\mathrm{Im}(\phi)$ and a smooth left inverse $r : U \to N$ for $\phi : N \to U$. Then, by Proposition 2.14, $\mathrm{LC}(r)$ is a smooth left inverse of $\mathrm{LC}(\phi) : \mathfrak{M}(\Omega,N) \to \mathfrak{M}(\Omega,U)$. This implies that $\mathrm{LC}(\phi)$ is a smooth embedding in $\mathfrak{M}(\Omega,U)$. The conclusion now follows from Proposition 2.13. □

We finish the section by showing the usual identification between the tangent bundle $T\mathfrak{M}(\Omega,M)$ and $\mathfrak{M}(\Omega,TM)$.

2.17**. Proposition.** *For every manifold M and every $a \in \Omega$, the evaluation map* (2.5) *is smooth. Given $f \in \mathfrak{M}(\Omega,M)$ and $v \in T_f\mathfrak{M}(\Omega,M)$ then the map $\hat{v} : \Omega \to TM$ defined by:*

$$\hat{v}(a) = \mathrm{d}(\mathrm{eval}_a)(f)v \in T_{f(a)}M, \quad a \in \Omega,$$

is in $\mathfrak{M}(\Omega,TM)$. Moreover, the map:

$$T\mathfrak{M}(\Omega,M) \ni v \longmapsto \hat{v} \in \mathfrak{M}(\Omega,TM) \tag{2.18}$$

is a smooth diffeomorphism.

Proof. Let $f \in \mathfrak{M}(\Omega,M)$ be fixed and consider a chart of the form (2.12) on $\mathfrak{M}(\Omega,M)$ whose domain contains f. Given $a \in \Omega$ then the fiber preserving smooth diffeomorphism:

$$\varphi : X \times M \supset V \longrightarrow \widetilde{V} \subset \xi$$

induces a chart $\varphi\big(\sigma(a),\cdot\big)$ around $f(a)$ on M, taking values in $\xi_{\sigma(a)}$. The coordinate representation of eval_a with respect to the charts $\mathrm{LC}(\varphi;\sigma)$ and $\varphi\big(\sigma(a),\cdot\big)$ is simply the restriction to $\mathfrak{M}(\Omega,\widetilde{V};\sigma)$ of the evaluation map:

$$\mathfrak{M}(\Omega,\xi;\sigma) \ni s \longmapsto s(a) \in \xi_{\sigma(a)},$$

which is obviously linear and continuous by axiom (A6). Thus eval_a is smooth and for $v \in T_f\mathfrak{M}(\Omega,M)$ we have:

$$\partial_2\varphi\big(\sigma(a),f(a)\big)\hat{v}(a) = \big[\mathrm{d}\,\mathrm{LC}(\varphi;\sigma)(f)v\big](a), \tag{2.19}$$

where $\partial_2\varphi$ denotes differentiation of the map φ with respect to the variable in M.

Denoting by $\pi : TM \to M$ the canonical projection and by $\mathrm{pr}_1 : \xi \oplus \xi \to \xi$ the projection onto the first component then φ induces a fiber preserving smooth diffeomorphism:

$$\mathbb{F}\varphi : X \times TM \supset (\mathrm{Id} \times \pi)^{-1}(V) \longrightarrow \mathrm{pr}_1^{-1}(\widetilde{V}) \subset \xi \oplus \xi,$$

defined by:

$$\mathbb{F}\varphi(x, z) = \big(\varphi(x, m), \partial_2\varphi(x, m)z\big) \in \xi_x \oplus \xi_x,$$

for all $(x, z) \in X \times TM$ with $(x, m) \in V$, where $m = \pi(z) \in M$. The map $\mathbb{F}\varphi$ induces the chart:

$$\mathrm{LC}(\mathbb{F}\varphi; \sigma) : \mathfrak{M}\big(\Omega, TM; \sigma, (\mathrm{Id} \times \pi)^{-1}(V)\big) \longrightarrow \mathfrak{M}\big(\Omega, \mathrm{pr}_1^{-1}(\widetilde{V}); \sigma\big) \subset \mathfrak{M}(\Omega, \xi \oplus \xi; \sigma)$$

on $\mathfrak{M}(\Omega, TM)$. Moreover, the chart $\mathrm{LC}(\varphi; \sigma)$ on $\mathfrak{M}(\Omega, M)$ induces the chart:

$$\big(\mathrm{LC}(\varphi; \sigma) \circ \mathfrak{p}, \mathrm{d}\,\mathrm{LC}(\varphi; \sigma)\big) : T\mathfrak{M}(\Omega, M; \sigma, V) \longrightarrow \mathfrak{M}(\Omega, \widetilde{V}; \sigma) \times \mathfrak{M}(\Omega, \xi; \sigma)$$

on $T\mathfrak{M}(\Omega, M)$, where $\mathfrak{p} : T\mathfrak{M}(\Omega, M) \to \mathfrak{M}(\Omega, M)$ denotes the canonical projection. Identifying $\mathfrak{M}(\Omega, \xi \oplus \xi; \sigma)$ with $\mathfrak{M}(\Omega, \xi; \sigma) \oplus \mathfrak{M}(\Omega, \xi; \sigma)$ (recall (2.9)) then by (2.19) the coordinate representation of (2.18) with respect to the charts $\big(\mathrm{LC}(\varphi; \sigma) \circ \mathfrak{p}, \mathrm{d}\,\mathrm{LC}(\varphi; \sigma)\big)$ and $\mathrm{LC}(\mathbb{F}\varphi; \sigma)$ is simply the identity map of $\mathfrak{M}(\Omega, \widetilde{V}; \sigma) \times \mathfrak{M}(\Omega, \xi; \sigma)$. This shows at the same time that $\hat{v} \in \mathfrak{M}(\Omega, TM)$ and that (2.18) is a smooth diffeomorphism. □

2.18. *Remark.* It is possible to generalize the theory of this section to include spaces of maps $\mathfrak{M}(\Omega, M)$ where M is infinite-dimensional. For instance, one can allow M to belong to the class of Hausdorff paracompact Banach manifolds modelled on a class of Banach spaces that admit a nonzero real valued smooth map with bounded support (for instance, Hilbert spaces). In this case one has to strengthen axiom (A7), so that $\mathfrak{M}(\Omega, E)$ is Banachble for every Banach space E in the class under consideration.

