LIE ALGEBRAS AND FLOWS

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1. General abstract results

In this section, a map is called *differentiable* if it admits a derivative.

Lemma 1. *Let V be a real finite-dimensional vector space and $v_i : I \to V$, $i = 1, \ldots, k$, be differentiable maps defined in an interval $I \subset \mathbb{R}$. Denote by V_t the linear span of $\{v_i(t) : i = 1, \ldots, k\}$, for all $t \in I$. If $\big(v_i(t)\big)_{i=1}^k$ is linearly independent for every $t \in I$ and $v_i'(t)$ belongs to V_t for all $i = 1, \ldots, k$ and all $t \in I$ then V_t is independent of $t \in I$.*

Proof. This can be proven by noting that $I \ni t \mapsto V_t$ is a differentiable curve in the Grassmannian of k-dimensional subspaces of V whose derivative vanishes[1]. Here is a more elementary proof in case one wishes to avoid the theory of Grassmannians: let $t_0 \in I$ and let us prove that the map $t \mapsto V_t$ is constant near t_0. Pick a subspace W of V such that $V = V_{t_0} \oplus W$ and write $v_i(t) = x_i(t) + y_i(t)$, with $x_i(t) \in V_{t_0}$ and $y_i(t) \in W$, for all $t \in I$ and all $i = 1, \ldots, k$. By continuity, $\big(x_i(t)\big)_{i=1}^k$ is a basis of V_{t_0} for $t \in I$ near t_0 and for such t there exists a unique linear map $L_t : V_{t_0} \to W$ such that $L_t\big(x_i(t)\big) = y_i(t)$, for $i = 1, \ldots, k$; moreover, the maps $t \mapsto x_i(t)$, $t \mapsto y_i(t)$ and $t \mapsto L_t$ are all differentiable. Note that V_t is the graph of L_t, i.e., $V_t = \big\{x + L_t(x) : x \in V_{t_0}\big\}$. Thus, since $v_i'(t) = x_i'(t) + y_i'(t)$ and $v_i'(t) \in V_t$, we have $y_i'(t) = L_t\big(x_i'(t)\big)$, for all $i = 1, \ldots, k$. Now differentiating both sides of the equality $L_t\big(x_i(t)\big) = y_i(t)$ with respect to t we obtain $L_t'\big(x_i(t)\big) = 0$ and therefore the derivative of $t \mapsto L_t$ vanishes, so that $t \mapsto L_t$ and hence $t \mapsto V_t$ is constant for $t \in I$ near t_0. □

In what follows, if M is a set and $(E_x)_{x \in M}$ is a family of vector spaces then, for each subset F of M, we denote by $\rho_F : \prod_{x \in M} E_x \to \prod_{x \in F} E_x$ the restriction map given by $\rho_F(v) = v|_F$, for all $v \in \prod_{x \in M} E_x$.

Lemma 2. *Let M be a set and $(E_x)_{x \in M}$ be a family of vector spaces. If V is a finite-dimensional subspace of $\prod_{x \in M} E_x$ then there exists a finite subset F of M with at most* $\dim(V)$ *elements such that $\rho_F|_V$ is injective.*

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[1] The tangent space of the Grassmannian at V_t can be identified with the space of linear maps from V_t to V/V_t and the derivative V_t' is identified with the linear map that carries $v_i(t)$ to $v_i'(t) + V_t \in V/V_t$. See [1, Proposition 2.3.3].

Proof. By induction on the dimension of V. If V is null, take F to be the empty set. If V has codimension one in V' and F is a finite subset of M with at most $\dim(V)$ elements such that $\rho_F|_V$ is injective, then either $\rho_F|_{V'}$ is also injective or its kernel is spanned by a nonzero element $v \in V'$. In the latter case, we pick $x \in M$ with $v(x) \neq 0$ and we set $F' = F \cup \{x\}$, so that F' has at most $\dim(V')$ elements and $\rho_{F'}|_{V'}$ is injective. □

Lemma 3. *Let M be a set, $(E_x)_{x\in M}$ be a family of vector spaces and V_1 and V_2 be subspaces of $\prod_{x\in M} E_x$. Assume that F is a subset of M such that $\rho_F|_{V_2}$ is injective. If $\rho_{F\cup\{x\}}[V_1] \subset \rho_{F\cup\{x\}}[V_2]$ for all $x \in M$, then $V_1 \subset V_2$. In particular, if $\rho_F|_{V_1}$ is also injective and if $\rho_{F\cup\{x\}}[V_1] = \rho_{F\cup\{x\}}[V_2]$ for all $x \in M$, then $V_1 = V_2$.*

Proof. Note that our assumptions imply that $\rho_F[V_1] \subset \rho_F[V_2]$. Given $v \in V_1$, we can thus find $w \in V_2$ with $v|_F = w|_F$. To conclude the proof, we show that $v(x) = w(x)$ for arbitrary $x \in M$. Since $\rho_{F\cup\{x\}}[V_1] \subset \rho_{F\cup\{x\}}[V_2]$, there exists $w' \in V_2$ with $v|_{F\cup\{x\}} = w'|_{F\cup\{x\}}$. We then have $w|_F = w'|_F$ and the injectivity of $\rho_F|_{V_2}$ implies that $w = w'$. □

