EXTERIOR MEASURES

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Definition 1. Let X be a set. By an *exterior measure* on X we mean a map $\phi : \wp(X) \to [0, +\infty]$ satisfying the following conditions:

(i) $\phi(\emptyset) = 0$;

(ii) for all $A, B \subset X$, if $A \subset B$ then $\phi(A) \le \phi(B)$;

(iii) for every sequence $(A_n)_{n\ge1}$ of subsets of X, we have:

$$\phi\Big(\bigcup_{n=1}^{\infty} A_n\Big) \le \sum_{n=1}^{\infty} \phi(A_n).$$

Because of (i), we have that (iii) also holds for finite unions.

Definition 2. Let ϕ be an exterior measure on a set X. We say that a subset A of X is *ϕ-measurable* if

$$\phi(S) = \phi(S \cap A) + \phi(S \setminus A), \tag{1}$$

for every $S \subset X$. We denote by $\mathcal{M}_\phi$ the collection of all ϕ-measurable subsets of X.

Clearly (1) is equivalent to

$$\phi(S) \ge \phi(S \cap A) + \phi(S \setminus A) \tag{2}$$

and thus it is sufficient to consider subsets S of X with $\phi(S) < +\infty$ when checking that A is ϕ-measurable.

Definition 3. Given a set X, a collection $\mathcal{A}$ of subsets of X is called a *σ-algebra* of subsets of X if it is nonempty, closed under countable unions and closed under complementation in X. A map $\mu : \mathcal{A} \to [0, +\infty]$ is called a *measure* if $\mu(\emptyset) = 0$ and

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n), \tag{3}$$

for every sequence $(A_n)_{n\ge1}$ in $\mathcal{A}$ of pairwise disjoint sets.

Clearly (3) also holds for finite unions.

It is well-known that if ϕ is an exterior measure on X then $\mathcal{M}_\phi$ is a σ-algebra of subsets of X and that

$$\phi\Big(S \cap \bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \phi(S \cap A_n),$$

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for every $S \subset X$ and every sequence $(A_n)_{n\geq 1}$ in $\mathcal{M}_\phi$ of pairwise disjoint sets. In particular, the restriction of ϕ to $\mathcal{M}_\phi$ is a measure.

Definition 4. Let X be a set, $\mathcal{A}$ be a σ-algebra of subsets of X and let $\mu : \mathcal{A} \to [0, +\infty]$ be a measure. We say that μ is *complete* if for every $A \in \mathcal{A}$ with $\mu(A) = 0$ it holds that every subset of A is in $\mathcal{A}$. The *completion* of the measure μ is the measure $\bar{\mu}$ in the σ-algebra $\overline{\mathcal{A}}$ of subsets of X defined by

$$\overline{\mathcal{A}} = \big\{A \cup N : A \in \mathcal{A},\ N \subset M,\ M \in \mathcal{A} \text{ and } \mu(M) = 0\big\}$$

and

$$\bar{\mu}(A \cup N) = \mu(A),$$

for every $A \in \mathcal{A}$ and every N that is contained in some $M \in \mathcal{A}$ with $\mu(M) = 0$.

It is easy to see that $\bar{\mu}$ is the smallest complete extension of μ, i.e., $\bar{\mu}$ is a complete extension of μ and every complete extension of μ extends $\bar{\mu}$. If ϕ is an exterior measure on a set X, then every $A \subset X$ with $\phi(A) = 0$ is ϕ-measurable. It follows that the restriction of ϕ to $\mathcal{M}_\phi$ is a complete measure.

Definition 5. Let X be a set, ϕ be an exterior measure on X and $\mathcal{A}$ be a σ-algebra of subsets of X. We say that ϕ is $\mathcal{A}$-*regular* if $\mathcal{A} \subset \mathcal{M}_\phi$ and for every $B \subset X$ there exists $A \in \mathcal{A}$ with $B \subset A$ and $\phi(B) = \phi(A)$. We say that ϕ is *regular* if it is $\mathcal{M}_\phi$-regular.