2.19. *Remark.* Given a rule $\mathfrak{M}$ satisfying axioms (A0)—(A7) then the manifold structure in the topological spaces $\mathfrak{M}(\Omega, M)$ is unique if one assumes the validity of Propositions 2.12, 2.13 and 2.14. Namely, the validity of such propositions implies the validity of Corollary 2.16; thus, if one chooses a smooth embedding $\phi : M \to I\!R^n$ into Euclidean space then $\mathrm{LC}(\phi)$ must be a smooth embedding of $\mathfrak{M}(\Omega, M)$ into the Banachble space $\mathfrak{M}(\Omega, I\!R^n)$. But there can be at most one manifold structure on $\mathfrak{M}(\Omega, M)$ for which $\mathrm{LC}(\phi)$ is a smooth embedding.

3. Concrete Examples

In this section we present several concrete examples of rules $\mathfrak{M}$ that satisfy axioms (A0)—(A7) of Section 2. We start by listing some simple examples where the topological space $\mathfrak{M}(\Omega, M)$ can be easily described for every manifold M. Then, we present a method for obtaining the spaces $\mathfrak{M}(\Omega, M)$ from a prescribed Banachble space $\mathcal{E} = \mathfrak{M}(\Omega, I\!R)$ satisfying two simple properties.

3.1**. Example.** Let Ω be an arbitrary set and for each manifold M let:

$$\mathfrak{M}(\Omega, M) = \mathfrak{B}(\Omega, M)$$

be the space of all maps $f : \Omega \to M$ with relatively compact image. Choose a metric d on the manifold M compatible with its topology and consider $\mathfrak{B}(\Omega, M)$ endowed with the uniform convergence topology. If $\phi : (M, d) \to (N, d')$ is a continuous map then it is easy to see that the map $\mathrm{LC}(\phi) : \mathfrak{B}(\Omega, M) \to \mathfrak{B}(\Omega, N)$ is continuous. This implies in particular that the topology on $\mathfrak{B}(\Omega, M)$ does not depend on the metric d, so that we indeed have a topological space $\mathfrak{B}(\Omega, M)$ associated to each manifold M. The verification of the axioms (A0)—(A7) is straightforward. For axiom (A7), observe that $\mathfrak{B}(\Omega, I\!\!R)$ is simply the space of bounded maps $f : \Omega \to I\!\!R$, whose topology can be induced by the usual sup norm $\|f\|_{\sup} = \sup_{a\in\Omega} \big|f(a)\big|$.

3.2**. Example.** Let $(\Omega, \mathcal{A})$ be a *measurable space*, i.e., Ω is a set and $\mathcal{A}$ is a σ-algebra of subsets of Ω. A map defined on Ω and taking values in a given topological space is called *measurable* if the inverse image of every open set is in $\mathcal{A}$. For each manifold M, let $\mathfrak{M}(\Omega, M) = \mathcal{M}_{\rm b}(\Omega, M)$ be the set of all measurable maps $f : \Omega \to M$ with relatively compact image; consider $\mathcal{M}_{\rm b}(\Omega, M)$ endowed with the topology induced by $\mathfrak{B}(\Omega, M)$. One easily checks axioms (A0)—(A7). For axiom (A7), observe that $\mathcal{M}_{\rm b}(\Omega, I\!\!R)$ is a closed subspace of $\mathfrak{B}(\Omega, I\!\!R)$.

3.3**. Example.** Let Ω be an arbitrary topological space and let $\mathfrak{M}(\Omega, M) = C^0_{\rm b}(\Omega, M)$ be the set of all continuous maps $f : \Omega \to M$ with relatively compact image; consider $C^0_{\rm b}(\Omega, M)$ endowed with the topology induced by $\mathfrak{B}(\Omega, M)$. Axioms (A0)—(A7) are easily checked. Observe that $C^0_{\rm b}(\Omega, I\!\!R)$ is the Banachble space of bounded real valued continuous maps on Ω, whose topology is induced by the sup norm.

3.4**. Example.** Let Ω be an open subset of $I\!\!R^m$ and $k \geq 1$ be fixed. Given a manifold M, the *k-th jet bundle* $J^k(I\!\!R^m, M)$ is a fiber bundle over M constructed as follows; for each $x \in M$, the fiber of $J^k(I\!\!R^m, M)$ over x is the set of equivalence classes of M-valued maps f of class C^k defined in a neighborhood of the origin in $I\!\!R^m$ and with $f(0) = x$. The equivalence relation is $f_1 \sim f_2$ iff f_1 and f_2 have the same Taylor polynomial of order k at the origin, when some coordinate chart of M is used around x. There are well-known natural local trivializations of $J^k(I\!\!R^m, M)$ induced by local charts of M. Given a map $f : \Omega \to M$ of class C^k we define the *k-th jet* of f as the continuous map $J^k(f) : \Omega \to J^k(I\!\!R^m, M)$ such that $J^k(f)(a) \in J^k(I\!\!R^m, M)$ is the equivalence class of $f \circ \mathfrak{t}_a$, where $\mathfrak{t}_a : I\!\!R^m \to I\!\!R^m$ denotes the translation by a. We define $C^k_{\rm b}(\Omega, M)$ to be the set of maps $f : \Omega \to M$ of class C^k such that $J^k(f)$ has relatively compact image in $J^k(I\!\!R^m, M)$. The topology on $C^k_{\rm b}(\Omega, M)$ will be induced from $\mathfrak{B}\big(\Omega, J^k(I\!\!R^m, M)\big)$ by the map $f \mapsto J^k(f)$. It is easy to see that $\mathfrak{M} = C^k_{\rm b}$ satisfies axioms (A0)—(A7). Observe that $C^k_{\rm b}(\Omega, I\!\!R)$ is the Banachble space of bounded maps $f : \Omega \to I\!\!R$ of class C^k having bounded partial derivatives up to order k, whose topology is induced by the standard C^k norm:

$$\|f\| = \sum_{|\lambda|\leq k} \|\partial_\lambda f\|_{\sup},$$

where $\lambda = (\lambda_1, \ldots, \lambda_m) \in I\!\!N^m$ denotes a multi-index and $|\lambda| = \sum_{i=1}^m \lambda_i$.

This example *do not* generalize directly to the case that Ω is a manifold. In order to make sense of the space $C^k_{\rm b}(\Omega, M)$ in this case, one needs for instance a connection and a Riemannian metric on Ω (this will be dealt with in Example 3.11 below).

We now present a more systematic method for producing examples of rules $\mathfrak{M}$ satisfying axioms (A0)—(A7). The idea is the following. We start with an arbitrary set Ω and a Banachble space $\mathcal{E}$ of maps $f : \Omega \to I\!\!R$ satisfying a suitable property and we will then show how to construct a rule $\mathfrak{M}$ satisfying axioms (A0)—(A7) for which $\mathfrak{M}(\Omega, I\!\!R) = \mathcal{E}$.

