SEPARATION OF SETS BY JORDAN MEASURABLE SETS

DANIEL V. TAUSK

The following result is obtained by a simple adaptation of the standard proof that a locally compact Hausdorff σ-compact topological space is paracompact. Recall that a space is *σ-compact* if it is a countable union of compact sets.

Lemma 1. *Let M be a locally compact metric space. If U is a σ-compact open subset of M then U is the union of a countable family $(B_i)_{i\in I}$ of open balls of M such that:*

(i) *$(B_i)_{i\in I}$ is locally finite in U, i.e., every point of U has a neighborhood that intersects B_i only for finitely many $i \in I$;*

(ii) *the diameter of the balls B_i converge to zero, i.e., for every $\varepsilon > 0$ there are only finitely many $i \in I$ such that the diameter of B_i is greater than ε.*

Proof. Since M is locally compact and U is σ-compact, there exists a sequence $(K_n)_{n\geq 1}$ of compact subsets of U such that $U = \bigcup_{n=1}^{\infty} K_n$ and $K_n \subset \mathrm{int}(K_{n+1})$ for all n, where $\mathrm{int}(A)$ denotes the interior of a subset A of M. Set $K_n = \emptyset$, for all $n \leq 0$. For each $n \geq 1$ consider the compact set $C_n = K_n \setminus \mathrm{int}(K_{n-1})$. Clearly $U = \bigcup_{n=1}^{\infty} C_n$. Now cover each C_n with a finite family of open balls $(B_i)_{i\in I_n}$ contained in U such that each B_i is disjoint from K_{n-2} and the diameter of each B_i is less than $\frac{1}{n}$. To conclude the proof, simply let I be equal to the disjoint union of the index sets I_n. Notice that (i) holds because if $x \in K_n$, then $V = \mathrm{int}(K_{n+1})$ is an open neighborhood of x and V is disjoint from B_i for all $i \in I_m$ with $m \geq n + 3$. □

Definition 2. A subset A of $\mathbb{R}^n$ is called *Jordan measurable* if its boundary ∂A has null Lebesgue measure.

Clearly, the collection of Jordan measurable sets is closed under finite unions, finite intersections and set differences.

Lemma 3. *Let U be an open subset of $\mathbb{R}^n$. If S is a subset of U such that $\overline{S}\setminus U$ has null Lebesgue measure then there exists an open Jordan measurable subset A of U containing S.*

Proof. Let $(B_i)_{i\in I}$ be a countable family of (say, Euclidean) open balls of $\mathbb{R}^n$ such that $U = \bigcup_{i\in I} B_i$ and (i) and (ii) of Lemma 1 hold. We set $A = \bigcup_{i\in J} B_i$, with $J = \big\{ i \in I : B_i \cap S \neq \emptyset \big\}$. To conclude the proof we show

Date: April 2nd, 2019.

that A is Jordan measurable by proving that ∂A is contained in the union of $\bigcup_{i\in J}\partial B_i$ with $\overline{S}\setminus U$. Let $x\in\partial A$ and assume that x is not in $\bigcup_{i\in J}\partial B_i$. Notice that since A is open, we have $x\notin A$ and thus $x\notin\bigcup_{i\in J}\overline{B}_i$. If $x\in U$, then the local finiteness of $(B_i)_{i\in I}$ in U implies that x has a neighborhood disjoint from all B_i with $i\in J$, which contradicts $x\in\partial A$. It now remains to check that $x\in\overline{S}$. If x has a neighborhood that intersects B_i only for finitely many $i\in J$, then as before we obtain a contradiction by finding a smaller neighborhood of x that is disjoint from all B_i with $i\in J$. So assume that every neighborhood of x intersects B_i for infinitely many $i\in J$. If B is an open ball with center x and radius $r>0$, then the open ball B' with center x and radius $\frac{r}{2}$ intersects infinitely many B_i with $i\in J$ and by (ii) of Lemma 1 it intersects some such B_i having diameter less than $\frac{r}{2}$. This implies that B_i is contained in B and, since B_i intersects S, we conclude that B intersects S. □

Definition 4. If f is a map taking values in a metric space then its *oscillation* $\omega(f)$ is defined as the diameter of the image of f. If the domain of f is a topological space and x is a point in the domain of f, then the *oscillation* $\omega(f;x)$ *of* f *at the point* x is defined as the infimum of $\omega(f|_V)$, with V ranging over all neighborhoods of x.

Clearly, f is continuous at a point x if and only if $\omega(f;x)=0$. Moreover, for every $c>0$ the set of all x in the domain of f satisfying $\omega(f;x)<c$ is open.

Definition 5. A function f defined in a subset of $\mathbb{R}^n$ and taking values in a topological space will be called *almost everywhere continuous* if its set of discontinuities has null Lebesgue measure.

Notice that if X is a bounded Jordan measurable subset of $\mathbb{R}^n$ and if $f:X\to\mathbb{R}$ is a bounded function, then f is almost everywhere continuous if and only if f is Riemann integrable.

Lemma 6. *If X is a Jordan measurable subset of $\mathbb{R}^n$ and f is an almost everywhere continuous function defined in X, then for every $c>0$ it holds that every subset C of*

$$\big\{x\in X:\omega(f;x)\geq c\big\} \tag{1}$$

is Jordan measurable.

Proof. Since (1) is closed in X, it follows that the closure of C is contained in the union of (1) with the boundary of X. Hence the closure of C (and its boundary) has null Lebesgue measure. □

Proposition 7. *Let X be a Jordan measurable subset of $\mathbb{R}^n$ and $f:X\to\mathbb{R}$ be an almost everywhere continuous function. Given $a,b\in\mathbb{R}$ with $a<b$, there exists a Jordan measurable subset B of $\mathbb{R}^n$ such that B contains*

$$\big\{x\in X:f(x)\leq a\big\} \tag{2}$$

and it is contained in

$$\big\{x \in X : f(x) < b\big\}. \tag{3}$$

Proof. Let S denote the set of all points $x \in X$ which are in the closure of (2) and such that $\omega(f;x) < c$, where $c = b - a$. Given $x \in S$, we have that x has an open neighborhood V_x in $\mathbb{R}^n$ such that $\omega(f|_{V_x \cap X}) < c$; since V_x intersects (2), it follows that $V_x \cap X$ is contained in (3). By setting $U = \bigcup_{x \in S} V_x$, we obtain an open subset U of $\mathbb{R}^n$ containing S such that $U \cap X$ is contained in (3).

Now if $x \in \overline{S}$ is not in S, then either x is not in X (in which case $x \in \partial X$) or $x \in X$ and $\omega(f;x) \geq c$ (in which case x is a discontinuity point of f). It follows that $\overline{S} \setminus S$ and thus $\overline{S} \setminus U$ has null Lebesgue measure. Using Lemma 3 we then obtain an open Jordan measurable subset A of U containing S. We now obtain B by taking the union of $A \cap X$ with

$$\big\{x \in X : \omega(f;x) \geq c \text{ and } f(x) < b\big\}$$

which is Jordan measurable by Lemma 6. ∎

Departamento de Matemática,
Universidade de São Paulo, Brazil
E-mail address: tausk@ime.usp.br
URL: http://www.ime.usp.br/~tausk