LOCALIZING THE CONNECTIVITY OF THE COMPLEMENT

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If X is a connected topological space we sometimes want to know if X remains connected after we remove a “small” subset S of X. For example, if we remove a line from a plane then we disconnect the plane, but if we remove just a point the plane remains connected. More generally, it can be shown[1] using transversality arguments that the complement of an immersed (second countable) submanifold of a connected differentiable manifold is path connected if the codimension of the submanifold is at least 2.

Here we are not interested in proving results establishing the connectivity of the complement of a subset, we will be merely showing conditions under which such problem is local. We start with the following result.

Proposition 1. *Let X be a connected topological space and S be a subset of X with empty interior. If every $x \in S$ has a neighborhood V in X such that $V \setminus S$ is connected then $X \setminus S$ is connected.*

Proof. Let C be nonempty and clopen with respect to $X \setminus S$. We will show that the closure $\overline{C}$ of C in X is open in X and this will imply that $\overline{C} = X$ and hence that $C = \overline{C} \cap (X \setminus S) = X \setminus S$. So let $x \in \overline{C}$ be given and let us check that x is an interior point of $\overline{C}$. If $x \in S$, we take V as in the statement of the proposition and we note that $C \cap (V \setminus S)$ is nonempty and clopen with respect to $V \setminus S$, so that C contains $V \setminus S$. Now since $X \setminus S$ is dense in X, if U is an open neighborhood of x contained in V we have that $U \cap (X \setminus S) = U \setminus S$ is dense in U and thus U is contained in $\overline{C}$. Now assume $x \notin S$, so that $x \in \overline{C} \cap (X \setminus S) = C$. Since C is open in $X \setminus S$, there exists A open in X with $C = A \cap (X \setminus S) = A \setminus S$ and since $A \setminus S$ is dense in A we have that A is contained in $\overline{C}$. But $x \in C \subset A$, so x is an interior point of $\overline{C}$. ☐

Can we get a path connected version of Proposition 1? Yes we can, but the obvious adaptation doesn't work as we will see later (Example 4). We need the following simple lemma.

Lemma 2. *Let X be a connected topological space and $\mathcal{U}$ be an open cover of X. Given $p, q \in X$, there exists a finite nonempty sequence $(U_i)_{i=1}^n$ in $\mathcal{U}$ such that $p \in U_1$, $q \in U_n$ and $U_i \cap U_{i+1} \neq \emptyset$ for $i = 1, \dots, n-1$.*

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[1]For two points p and q not on the submanifold, one shows that every smooth path connecting p and q admits a small perturbation that is transversal to the submanifold. If the codimension is at least 2, transversality implies disjointness.

Proof. Consider the binary relation $\sim$ on X such that $p \sim q$ if and only if a sequence $(U_i)_{i=1}^n$ as in the statement of the lemma exists. It is readily checked that $\sim$ is an equivalence relation whose equivalence classes are open. It follows that the equivalence classes are also closed, as the complement of an equivalence class is the union of the other equivalence classes. The connectivity of X then implies that there is at most one equivalence class. □

Proposition 3. *Let X be a connected topological space and S be a subset of X with empty interior. If every $x \in X$ has a neighborhood V in X such that $V \setminus S$ is path connected then $X \setminus S$ is path connected.*

Proof. Let $\mathcal{U}$ be the collection consisting of the interiors of all the subsets V of X such that $V \setminus S$ is path connected. By our assumptions, $\mathcal{U}$ is a covering of X. Given $p, q \in X \setminus S$, let $(U_i)_{i=1}^n$ be a sequence in $\mathcal{U}$ as in the statement of Lemma 2. Since S has empty interior, for $i = 1, \ldots, n-1$, there exists x_i in the nonempty open set $U_i \cap U_{i+1}$ not in S. Setting $x_0 = p$ and $x_n = q$, we have that x_i and x_{i+1} are both in $U_{i+1} \setminus S$ for $i = 0, \ldots, n-1$ and therefore x_i and x_{i+1} can be connected by a continuous path in $X \setminus S$. The conclusion follows. □

The simplest attempted adaptation of Proposition 1 to the path connected case would assume that every $x \in S$ has a neighborhood V such that $V \setminus S$ is path connected, but in Proposition 3 we assumed this condition to hold for every $x \in X$. If X is locally path connected then obviously such condition will hold automatically for $x \in X$ outside of the closure of S, as for such x we could pick a path connected neighborhood V of x that is disjoint from S. Would it be sufficient in Proposition 3 to assume the existence of the neighborhood V for every $x \in S$ if we knew that X is locally path connected? The answer is no, as the next example shows.

Example 4. Let $X = \big(\,]0, +\infty[\, \times \mathbb{R}\big) \cup \big(\mathbb{R} \times \,]-1, 1[\,\big)$ be endowed with the topology induced by the Euclidean plane $\mathbb{R}^2$ and set:

$$S = \left\{ \frac{1}{n} : n = 1, 2, \ldots \right\} \times \,]-1, 1[\,.$$

We have that X is a connected open subset of $\mathbb{R}^2$ and therefore it is also locally path connected and path connected. Moreover, $V = \,]0, +\infty[\, \times \mathbb{R}$ is an open subset of X containing S and it is easily checked that $V \setminus S$ is path connected, so that for every $x \in S$ we have that V is a neighborhood of x in X such that $V \setminus S$ is path connected. We show that $X \setminus S$ is not path connected. Assume by contradiction that there exists a continuous path $(x, y) : [0, 1] \to X \setminus S \subset \mathbb{R}^2$ such that $x(0) = y(0) = 0$ and $x(1) > 0$. Let $t \in [0, 1]$ be the infimum of $x^{-1}\big[\,]0, +\infty[\,\big]$ which by continuity must satisfy $x(t) = 0$ and $t < 1$. Now since $\big(x(t), y(t)\big) \in X$ we must have $y(t) \in \,]-1, 1[$ and therefore $[t, t+\varepsilon] \subset y^{-1}\big[\,]-1, 1[\,\big]$ for some $\varepsilon > 0$. But by the definition of t the map x must assume some positive value on $]t, t + \varepsilon]$ and thus $x(s) = \frac{1}{n}$

for some $s \in \,]t, t+\varepsilon]$ and some positive integer n. Since also $y(s) \in \,]-1, 1[$ we get $\big(x(s), y(s)\big) \in S$, contradicting our assumptions on the path (x, y).

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