Let Ω be an arbitrary set. As in Example 3.1, we denote by $\mathfrak{B}(\Omega, I\!\!R^n)$ the Banach space of all bounded maps $f : \Omega \to I\!\!R^n$ endowed with the sup norm. We give the following:

3.5**. Definition.** A *Banachble space of bounded maps on* Ω is a Banachble space $\mathcal{E}$ which is a vector subspace of $\mathfrak{B}(\Omega, I\!\!R)$ and such that the inclusion map $\mathcal{E} \hookrightarrow \mathfrak{B}(\Omega, I\!\!R)$ is continuous. For all $n \geq 1$, we identify the Banachble space $\mathcal{E}^n = \bigoplus_n \mathcal{E}$ with the subspace of

$\mathfrak{B}(\Omega, I\!\!R^n)$ consisting of those maps $f : \Omega \to I\!\!R^n$ all of whose coordinates $f_i : \Omega \to I\!\!R$ are in $\mathcal{E}$. We say that $\mathcal{E}$ has the *left composition property* if for every $n \geq 1$ and every smooth map $\phi : I\!\!R^n \to I\!\!R$ the left composition map:

$$\mathrm{LC}(\phi) : \mathcal{E}^n \ni f \longmapsto \phi \circ f \in \mathcal{E}$$

is well-defined[3] and continuous.

We insist on calling $\mathcal{E}$ a Banachble space, rather than a Banach space, to emphasize that only the topology of $\mathcal{E}$ is relevant, and not a particular choice of norm.

Let Ω be a set and $\mathcal{E}$ be a fixed Banachble space of bounded maps on Ω satisfying the left composition property. Given manifolds M, N, we will denote by $C^\infty(M, N)$ the set of all smooth maps from M to N. For an arbitrary manifold M, we set:

$$\mathfrak{M}(\Omega, M) = \big\{ f \in M^\Omega : \alpha \circ f \in \mathcal{E}, \text{ for all } \alpha \in C^\infty(M, I\!\!R) \big\}.$$

We endow $\mathfrak{M}(\Omega, M)$ with the topology induced by the left composition maps:

$$\mathrm{LC}(\alpha) : \mathfrak{M}(\Omega, M) \longrightarrow \mathcal{E},$$

where α runs over the set $C^\infty(M, I\!\!R)$. More explicitly, $\mathfrak{M}(\Omega, M)$ has the coarsest topology for which $\mathrm{LC}(\alpha)$ is continuous for every $\alpha \in C^\infty(M, I\!\!R)$. Thus, if ρ is an $\mathfrak{M}(\Omega, M)$-valued map defined on an arbitrary topological space, then ρ is continuous if and only if $\mathrm{LC}(\alpha) \circ \rho$ is an $\mathcal{E}$-valued continuous map for every $\alpha \in C^\infty(M, I\!\!R)$.

Now we prove that the rule $\mathfrak{M}$ defined above satisfies axioms (A0)—(A7). First, we have the following:

3.6. **Lemma.** *If $\mathcal{E}$ is a Banachble space of bounded maps on a set Ω satisfying the left composition property and if the rule $\mathfrak{M}$ is defined as above then, for every $n \geq 1$, the topological spaces $\mathfrak{M}(\Omega, I\!\!R^n)$ and $\mathcal{E}^n$ are equal.*

Proof. The fact that $\mathcal{E}$ satisfies the left composition property implies that $\mathcal{E}^n$ is contained in $\mathfrak{M}(\Omega, I\!\!R^n)$ and that the inclusion map $\mathcal{E}^n \hookrightarrow \mathfrak{M}(\Omega, I\!\!R^n)$ is continuous. If $\pi_i : I\!\!R^n \to I\!\!R$ denotes projection onto the i-th coordinate, then the fact that $\mathrm{LC}(\pi_i) : \mathfrak{M}(\Omega, I\!\!R^n) \to \mathcal{E}$ is (well-defined and) continuous for all $i = 1, \ldots, n$ implies that $\mathfrak{M}(\Omega, I\!\!R^n)$ is contained in $\mathcal{E}^n$ and that the inclusion map $\mathfrak{M}(\Omega, I\!\!R^n) \hookrightarrow \mathcal{E}^n$ is continuous. $\square$

3.7**. Theorem.** *If Ω is a set and $\mathcal{E}$ is a Banachble space of bounded maps on Ω satisfying the left composition property then the rule $\mathfrak{M}$ defined above satisfies axioms (A0)—(A7) of Section 2.*

Proof. Axioms (A0) and (A7) follow from Lemma 3.6. We now prove the other axioms.

Proof of axiom (A1). It is easy to see that $\mathrm{LC}(\phi)$ carries $\mathfrak{M}(\Omega, M)$ to $\mathfrak{M}(\Omega, N)$. For the continuity, it suffices to show that $\mathrm{LC}(\alpha) \circ \mathrm{LC}(\phi)$ is continuous for all $\alpha \in C^\infty(N, I\!\!R)$; but $\mathrm{LC}(\alpha) \circ \mathrm{LC}(\phi) = \mathrm{LC}(\alpha \circ \phi)$ and $\alpha \circ \phi \in C^\infty(M, I\!\!R)$.

Proof of axiom (A3). If there were some $f \in \mathfrak{M}(\Omega, M)$ with non relatively compact image, we could find a smooth map $\alpha : M \to I\!\!R$ which is unbounded on $\mathrm{Im}(f)$ (for instance, there exists a smooth proper map $\alpha : M \to I\!\!R$). But this contradicts the fact that $\alpha \circ f \in \mathcal{E}$ and $\mathcal{E} \subset \mathfrak{B}(\Omega, I\!\!R)$.

Proof of axiom (A4). Choose $f \in \mathfrak{M}(\Omega, M)$ with $\overline{\mathrm{Im}(f)} \subset U$ and $\lambda \in C^\infty(M, I\!\!R)$ with $\lambda \equiv 1$ on $\overline{\mathrm{Im}(f)}$ and $\mathrm{supp}(\lambda) \subset U$. Given $\alpha \in C^\infty(U, I\!\!R)$, then $\overline{\alpha} = \lambda\alpha$ extends to a smooth map on M that vanishes outside U; moreover, $\alpha \circ f = \overline{\alpha} \circ f \in \mathcal{E}$. This proves that $f \in \mathfrak{M}(\Omega, U)$.