Proposition 6. *Let X be a set, $\mathcal{A}$ be a σ-algebra of subsets of X and $\mu : \mathcal{A} \to [0, +\infty]$ be a measure. If $\mu^* : \wp(X) \to [0, +\infty]$ is defined by setting*

$$\mu^*(B) = \inf\big\{\mu(A) : A \in \mathcal{A} \text{ and } B \subset A\big\}$$

then μ^* is the unique $\mathcal{A}$-regular exterior measure on X that extends μ

Proof. It is clear that an $\mathcal{A}$-regular exterior measure on X that extends μ must be equal to μ^*, that μ^* does extend μ and that $B_1 \subset B_2 \subset X$ implies $\mu^*(B_1) \leq \mu^*(B_2)$. Given a sequence $(B_n)_{n\geq 1}$ of subsets of X and $\varepsilon > 0$, we pick $A_n \in \mathcal{A}$ containing B_n with $\mu(A_n) \leq \mu^*(B_n) + \frac{\varepsilon}{2^n}$ and note that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ contains $\bigcup_{n=1}^{\infty} B_n$, so that:

$$\mu^*\Big(\bigcup_{n=1}^{\infty} B_n\Big) \leq \mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} \mu(A_n) \leq \Big(\sum_{n=1}^{\infty} \mu^*(B_n)\Big) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves that μ^* is an exterior measure. Now let $A \in \mathcal{A}$ be given and let us check that A is μ^*-measurable. Given $S \subset X$, to verify that (2) holds with $\phi = \mu^*$ it suffices to check that

$$\mu(E) \geq \mu^*(S \cap A) + \mu^*(S \setminus A),$$

for every $E \in \mathcal{A}$ containing S. This follows from:

$$\mu(E) = \mu(E \cap A) + \mu(E \setminus A) \geq \mu^*(S \cap A) + \mu^*(S \setminus A).$$

Finally, given $B \subset X$, for each $n \geq 1$ we pick $A_n \in \mathcal{A}$ containing B with $\mu(A_n) \leq \mu^*(B) + \frac{1}{n}$ and we set $A = \bigcap_{n=1}^{\infty} A_n$, so that $A \in \mathcal{A}$ contains B and $\mu^*(B) = \mu(A) = \mu^*(A)$. □

Corollary 7. *Let X be a set and $\mathcal{A}$ be a σ-algebra of subsets of X. The correspondences $\mu \mapsto \mu^*$ and $\phi \mapsto \phi|_{\mathcal{A}}$ define mutually inverse bijections between the set of all measures $\mu : \mathcal{A} \to [0, +\infty]$ and the set of all $\mathcal{A}$-regular exterior measures $\phi : \wp(X) \to [0, +\infty]$.* □

Proposition 8. *Let X be a set and ϕ be an exterior measure on X. Given $A \subset X$, if every $S \subset X$ with $\phi(S) < +\infty$ is contained in some $T \subset X$ such that $A \cap T$ is ϕ-measurable, then A is ϕ-measurable. In particular, if ϕ is $\mathcal{A}$-regular for some σ-algebra $\mathcal{A}$ of subsets of X, then $A \subset X$ is ϕ-measurable if and only if $A \cap T$ is ϕ-measurable for every $T \in \mathcal{A}$ with $\phi(T) < +\infty$.*

Proof. We have to verify that (2) holds for $S \subset X$ with $\phi(S) < +\infty$. Pick $T \subset X$ containing S such that $A \cap T$ is ϕ-measurable. It then follows that (2) holds with A replaced with $A \cap T$, i.e.:

$$\phi(S) \geq \phi(S \cap A \cap T) + \phi\big(S \setminus (A \cap T)\big).$$

But clearly $S \subset T$ implies $S \cap A \cap T = S \cap A$ and $S \setminus (A \cap T) = S \setminus A$. □

Proposition 9. *Let X be a set, $\mathcal{A}$ be a σ-algebra of subsets of X and ϕ be an $\mathcal{A}$-regular exterior measure on X. Denote by $\overline{\mathcal{A}}$ the domain of the completion of $\phi|_{\mathcal{A}}$ (then automatically every element of $\overline{\mathcal{A}}$ is ϕ-measurable). If $A \subset X$ is ϕ-measurable and $\phi(A) < +\infty$, then $A \in \overline{\mathcal{A}}$. Moreover, if $A \subset X$ is ϕ-measurable and $A \subset \bigcup_{n=1}^{\infty} A_n$, with each $A_n \subset X$ satisfying $\phi(A_n) < +\infty$, then $A \in \overline{\mathcal{A}}$.*

