SOME INEQUALITIES WITH PROBABILITIES

DANIEL V. TAUSK

1. The inequalities

Let C denote the set of all maps $\alpha : \{1,2,3\} \to \{0,1\}$. We think of 0, 1 as *colors* (say, 0 for green and 1 for red) and of an element $\alpha \in C$ as a triple of colors. Given $\alpha, \beta \in C$, we set:

$$n(\alpha,\beta) = \big|\big\{(i,j) \in \{1,2,3\}^2 : i \neq j \text{ and } \alpha(i) = \beta(j)\big\}\big|,$$
$$m(\alpha,\beta) = \big|\big\{i \in \{1,2,3\} : \alpha(i) = \beta(i)\big\}\big|,$$

where $|X|$ denotes the number of elements of a finite set X.

1.1. **Lemma.** *For any $\alpha, \beta \in C$, we have:*

$$1 + n(\alpha,\beta) \geq m(\alpha,\beta).$$

Proof. The inequality is trivial if $m(\alpha,\beta)$ is either 0 or 1. If $m(\alpha,\beta) = 2$, a simple inspection of possibilities shows that $n(\alpha,\beta) \geq m(\alpha,\beta)$. Finally, if $m(\alpha,\beta) = 3$ then $\alpha = \beta$ and $n(\alpha,\beta) = n(\alpha,\alpha) \geq 2$. □

By a *probability distribution* on C we mean a map $p : C \to [0,1]$ such that:

$$\sum_{\alpha \in C} p(\alpha) = 1.$$

We denote by Δ the set of all probability distributions on C. Given $p, q \in \Delta$, $i, j \in \{1,2,3\}$, we set:

$$P_{ij}(p,q) = \sum_{\substack{\alpha,\beta \in C \\ \alpha(i) = \beta(j)}} p(\alpha)q(\beta). \tag{1.1}$$

1.2. **Lemma.** *Given $p, q \in \Delta$ then:*

$$1 + \sum_{\substack{i,j=1 \\ i \neq j}}^{3} P_{ij}(p,q) \geq \sum_{i=1}^{3} P_{ii}(p,q). \tag{1.2}$$

Proof. We have:

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{3} P_{ij}(p,q) = \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \sum_{\substack{\alpha,\beta \in C \\ \alpha(i) = \beta(j)}} p(\alpha)q(\beta) = \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \sum_{\alpha,\beta \in C} \chi(i,j,\alpha,\beta)p(\alpha)q(\beta),$$

Date: February 17th, 2008.

where $\chi(i,j,\alpha,\beta) = 1$ for $\alpha(i) = \beta(j)$ and $\chi(i,j,\alpha,\beta) = 0$ for $\alpha(i) \neq \beta(j)$. But:

$$\sum_{\substack{i,j=1\\i\neq j}}^{3} \sum_{\alpha,\beta\in C} \chi(i,j,\alpha,\beta)p(\alpha)q(\beta) = \sum_{\alpha,\beta\in C} \sum_{\substack{i,j=1\\i\neq j}}^{3} \chi(i,j,\alpha,\beta)p(\alpha)q(\beta)$$
$$= \sum_{\alpha,\beta\in C} n(\alpha,\beta)p(\alpha)q(\beta),$$

and therefore:

$$\sum_{\substack{i,j=1\\i\neq j}}^{3} P_{ij}(p,q) = \sum_{\alpha,\beta\in C} n(\alpha,\beta)p(\alpha)q(\beta). \tag{1.3}$$

A similar computation gives:

$$\sum_{i=1}^{3} P_{ii}(p,q) = \sum_{i=1}^{3} \sum_{\substack{\alpha,\beta\in C\\ \alpha(i)=\beta(i)}} p(\alpha)q(\beta) = \sum_{\alpha,\beta\in C} m(\alpha,\beta)p(\alpha)q(\beta). \tag{1.4}$$

Hence:

$$1 + \sum_{\substack{i,j=1\\i\neq j}}^{3} P_{ij}(p,q) \overset{(1.3)}{=} 1 + \sum_{\alpha,\beta\in C} n(\alpha,\beta)p(\alpha)q(\beta)$$
$$= \sum_{\alpha,\beta\in C} p(\alpha)q(\beta) + \sum_{\alpha,\beta\in C} n(\alpha,\beta)p(\alpha)q(\beta)$$
$$= \sum_{\alpha,\beta\in C} \big(1 + n(\alpha,\beta)\big)p(\alpha)q(\beta) \geq \sum_{\alpha,\beta\in C} m(\alpha,\beta)p(\alpha)q(\beta) \overset{(1.4)}{=} \sum_{i=1}^{3} P_{ii}(p,q),$$

where the last inequality follows from Lemma 1.1. □

Now let μ be an arbitrary probability measure[1] on the cartesian product $\Delta^2 = \Delta \times \Delta$. We set:

$$\mathcal{P}_{ij}(\mu) = \int_{\Delta^2} P_{ij}(p,q)\,\mathrm{d}\mu(p,q), \tag{1.5}$$

for all $i,j \in \{1,2,3\}$.

[1] We assume that the domain of μ is a σ-algebra of subsets of $\Delta \times \Delta$ that is large enough to make the map $(p,q) \mapsto p(\alpha)q(\beta)$ measurable, for all $\alpha,\beta \in C$. Notice that Δ can be identified with a subset of $\mathbb{R}^8$ (an affine simplex) and that such condition holds if we endow $\Delta \times \Delta$ with the Borel σ-algebra.