[3]Obviously *well-defined* means that $f \in \mathcal{E}^n$ implies $\phi \circ f \in \mathcal{E}$.

Proof of axiom (A5). Given $f \in \mathfrak{M}(\Omega, U)$, choose $\lambda \in C^\infty(M, I\!\!R)$ with $\lambda \equiv 1$ on $\overline{\mathrm{Im}(f)}$ and $\mathrm{supp}(\lambda) \subset U$. If $\mathcal{P}$ denotes the open subset of $\mathfrak{B}(\Omega, I\!\!R)$ consisting of bounded maps $u : \Omega \to I\!\!R$ with $\inf_{a \in \Omega} u(a) > 0$ then:

$$f \in \mathrm{LC}(\lambda)^{-1}(\mathcal{P} \cap \mathcal{E}) \subset \mathfrak{M}(\Omega, U),$$

and therefore $\mathfrak{M}(\Omega, U)$ is open in $\mathfrak{M}(\Omega, M)$. Now let $\mathfrak{M}(\Omega, U)_\diamondsuit$ temporarily denote the space $\mathfrak{M}(\Omega, U)$ endowed with the topology induced from $\mathfrak{M}(\Omega, M)$ and let us show that $\mathfrak{M}(\Omega, U)_\diamondsuit$ equals $\mathfrak{M}(\Omega, U)$ as a topological space. By axiom (A1), the inclusion map of $\mathfrak{M}(\Omega, U)$ in $\mathfrak{M}(\Omega, M)$ is continuous and hence the identity map:

$$\mathrm{Id} : \mathfrak{M}(\Omega, U) \longrightarrow \mathfrak{M}(\Omega, U)_\diamondsuit$$

is continuous. To prove the continuity of $\mathrm{Id} : \mathfrak{M}(\Omega, U)_\diamondsuit \to \mathfrak{M}(\Omega, U)$, we show that $\mathrm{LC}(\alpha) : \mathfrak{M}(\Omega, U)_\diamondsuit \to \mathcal{E}$ is continuous for every $\alpha \in C^\infty(U, I\!\!R)$. Let $f \in \mathfrak{M}(\Omega, U)$ be fixed and choose an open subset $V \subset M$ with $\overline{\mathrm{Im}(f)} \subset V \subset \overline{V} \subset U$. Choose a map $\overline{\alpha} \in C^\infty(M, I\!\!R)$ that equals α on $\overline{V}$. Then the map $\mathrm{LC}(\overline{\alpha})$ is continuous on $\mathfrak{M}(\Omega, M)$ and hence on $\mathfrak{M}(\Omega, U)_\diamondsuit$; moreover, $\mathrm{LC}(\overline{\alpha})$ agrees with $\mathrm{LC}(\alpha)$ on $\mathfrak{M}(\Omega, V)$, which is an open neighborhood of f in $\mathfrak{M}(\Omega, M)$. This proves the continuity of $\mathrm{LC}(\alpha)$ on $\mathfrak{M}(\Omega, U)_\diamondsuit$.

Proof of axiom (A6). Let $a \in \Omega$ be fixed. We show that $\mathrm{eval}_a^{-1}(U)$ is open in $\mathfrak{M}(\Omega, M)$ for every open set $U \subset M$. Choose $f \in \mathrm{eval}_a^{-1}(U)$; then $f(a) \in U$. Now consider the open subset $\mathcal{P}_a \subset \mathfrak{B}(\Omega, I\!\!R)$ of bounded maps $u : \Omega \to I\!\!R$ with $u(a) > 0$ and choose $\alpha \in C^\infty(M, I\!\!R)$ with $\alpha\big(f(a)\big) = 1$ and $\mathrm{supp}(\alpha) \subset U$; we have:

$$f \in \mathrm{LC}(\alpha)^{-1}(\mathcal{P}_a \cap \mathcal{E}) \subset \mathrm{eval}_a^{-1}(U),$$

which proves that $\mathrm{eval}_a^{-1}(U)$ is open.

Proof of axiom (A2). In order to prove that (2.1) is surjective and that its inverse is continuous, we have to show that for every $\alpha \in C^\infty(M_1 \times M_2, I\!\!R)$, the map:

$$\mathfrak{M}(\Omega, M_1) \times \mathfrak{M}(\Omega, M_2) \ni (f, g) \longmapsto \alpha \circ (f, g) \tag{3.1}$$

takes values in $\mathcal{E}$ and is continuous. Denote by $S \subset C^\infty(M_1 \times M_2, I\!\!R)$ the set of those α for which (3.1) takes values in $\mathcal{E}$ and is continuous. It is obvious that S is a subspace of $C^\infty(M_1 \times M_2, I\!\!R)$. In fact, S is a subalgebra (under pointwise multiplication) of $C^\infty(M_1 \times M_2, I\!\!R)$, because the map $\mathrm{LC}(\mathfrak{m}) : \mathcal{E}^2 \to \mathcal{E}$ of left composition with the multiplication map $\mathfrak{m} : I\!\!R \times I\!\!R \to I\!\!R$ is continuous. It is also obvious that S contains those α that are independent of one of the two variables. Denote by n_i the dimension of M_i, $i = 1, 2$. For the rest of the proof, we will say that a map $\alpha \in C^\infty(M_1 \times M_2, I\!\!R)$ has *small support* if there exists closed sets $F_i \subset M_i$, open sets $U_i \subset M_i$ and diffeomorphisms $\varphi_i : U_i \to I\!\!R^{n_i}$ with $F_i \subset U_i$, $i = 1, 2$ and $\mathrm{supp}(\alpha) \subset F_1 \times F_2$. We claim that S contains all $\alpha \in C^\infty(M_1 \times M_2, I\!\!R)$ with small support. Namely, for $i = 1, 2$, choose an open subset $V_i \subset M_i$ with $F_i \subset V_i \subset \overline{V_i} \subset U_i$, a map $\overline{\varphi}_i \in C^\infty(M_i, I\!\!R^{n_i})$ that equals φ_i on $\overline{V_i}$ and a map $\lambda_i \in C^\infty(M_i, I\!\!R)$ with $\lambda_i \equiv 1$ on F_i and $\mathrm{supp}(\lambda_i) \subset V_i$. Set:

$$\widetilde{\alpha} = \alpha \circ (\varphi_1 \times \varphi_2)^{-1} \in C^\infty(I\!\!R^{n_1} \times I\!\!R^{n_2}, I\!\!R), \quad \overline{\alpha} = \widetilde{\alpha} \circ (\overline{\varphi}_1 \times \overline{\varphi}_2) \in C^\infty(M_1 \times M_2, I\!\!R).$$

It is easy to see that $\alpha(x, y) = \lambda_1(x)\lambda_2(y)\overline{\alpha}(x, y)$, for all $x \in M_1$, $y \in M_2$. Thus, to prove that $\alpha \in S$, it suffices to show that $\overline{\alpha}$ is in S. But this follows by observing that the map $(f, g) \mapsto \overline{\alpha} \circ (f, g)$ is equal to the composite $\mathrm{LC}(\widetilde{\alpha}) \circ \big(\mathrm{LC}(\overline{\varphi}_1) \times \mathrm{LC}(\overline{\varphi}_2)\big)$.