If M is a set and $(E_x)_{x\in M}$ is a family of finite-dimensional real vector spaces, we call a map $v : I \to \prod_{x\in M} E_x$ defined in an interval $I \subset \mathbb{R}$ *pointwise differentiable* if the map $I \ni t \mapsto v(t)(x) \in E_x$ is differentiable, for every $x \in M$. If v is pointwise differentiable, we define the *pointwise derivative* $v' : I \to \prod_{x\in M} E_x$ by letting $I \ni t \mapsto v'(t)(x) \in E_x$ be the derivative of the map $I \ni t \mapsto v(t)(x) \in E_x$, for all $x \in M$.

Lemma 4. *Let M be a set, $(E_x)_{x\in M}$ be a family of finite-dimensional real vector spaces and $v_i : I \to \prod_{x\in M} E_x$, $i = 1, \ldots, k$, be pointwise differentiable maps defined in an interval $I \subset \mathbb{R}$. Denote by $V_t \subset \prod_{x\in M} E_x$ the linear span of $\{v_i(t) : i = 1, \ldots, k\}$, for all $t \in I$. If $\big(v_i(t)\big)_{i=1}^k$ is linearly independent for every $t \in I$ and $v_i'(t)$ belongs to V_t for all $i = 1, \ldots, k$ and all $t \in I$ then V_t is independent of $t \in I$.*

Proof. Fix $t_0 \in I$ and let us show that V_t is independent of t for $t \in I$ near t_0. By Lemma 2, there exists a finite subset F of M such that ρ_F is injective on V_{t_0}. Thus $\big(v_i(t_0)|_F\big)_{i=1}^k$ is linearly independent and by continuity we have that $\big(v_i(t)|_F\big)_{i=1}^k$ is linearly independent for all t in a connected neighborhood J of t_0 in I. This implies that ρ_F is injective on V_t for all $t \in J$. By Lemma 3, to prove that V_t is independent of t for $t \in J$ it is sufficient to check that $\rho_G[V_t]$ is independent of $t \in J$ for an arbitrary finite subset G of M containing F. Since $\big(v_i(t)|_G\big)_{i=1}^k$ is linearly independent for all $t \in J$ and $\prod_{x\in G} E_x$ is finite-dimensional, the latter statement follows from Lemma 1. □

Lemma 5. *Let M be a set and $(E_x)_{x\in M}$ be a family of finite-dimensional real vector spaces. If V is a finite-dimensional subspace of $\prod_{x\in M} E_x$ then V is closed with respect to the pointwise convergence topology.*

Proof. This follows from the general fact that a finite-dimensional subspace is always closed in a Hausdorff topological vector space. Here is a more elementary proof: pick $w \in \prod_{x\in M} E_x$ in the closure of V with respect to the pointwise convergence topology. By Lemma 2, there exists a finite subset F of M such that ρ_F is injective on the linear span of $V \cup \{w\}$. Since ρ_F is continuous with respect to the pointwise convergence topology, we have that $\rho_F(w)$ is in the closure of $\rho_F[V]$. But the pointwise convergence topology in the finite-dimensional vector space $\prod_{x\in F} E_x$ coincides with the canonical topology (induced by an arbitrary norm) and thus $\rho_F[V]$ is closed in $\prod_{x\in F} E_x$. Hence $\rho_F(w)$ is in $\rho_F[V]$ and the injectivity of ρ_F on $V \cup \{w\}$ implies that $w \in V$. □

2. Results on manifolds and vector fields

Given a differentiable manifold M, we denote by $\mathfrak{X}(M)$ the space of all smooth vector fields over M, where *smooth* means "of class C^∞". If V is a subspace of $\mathfrak{X}(M)$ and A is an open subset of M, we denote by $V|_A$ the subspace of $\mathfrak{X}(A)$ given by $V|_A = \big\{X|_A : X \in V\big\}$. If $\phi : N \to M$ is a smooth local diffeomorphism between differentiable manifolds, we denote by $\phi^* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ the pull-back linear map induced by ϕ defined by $(\phi^* X)(x) = \mathrm{d}\phi_x^{-1}\big(X(\phi(x))\big)$, for all $x \in N$. If V is a subspace of $\mathfrak{X}(M)$ then $\phi^*[V]$ denotes the subspace of $\mathfrak{X}(N)$ which is the image of V under ϕ^*.

Given $X \in \mathfrak{X}(M)$, recall that the *maximal flow* of the vector field X is the map $F : D \to M$, with $D \subset \mathbb{R} \times M$, such that for every $x \in M$ we have that $\big\{t \in \mathbb{R} : (t, x) \in D\big\} \ni t \mapsto F(t, x) \in M$ is the maximal integral curve of X passing through x at $t = 0$. It holds that D is open in $\mathbb{R}\times M$ and F is smooth. Moreover, for every $t \in \mathbb{R}$, the map $F_t : D_t \to D_{-t}$ defined by $F_t(x) = F(t, x)$ is a smooth diffeomorphism, where $D_t = \big\{x \in M : (t, x) \in D\big\}$. We note also that the domain D_t of F_t is decreasing in t for $t \geq 0$, i.e., $D_s \subset D_t$ if $0 \leq t \leq s$.