1.3. **Proposition.** *Given a probability measure μ on Δ^2, the following inequality holds:*

$$1 + \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \mathcal{P}_{ij}(\mu) \geq \sum_{i=1}^{3} \mathcal{P}_{ii}(\mu).$$

Proof. Follows immediately by taking the integral with respect to μ on both sides of inequality (1.2). □

2. An experiment

Assume that we have a pair of boxes, call them A and B. Each box has three buttons, numbered 1, 2, 3 and two small lamps; one lamp is colored green and the other is colored red. If we press one of the buttons in a box then one (and only one) of the colored lamps lights up. The two boxes are manufactured by the same source (let's call it the *factory*); then, box A is sent to a far away location and box B is sent to another far away location. Each box will be received by an experimenter that will randomly choose one of the three buttons and press it. The experiment is repeated a large number of times, each time with a new pair of boxes. Each time the pair of boxes is prepared, the factory chooses a pair $p, q \in \Delta$ of probability distributions on the set C of all possible triples of colors. The probability distribution p is programmed into the box A as follows: when button number $i \in \{1, 2, 3\}$ is pressed, the box randomly chooses a triple of colors $\alpha \in C$ according to the probability distribution p and then it lights up the lamp[2] colored $\alpha(i)$. Similarly, the probability distribution q is programmed into the box B. The factory chooses the pairs (p, q) of probability distributions randomly according to some probability measure μ on $\Delta^2 = \Delta \times \Delta$. It seems that this procedure is general enough so that essentially any manufacturing strategy used by the factory is covered, *as long as the boxes A and B are not allowed to interact after a button is pressed by one of the experimenters.* This "impossibility of interaction" hypothesis — let's call it *locality* — is encoded in our formalism in the hypothesis that the lottery p used by box A to associate a button to a lamp is independent of the lottery q used by box B to associate a button to a lamp. Notice that, since p and q are chosen by the same source, we allow the choice of q to be dependent on the choice of p; this is encoded in our formalism in the fact that the probability measure μ is *arbitrary*, i.e., it is not necessarily a product $\mu_1 \times \mu_2$ of two probability measures μ_1, μ_2 on Δ.

Now, let $p, q \in \Delta$ be given and assume that the boxes A and B are programmed respectively with the probability distributions p and q. Given, $i, j \in \{1, 2, 3\}$, if button number i is pressed in box A and button number j

[2] The factory could choose a triple $p_1, p_2, p_3 \in [0, 1]$ and programme box A to behave as follows: if button number $i \in \{1, 2, 3\}$ is pressed then the green lamp lights up with probability p_i (and the red lamp lights up with probability $1 - p_i$). This would amount to defining $p \in \Delta$ by the formula: $p(\alpha) = \prod_{i=1}^{3} \big(p_i + \alpha(i)(1 - 2p_i)\big)$, for all $\alpha \in C$.

is pressed in box B, the probability that both boxes will light up the same color is $P_{ij}(p, q)$ (recall (1.1)). Now recall that the experiment is repeated a large number N of times. Given $i, j \in \{1, 2, 3\}$, let N_{ij} denote the number of times that button number i was pressed in box A and button number j was pressed in box B. Let N_{ij}^* denote the number of times that button number i was pressed in box A, button number j was pressed in box j and the same color lit up in both boxes. *We now assume that the experimenters choose the buttons they press using a lottery that is independent*[3] *of the lottery* μ *used by the factory to choose the pair* (p, q). This hypothesis implies that the quotient of N_{ij}^* by N_{ij} is approximately equal to $\mathcal{P}_{ij}(\mu)$ (recall (1.5)); such approximation gets better as N gets larger.

Now, assume that the experiment has been performed a large number N of times and that we have observed:

$$\frac{N_{ij}^*}{N_{ij}} \approx \frac{1}{4}, \quad \text{for } i, j \in \{1, 2, 3\},\ i \neq j, \tag{2.1}$$

$$\frac{N_{ii}^*}{N_{ii}} \approx 1, \quad \text{for } i \in \{1, 2, 3\}, \tag{2.2}$$

where $\approx$ denotes[4] "approximately equal". Is this possible? We have:

$$\mathcal{P}_{ij}(\mu) \approx \frac{1}{4}, \text{ for } i \neq j, \quad \mathcal{P}_{ii}(\mu) \approx 1,$$

so that:

$$1 + \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \mathcal{P}_{ij}(\mu) \approx 1 + \frac{6}{4} = \frac{5}{2},$$

$$\sum_{i=1}^{3} \mathcal{P}_{ii}(\mu) \approx 3.$$

But this contradicts Proposition 1.3.

Now, it is a well-known fact that such an experiment can be performed[5] and that both (2.1) and (2.2) are observed. What is going on? Since these observations contradict Proposition 1.3, we have to drop one of our hypotheses: either the "non clairvoyant factory hypothesis" or the "locality hypothesis". It seems that locality is gone for good!

---

[3]We call this the *non clairvoyant factory* hypothesis: the factory cannot predict how the buttons to be pressed (far away in space and in the future) will be chosen.

[4]How good the approximation must be? Good enough so that I can get to the contradiction with Proposition 1.3 explained below.

[5]See [1, 2, 3]. The experiments confirm the quantum predictions for measurements of polarization of entangled pairs of photons. The theoretical prediction gives $N_{ii}^* = N_{ii}$ and $\mathcal{P}_{ii}(\mu) = 1$, but we can never count on real life experiments to give us perfect correlations, so my argument allows $\mathcal{P}_{ii}(\mu) \approx 1$.

References

[1] A. Aspect, P. Grangier, G. Roger, Phys. Rev. Lett. **47**, 460 (1981).
[2] A. Aspect, P. Grangier, G. Roger, Phys. Rev. Lett. **49**, 91 (1982).
[3] A. Aspect, J. Dalibard, G. Roger, Phys. Rev. Lett. **49**, 1804 (1982).