We will now conclude the proof by showing that every $\alpha \in C^\infty(M_1 \times M_2, I\!\!R)$ is in S. Let $f_0 \in \mathfrak{M}(\Omega, M_1)$ and $g_0 \in \mathfrak{M}(\Omega, M_2)$ be fixed. Since $\mathrm{Im}(f_0)$ and $\mathrm{Im}(g_0)$ are relatively compact, a simple argument using a partition of unity shows that we can find maps $\alpha_1, \ldots, \alpha_r \in C^\infty(M_1 \times M_2, I\!\!R)$ with small support such that α equals $\sum_{j=1}^r \alpha_j$

on an open subset of the form $U_1 \times U_2 \subset M_1 \times M_2$ containing $\overline{\mathrm{Im}(f_0)} \times \overline{\mathrm{Im}(g_0)}$. Then $\sum_{j=1}^r \alpha_j$ is in S and thus the map $(f,g) \mapsto \alpha \circ (f,g)$ is $\mathcal{E}$-valued and continuous on the neighborhood $\mathfrak{M}(\Omega, U_1) \times \mathfrak{M}(\Omega, U_2)$ of (f_0, g_0). Since f_0 and g_0 are arbitrary, we have $\alpha \in S$. This concludes the proof. □

Let us now give some examples of Banachble spaces of bounded maps satisfying the left composition property.

3.8**. Example.** Let Ω be a metric space (or, more generally, a uniform space). The space $\mathcal{E} = C^0_{\mathrm{bu}}(\Omega, I\!R)$ of uniformly continuous bounded maps $f : \Omega \to I\!R$ endowed with the sup norm is a Banach space of bounded maps. It is easy to see that $\mathcal{E}$ has the left composition property.

3.9**. Example.** Let (Ω, d) be a metric space, $\alpha \in \left]0,1\right]$ and let $\mathcal{E} = C^{0,\alpha}(\Omega, I\!R)$ be the space of all α-Hölderian bounded maps $f : \Omega \to I\!R$. We define a norm on $\mathcal{E}$ by:

$$\|f\| = \|f\|_{\mathrm{sup}} + \sup_{\substack{a,b\in\Omega \\ a\neq b}} \frac{\big|f(a) - f(b)\big|}{d(a,b)^\alpha}.$$

Then $\mathcal{E}$ is a Banach space of bounded maps with the left composition property.

3.10**. Example.** Let Ω be an open subset of $I\!R^m$, $\alpha \in \left]0,1\right]$, $k \geq 1$ and let $\mathcal{E} = C^{k,\alpha}(\Omega, I\!R)$ be the space of all maps $f : \Omega \to I\!R$ of class C^k such that f and its partial derivatives up to order k are bounded and α-Hölderian. We define a norm on $\mathcal{E}$ by:

$$\|f\| = \sum_{|\lambda|\leq k} \|\partial_\lambda f\|_{\mathrm{sup}} + \sum_{|\lambda|\leq k} \sup_{\substack{a,b\in\Omega \\ a\neq b}} \frac{\big|\partial_\lambda f(a) - \partial_\lambda f(b)\big|}{\|a-b\|^\alpha},$$

where $\lambda = (\lambda_1, \ldots, \lambda_m) \in I\!N^m$ denotes a multi-index. Then $\mathcal{E}$ is a Banach space of bounded maps with the left composition property.

3.11**. Example.** Let Ω be a manifold endowed with a connection on the tangent bundle $T\Omega$ and with a Riemannian metric. If $f : \Omega \to I\!R$ is a map of class C^k then for $i = 1, \ldots, k$, the i-th covariant derivative $\nabla^i f$ of f is a section of the tensor bundle $\bigotimes_i T^*\Omega$, on which there is a natural Riemannian structure induced by the Riemannian metric of Ω. We denote by $\mathcal{E} = C^k_{\mathrm{b}}(\Omega, I\!R)$ the space of all bounded maps $f : \Omega \to I\!R$ of class C^k for which $\|\nabla^i f\|$ is bounded for all $i = 1, \ldots, k$. We define a norm on $\mathcal{E}$ by:

$$\|f\| = \|f\|_{\mathrm{sup}} + \sum_{i=1}^k \big\|\nabla^i f\big\|_{\mathrm{sup}}.$$

Then $\mathcal{E}$ is a Banach space of bounded maps with the left composition property.

3.12**. Example.** Let Ω be a set and let $\mathcal{E}_1$, $\mathcal{E}_2$ be Banachble spaces of bounded maps in Ω with the left composition property. Set $\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2$ and consider $\mathcal{E}$ endowed with the topology induced by the two inclusion maps $\mathcal{E} \to \mathcal{E}_i$, $i = 1, 2$. If $\|\cdot\|_i$ is a norm for $\mathcal{E}_i$, $i = 1, 2$, then $\|\cdot\| = \|\cdot\|_1|_{\mathcal{E}} + \|\cdot\|_2|_{\mathcal{E}}$ is a norm for $\mathcal{E}$. Obviously $\mathcal{E}$ is a Banach space with continuous inclusion in both $\mathcal{E}_1$ and $\mathcal{E}_2$ (and hence in $\mathfrak{B}(\Omega, I\!R)$). Moreover, it is easy to see that $\mathcal{E}$ also has the left composition property.

3.13. *Remark.* If $\mathcal{E}$ is a Banachble space of bounded maps on a set Ω satisfying the left composition property then there exists a *unique* way of defining the topological spaces $\mathfrak{M}(\Omega, M)$ so that axioms (A0)—(A7) are satisfied and $\mathfrak{M}(\Omega, I\!R) = \mathcal{E}$. Namely, if M is a manifold and $\phi : M \to I\!R^n$ is an embedding into Euclidean space then, by Lemma 2.2, $\mathrm{LC}(\phi)$ has to be a homeomorphism onto the space:

$$\big\{f \in \mathfrak{M}(\Omega, I\!R^n) : \overline{\mathrm{Im}(f)} \subset \mathrm{Im}(\phi)\big\},$$

with the topology induced from $\mathfrak{M}(\Omega, I\!R^n)$. But axiom (A2) implies that $\mathfrak{M}(\Omega, I\!R^n) = \mathcal{E}^n$.