Theorem 6. *Let M be a differentiable manifold, V be a finite-dimensional subspace of $\mathfrak{X}(M)$ and $X \in \mathfrak{X}(M)$ be such that the Lie bracket $[X, Y]$ belongs to V, for all $Y \in V$. If $F : D \to M$ denotes the maximal flow of X then $F_t^*[V] = V|_{D_t}$, for all $t \in \mathbb{R}$.*

As an immediate consequence of Theorem 6 we have the following result.

Corollary 7. *Let M be a differentiable manifold and $\mathfrak{g}$ be a finite-dimensional Lie subalgebra of $\mathfrak{X}(M)$. For every $X \in \mathfrak{g}$, if $F : D \to M$ denotes the maximal flow of X, then $F_t^*[\mathfrak{g}] = \mathfrak{g}|_{D_t}$, for all $t \in \mathbb{R}$.* □

We start by proving a particular case of Theorem 6.

Lemma 8. *Under the assumptions of Theorem 6, it holds that $F_t^*[V]$ is contained in $V|_{D_t}$, for all $t \geq 0$.*

Proof. Set

$$I = \big\{t \in [0, +\infty[\, : F_s^*[V] \subset V|_{D_s}, \text{ for all } s \in [0, t]\big\}$$

so that $0 \in I$ and $t \in I$ implies $[0,t] \subset I$. To conclude the proof, we show that I must be unbounded from above by establishing the following facts:

(a) if $t > 0$ and $[0,t[\subset I$, then $t \in I$;

(b) if $t \in I$, then $[t, t+\varepsilon] \subset I$, for some $\varepsilon > 0$.

Note that if I were bounded from above then (a) would imply that $t = \sup I$ belongs to I and this would contradict (b). Now let us proceed with the proof of (a) and (b). To prove (a), simply note that for all $Y \in V$ we have that $\lim_{s\to t^-} F_s^*(Y)|_{D_t} = F_t^*(Y)$ with respect to the pointwise convergence topology of $\mathfrak{X}(D_t)$ and that $V|_{D_t}$ is closed in $\mathfrak{X}(D_t)$ with respect to such topology, being a finite-dimensional subspace (Lemma 5). To prove (b), let $t \in I$ be fixed and pick $Y_1, \ldots, Y_k \in V$ such that $\big(F_t^*(Y_i)\big)_{i=1}^k$ is a basis of $F_t^*[V]$. By Lemma 2, there exists a finite subset A of D_t such that $\big(F_t^*(Y_i)|_A\big)_{i=1}^k$ is linearly independent. Since D is open and F is continuous, we can then find $\varepsilon > 0$ such that $A \subset D_{t+\varepsilon}$ and $\big(F_s^*(Y_i)|_A\big)_{i=1}^k$ is linearly independent for all $s \in [t, t+\varepsilon]$. Fix $t' \in \,]t, t+\varepsilon]$ and let us prove that $F_{t'}^*[V] \subset V|_{D_{t'}}$. To this aim, we will apply Lemma 4 to the pointwise differentiable maps

$$[t,t'] \ni s \longmapsto F_s^*(Y_i)|_{D_{t'}} \in \prod_{x\in D_{t'}} T_xM, \quad i = 1, \ldots, k.$$

Since A is contained in $D_{t'}$, we have that $\big(F_s^*(Y_i)|_{D_{t'}}\big)_{i=1}^k$ is linearly independent for $s \in [t,t']$. Moreover, for $s \geq t$, since $F_s = F_t \circ (F_{s-t}|_{D_s})$ we have that $\big\{F_s^*(Y_i) : i = 1, \ldots, k\big\}$ spans $F_s^*[V]$. Now $\frac{\mathrm{d}}{\mathrm{d}s} F_s^*(Y_i) = F_s^*\big([X, Y_i]\big) \in F_s^*[V]$ for $i = 1, \ldots, k$ and thus the assumptions of Lemma 4 are verified. The lemma now yields $F_{t'}^*[V] = F_t^*[V]|_{D_{t'}} \subset V|_{D_{t'}}$ and we are done. □

Proof of Theorem 6. Replacing X with $-X$ and using Lemma 8 we obtain that $F_t^*[V] \subset V|_{D_t}$, for all $t \in \mathbb{R}$. Moreover, since F_{-t} is the inverse of F_t, $F_t^*[V] \subset V|_{D_t}$ implies $V|_{D_{-t}} \subset F_{-t}^*[V]$, for all $t \in \mathbb{R}$ and this concludes the proof. □

References

[1] P. Piccione & D. V. Tausk, A Student's Guide to Symplectic Spaces, Grassmannians and Maslov Index, https://www.ime.usp.br/ piccione/Downloads/MaslovBook.pdf

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