3.14. *Remark.* If $\mathfrak{M}$ is a rule satisfying axioms (A0)—(A7) then $\mathcal{E} = \mathfrak{M}(\Omega, I\!R)$ must be a Banachble space contained in $\mathfrak{B}(\Omega, I\!R)$, by axioms (A3) and (A7). By axiom (A6), the inclusion map $\mathcal{E} \to \mathfrak{B}(\Omega, I\!R)$ has closed graph and it is therefore continuous. Moreover, axioms (A1) and (A2) imply that $\mathcal{E}$ must have the left composition property. Thus (keeping in mind Lemma 3.6, Theorem 3.7 and Remark 3.13), $\mathfrak{M} \mapsto \mathcal{E} = \mathfrak{M}(\Omega, I\!R)$ gives a one to one correspondence between rules $\mathfrak{M}$ satisfying axioms (A0)—(A7) and Banachble spaces $\mathcal{E}$ of bounded maps satisfying the left composition property.

3.1. **When constant maps are missing.** Let Ω be a set and let $\mathcal{E}$ be a Banachble space of bounded maps on Ω. If $\mathcal{E}$ satisfies the left composition property, then $\mathcal{E}$ must contain the constant maps; namely, if $\phi : I\!R \to I\!R$ is a constant map then $\mathrm{LC}(\phi)$ must take values in $\mathcal{E}$. Thus, if a Banachble space of bounded maps $\mathcal{E}$ does not contain the constants, one cannot hope that $\mathcal{E}$ could have the left composition property. But there are important natural function spaces that do not contain the constants. We will deal with this problem now.

3.15**. Definition.** Let Ω be a set and let $\mathcal{E}$ be a Banachble space of bounded maps on Ω. We say that $\mathcal{E}$ has the *left composition property of type zero* if for every $n \geq 1$ and every smooth map $\phi : I\!R^n \to I\!R$ with $\phi(0) = 0$ the left composition map $\mathrm{LC}(\phi) : \mathcal{E}^n \to \mathcal{E}$ is well-defined and continuous.

We have a simple lemma.

3.16**. Lemma.** *Let $\mathcal{E}$ be a Banachble space of bounded maps on a set Ω satisfying the left composition property of type zero. If $\mathcal{E}$ contains the constant maps then $\mathcal{E}$ satisfies the left composition property.*

Proof. Given $\phi \in C^\infty(I\!R^n, I\!R)$, set $\psi = \phi - \phi(0) \in C^\infty(I\!R^n, I\!R)$ and observe that $\mathrm{LC}(\phi) = \mathrm{LC}(\psi) + c$, where $c : \mathcal{E}^n \to \mathcal{E}$ denotes the constant map equal to $\phi(0) \in \mathcal{E}$. $\square$

If a Banachble space of bounded maps $\mathcal{E}$ does not contain the constants, one may hope that, by adding the constants to $\mathcal{E}$, we may obtain a space that satisfies the left composition property. With this in mind, we give the following:

3.17**. Definition.** Let Ω be a set and let $\mathcal{E}$ be a Banachble space of bounded maps on Ω. We say that $\mathcal{E}$ has the *parametric left composition property of type zero* if for every $s, n \geq 1$ and every smooth map $\phi : I\!R^n \times I\!R^s \to I\!R$ such that $\phi(0, c) = 0$, for all $c \in I\!R^s$, the map:

$$\mathcal{E}^n \times I\!R^s \ni (f, c) \longmapsto \phi_c \circ f \in \mathcal{E},$$

is well-defined and continuous, where $\phi_c = \phi(\cdot, c) : I\!R^n \to I\!R$, for all $c \in I\!R^s$.

Obviously the parametric left composition property of type zero implies the left composition property of type zero.

If Ω is a set we denote by $\mathrm{Ct}(\Omega)$ the unidimensional space of constant maps $f : \Omega \to I\!R$; we identify $\mathrm{Ct}(\Omega)^n$ with the space of constant maps $f : \Omega \to I\!R^n$.

3.18**. Lemma.** *Let $\mathcal{E}_0$ be a Banachble space of bounded maps on a set Ω satisfying the parametric left composition property of type zero. If $\mathrm{Ct}(\Omega) \not\subset \mathcal{E}_0$ (or, equivalently, if $\mathrm{Ct}(\Omega)\cap\mathcal{E}_0 = \{0\}$) then $\mathcal{E} = \mathcal{E}_0 \oplus \mathrm{Ct}(\Omega)$ is a Banachble space of bounded maps satisfying the left composition property.*

Proof. Obviously $\mathcal{E}$ has continuous inclusion in $\mathfrak{B}(\Omega, I\!R)$. Let $\phi : I\!R^n \to I\!R$ be a smooth map. If we write the elements of $\mathcal{E}^n$ as pairs (f, c) with $f \in \mathcal{E}_0^n$, $c \in I\!R^n \cong \mathrm{Ct}(\Omega)^n$, then the left composition map $\mathrm{LC}(\phi)$ is given by:

$$\mathrm{LC}(\phi) : \mathcal{E}^n \ni (f, c) \longmapsto \big(\psi_c \circ f, \phi(c)\big) \in \mathcal{E},$$

where $\psi : I\!R^n \times I\!R^n \to I\!R$ is given by $\psi(x, c) = \phi(x + c) - \phi(c)$ and $\psi_c = \psi(\cdot, c)$. □

We now use the trick of "adding the constants" explained above to deal with some interesting examples.

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, i.e., Ω is a set, $\mathcal{A}$ is a σ-algebra of subsets of Ω and $\mu : \mathcal{A} \to [0, +\infty]$ is a countably additive measure. Choose a real number $p \in [1, +\infty[$ and consider the space $L^p_{\mathrm{b}}(\Omega, I\!R)$ consisting of bounded measurable maps $f : \Omega \to I\!R$ such that:

$$\|f\|_p = \Big(\int_\Omega |f|^p \,\mathrm{d}\mu\Big)^{\frac{1}{p}} < +\infty.$$

It is easy to see that $L^p_{\mathrm{b}}(\Omega, I\!R)$ endowed with the norm:

$$\|f\| = \|f\|_{\sup} + \|f\|_p$$

is a Banach space[4]. If $\mu(\Omega) < +\infty$ then $L^p_{\mathrm{b}}(\Omega, I\!R)$ coincides with the closed subspace $\mathcal{M}_{\mathrm{b}}(\Omega, I\!R)$ of $\mathfrak{B}(\Omega, I\!R)$ (see Example 3.2) and the norm above is equivalent to the sup norm. We will therefore focuss on the case $\mu(\Omega) = +\infty$; observe that in this case $L^p_{\mathrm{b}}(\Omega, I\!R)$ does not contain the constant maps.

3.19**. Lemma.** *The space $\mathcal{E} = L^p_{\mathrm{b}}(\Omega, I\!R)$ is a Banachble space of bounded maps satisfying the parametric left composition property of type zero.*

Proof. Obviously $\mathcal{E}$ has continuous inclusion in $\mathfrak{B}(\Omega, I\!R)$. Let $\phi : I\!R^n \times I\!R^s \to I\!R$ be a smooth map with $\phi(0, c) = 0$ for all $c \in I\!R^s$; let us show that the map:

$$\mathcal{E}^n \times I\!R^s \ni (f, c) \longmapsto \phi_c \circ f \in \mathcal{E} \tag{3.2}$$

is well-defined and continuous, where $\phi_c = \phi(\cdot, c)$. For fixed $c \in I\!R^s$, the map ϕ_c is Lipschitz on every bounded subset of $I\!R^n$ and, since $\phi_c(0) = 0$, we have:

$$\big|\phi_c\big(f(a)\big)\big| \le C\big\|f(a)\big\|, \quad a \in \Omega,$$

for some constant $C \ge 0$ dependent on $f \in \mathcal{E}^n$. This shows that $\|\phi_c \circ f\|_p < +\infty$ and obviously $\phi_c \circ f$ is measurable and bounded. Thus (3.2) is well-defined. To prove the continuity of (3.2), let $f \in \mathcal{E}^n$ and $c \in I\!R^s$ be fixed and choose a compact convex set $K \subset I\!R^n$ containing the origin and with $\overline{\mathrm{Im}(f)}$ contained in the interior of K. Then the set of those $g \in \mathcal{E}^n$ with $\mathrm{Im}(g) \subset K$ is a neighborhood of f in $\mathcal{E}^n$; for such g and any $d \in I\!R^s$, $a \in \Omega$, we compute:

$$\begin{aligned}\big|\phi\big(f(a), c\big) - \phi\big(g(a), d\big)\big| &\le \big|\phi\big(f(a), c\big) - \phi\big(g(a), c\big)\big| + \big|\phi\big(g(a), c\big) - \phi\big(g(a), d\big)\big| \\ &\le C\big\|f(a) - g(a)\big\| + \sup_{x \in K}\big\|\mathrm{d}\phi_c(x) - \mathrm{d}\phi_d(x)\big\|\,\big\|g(a)\big\|,\end{aligned}$$

where C is a Lipschitz constant for ϕ_c on K. From the inequalities above it is easy to see that $\phi_d \circ g$ tends to $\phi_c \circ f$ in $\mathcal{E}$ as d tends to c in $I\!R^s$ and g tends to f in $\mathcal{E}^n$. □

[4] We do not identify maps that are equal almost everywhere!

3.20**. Corollary.** *If $\mu(\Omega)=+\infty$ then the space $\mathcal{E}=L^p_{\mathrm{b}}(\Omega,I\!R)\oplus\mathrm{Ct}(\Omega)$ is a Banachble space of bounded maps satisfying the left composition property.*

Proof. Follows from Lemmas 3.18 and 3.19. □

If Ω is an open subset of $I\!R^m$ and $p\in[1,+\infty[$ is a real number, we denote as usual by $L^p(\Omega,I\!R)$ the space of (equivalence classes of almost everywhere equal) measurable maps $f:\Omega\to I\!R$ with $\|f\|_p<+\infty$. For every integer $k\ge 1$, we denote by $W^{k,p}(\Omega,I\!R)$ the Sobolev space of maps $f\in L^p(\Omega,I\!R)$ such that $\partial_\lambda f\in L^p(\Omega,I\!R)$ for every multi-index $\lambda=(\lambda_1,\ldots,\lambda_m)\in I\!N^m$ with $|\lambda|\le k$; the partial derivative ∂_λ is understood in the distributional sense. As usual, the norm on $W^{k,p}(\Omega,I\!R)$ is defined by:

$$\|f\|=\sum_{|\lambda|\le k}\|\partial_\lambda f\|_p$$

and $\|\cdot\|$ turns $W^{k,p}(\Omega,I\!R)$ into a Banach space. For all the results that we will use about Sobolev spaces we refer to [1]. If Ω is a bounded domain with smooth boundary then the theory of Banach manifolds modelled on the Sobolev spaces $W^{k,p}(\Omega,I\!R)$ is well-known. For instance, if $k>\frac{m}{p}$ then $W^{k,p}(\Omega,I\!R)$ has continuous inclusion on $C^0_{\mathrm{b}}(\Omega,I\!R)$ and it satisfies the left composition property (see [4, Lemma 9.9]). We will therefore focuss only on the case that Ω is unbounded and, for simplicity, we take $\Omega=I\!R^m$. Obviously, $W^{k,p}(I\!R^m,I\!R)$ doesn't contain the constant maps because $L^p(I\!R^m,I\!R)$ doesn't. If $k>\frac{m}{p}$, the space $W^{k,p}(I\!R^m,I\!R)$ has continuous inclusion in $C^0_{\mathrm{b}}(I\!R^m,I\!R)$. Since the space $C^\infty_{\mathrm{c}}(I\!R^m,I\!R)$ of smooth real valued maps with compact support on $I\!R^m$ is dense in $W^{k,p}(I\!R^m,I\!R)$, it follows that the elements of $W^{k,p}(I\!R^m,I\!R)$ tend to zero at infinity. We are now going to look for conditions under which the space $W^{k,p}(I\!R^m,I\!R)$ satisfies the parametric left composition property of type zero. Unfortunately, contrary to the case of bounded Ω, the condition $k>\frac{m}{p}$ alone does not imply such property[5]. Not all is lost, though, as it is shown in the following:

3.21**. Lemma.** *If $m<p$ and $k\ge 1$ then the Sobolev space $W^{k,p}(I\!R^m,I\!R)$ is a Banachble space of bounded maps in $I\!R^m$ satisfying the parametric left composition property of type zero.*

Proof. Since $k\ge 1>\frac{m}{p}$, the space $\mathcal{E}=W^{k,p}(I\!R^m,I\!R)$ has continuous inclusion in $C^0_{\mathrm{b}}(I\!R^m,I\!R)$ and thus in $\mathfrak{B}(I\!R^m,I\!R)$. Let $\phi:I\!R^n\times I\!R^s\to I\!R$ be a smooth map with $\phi(0,c)=0$ for all $c\in I\!R^s$; let us show that the map $\mathcal{E}^n\times I\!R^s\ni(f,c)\mapsto\phi_c\circ f\in\mathcal{E}$ is well-defined and continuous, where $\phi_c=\phi(\cdot,c)$. Since $C^\infty_{\mathrm{c}}(I\!R^m,I\!R^n)$ is dense in $\mathcal{E}^n$, by standard arguments, it suffices to show that the map:

$$C^\infty_{\mathrm{c}}(I\!R^m,I\!R^n)\times I\!R^s\ni(f,c)\longmapsto\partial_\lambda(\phi_c\circ f)\in L^p(I\!R^m,I\!R),$$

has a continuous extension to $\mathcal{E}^n\times I\!R^s$ for every multi-index $\lambda=(\lambda_1,\ldots,\lambda_m)\in I\!N^m$ with $|\lambda|\le k$. The case $\lambda=0$ (i.e., $\partial_\lambda(\phi_c\circ f)=\phi_c\circ f$) follows from Lemma 3.19, observing that $\mathcal{E}$ has continuous inclusion in $L^p_{\mathrm{b}}(I\!R^m,I\!R)$. If $\lambda\ne 0$ then, for $a\in I\!R^m$, $\partial_\lambda(\phi_c\circ f)(a)$ is a linear combination of terms of the form:

$$(\partial_\beta\phi_c)\big(f(a)\big)\;(\partial_{\gamma_1}f_{i_1})(a)\;(\partial_{\gamma_2}f_{i_2})(a)\;\cdots\;(\partial_{\gamma_r}f_{i_r})(a),$$

where $f=(f_1,\ldots,f_n)$, $i_1,\ldots,i_r\in\{1,\ldots,n\}$ and $\beta\in I\!N^n$, γ_1, …, $\gamma_r\in I\!N^m$ are nonzero multi-indices with $|\beta|\le|\lambda|$ and $\sum_{i=1}^r|\gamma_i|=|\lambda|$. Since $\mathcal{E}^n$ has continuous

---

[5]The difficulty of the case $\Omega=I\!R^m$ lies on the fact that the continuous embedding $W^{k,p}(I\!R^m,I\!R)\hookrightarrow L^q(I\!R^m,I\!R)$ holds only for $\frac{1}{q}=\frac{1}{p}-\frac{k}{m}$ and not in general for $\frac{1}{q}\ge\frac{1}{p}-\frac{k}{m}$, as it does in the case of bounded domains. Thus the proof of [4, Theorem 9.4] doesn't work if the domain is unbounded.

inclusion in $C^0_{\mathrm{b}}(I\!R^m, I\!R^n)$, we have that $(f, c) \mapsto (\partial_\beta \phi_c) \circ f$ defines a $C^0_{\mathrm{b}}(I\!R^m, I\!R)$-valued continuous map on $\mathcal{E}^n \times I\!R^s$. Since the multiplication map:

$$C^0_{\mathrm{b}}(I\!R^m, I\!R) \times L^p(I\!R^m, I\!R) \ni (g_1, g_2) \longmapsto g_1 g_2 \in L^p(I\!R^m, I\!R) \tag{3.3}$$

is continuous, to complete the proof it suffices to show that $f \mapsto \prod_{u=1}^r \partial_{\gamma_u} f_{i_u}$ defines a continuous $L^p(I\!R^m, I\!R)$-valued map on $\mathcal{E}^n$. If $r = 1$ then $|\gamma_1| \le k$ and $f \mapsto \partial_{\gamma_1} f_{i_1}$ is a continuous $L^p(I\!R^m, I\!R)$-valued map on $\mathcal{E}^n$. Assume now that $r > 1$. Then $|\gamma_u| < k$ for all $u = 1, \ldots, r$ and then $f \mapsto \partial_{\gamma_u} f_{i_u}$ defines a continuous $W^{1,p}(I\!R^m, I\!R)$-valued map on $\mathcal{E}^n$. The hypothesis $m < p$ implies that $W^{1,p}(I\!R^m, I\!R)$ has continuous inclusion in $C^0_{\mathrm{b}}(I\!R^m, I\!R)$ and, since the multiplication map of $C^0_{\mathrm{b}}(I\!R^m, I\!R)$ is continuous, it follows that the map:

$$\mathcal{E}^n \ni f \longmapsto \prod_{u=1}^{r-1} \partial_{\gamma_u} f_{i_u} \in C^0_{\mathrm{b}}(I\!R^m, I\!R)$$

is continuous. Finally, since $\mathcal{E}^n \ni f \mapsto \partial_{\gamma_r} f_{i_r} \in L^p(I\!R^m, I\!R)$ is continuous, the conclusion follows from the continuity of (3.3). □

3.22**. Corollary.** *Under the hypothesis of Lemma 3.21,* $W^{k,p}(I\!R^m, I\!R) \oplus \mathrm{Ct}(I\!R^m)$ *is a Banachble space of bounded maps in* $I\!R^m$ *satisfying the left composition property.*

Proof. Follows from Lemmas 3.18 and 3.21. □

Observe that the important case $H^k(I\!R, I\!R) = W^{k,2}(I\!R, I\!R)$ satisfies the hypothesis of Lemma 3.21.

References

[1] R. A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, vol. 65. Academic Press, New York–London, 1975.

[2] A. Kriegl, P. Michor, *The Convenient Setting for Global Analysis*, M.S.M. vol. 53, Amer. Math. Soc., Providence, 1997.

[3] S. Lang, *Differential Manifolds*, Addison–Wesley Publishing Company, Massachusetts, 1972.

[4] R. Palais, *Foundations of Global Nonlinear Analysis*, W. A. Benjamin, 1968.

[5] P. Piccione, D. V. Tausk, *On the Banach Differentiable Structure for Sets of Maps on Non-Compact Domains*, Nonlinear Analysis. Theory, Methods & Applications, v. 2, n. 46, pgs. 245—265 (2001).

[6] M. Schwarz, *Morse Homology*, Progress in Mathematics vol. 111, Birkhäuser, 1993.

DEPARTAMENTO DE MATEMÁTICA,
UNIVERSIDADE DE SÃO PAULO, BRAZIL
E-mail address: `tausk@ime.usp.br`
URL: `http://www.ime.usp.br/~tausk`