Proof. First, note that every $N \subset X$ with $\phi(N) = 0$ is in $\overline{\mathcal{A}}$; namely, by $\mathcal{A}$-regularity there exists $M \in \mathcal{A}$ containing N with $\phi(M) = \phi(N) = 0$. Now given a ϕ-measurable subset A of X with $\phi(A) < +\infty$, by $\mathcal{A}$-regularity we obtain $B \in \mathcal{A}$ containing A with $\phi(A) = \phi(B)$. Since ϕ is additive on ϕ-measurable sets, it follows that $\phi(B) = \phi(A) + \phi(B \setminus A)$, so that $\phi(B \setminus A) = 0$ and thus $B \setminus A \in \overline{\mathcal{A}}$. Then $A = B \setminus (B \setminus A)$ is in $\overline{\mathcal{A}}$. Now assume that $A \subset \bigcup_{n=1}^{\infty} A_n$ with each $A_n \subset X$ satisfying $\phi(A_n) < +\infty$ and A being ϕ-measurable. For each $n \geq 1$ we pick $B_n \in \mathcal{A}$ containing A_n with $\phi(B_n) = \phi(A_n)$ and note that $A = \bigcup_{n=1}^{\infty} (A \cap B_n)$, where each $A \cap B_n$ is ϕ-measurable and satisfies $\phi(A \cap B_n) < +\infty$. It follows that $A \cap B_n \in \overline{\mathcal{A}}$ and hence that A is in $\overline{\mathcal{A}}$. □

Corollary 10. *Let X be a set, $\mathcal{A}$ be a σ-algebra of subsets of X and ϕ be an $\mathcal{A}$-regular exterior measure on X. Denote by $\overline{\mathcal{A}}$ the domain of the completion of $\phi|_{\mathcal{A}}$. We have:*

$$\begin{aligned} (4) \qquad \mathcal{M}_\phi &= \big\{ A \subset X : A \cap E \in \overline{\mathcal{A}}, \text{ for every } E \in \mathcal{A} \text{ with } \phi(E) < +\infty \big\} \\ &= \big\{ A \subset X : A \cap E \in \overline{\mathcal{A}}, \text{ for every } E \in \overline{\mathcal{A}} \text{ with } \phi(E) < +\infty \big\}. \end{aligned}$$

Moreover, $\phi(A) = +\infty$ for every $A \in \mathcal{M}_\phi \setminus \overline{\mathcal{A}}$.

Proof. If $A \in \mathcal{M}_\phi$ and $E \in \overline{\mathcal{A}}$ satisfies $\phi(E) < +\infty$, then $A \cap E$ is ϕ-measurable with $\phi(A \cap E) < +\infty$, so that $A \cap E \in \overline{\mathcal{A}}$. Now, if $A \subset X$ is such that $A \cap E \in \overline{\mathcal{A}}$ for every $E \in \mathcal{A}$ with $\phi(E) < +\infty$ then the ϕ-measurability of A follows from Proposition 8. □

Corollary 10 shows how one can recover $\phi|_{\mathcal{M}_\phi}$ from $\mu = \phi|_{\mathcal{A}}$ in case ϕ is $\mathcal{A}$-regular (avoiding the construction of $\mu^* = \phi$): starting with the measure μ defined in the σ-algebra $\mathcal{A}$, one first considers the completion $\bar{\mu}$ defined in $\overline{\mathcal{A}}$ and then one recovers $\mathcal{M}_\phi$ from μ using (4). To recover $\phi|_{\mathcal{M}_\phi}$, simply extend $\bar{\mu}$ to $\mathcal{M}_\phi$ by setting it equal to $+\infty$ in $\mathcal{M}_\phi \setminus \overline{\mathcal{A}}$.

Pushing and pulling with maps

Definition 11. Let X and Y be sets and $f : X \to Y$ be a map. Given an exterior measure ϕ on X, we define the *push-forward* $f_*\phi$ of ϕ by f to be the exterior measure on Y given by

$$(f_*\phi)(A) = \phi\big(f^{-1}[A]\big),$$

for every $A \subset Y$. Given an exterior measure ψ on Y, we define the *pull-back* $f^*\psi$ of ψ by f to be the exterior measure on X given by

$$(f^*\psi)(A) = \psi\big(f[A]\big),$$

for every $A \subset X$.

Example 12. Let Y be a subset of a set X and denote by $i : Y \to X$ the inclusion map. If ϕ is an exterior measure on X, then the pull-back $i^*\phi$ is simply the restriction of ϕ to $\wp(Y)$. If ψ is an exterior measure on Y, then the push-forward $i_*\psi$ is the *extension to zero* of ψ to X given by $(i_*\psi)(A) = \psi(A \cap Y)$, for all $A \subset X$. The exterior measure $i_*(i^*\phi)$ equals the *truncated measure* $\phi \llcorner Y$ on X given by $(\phi \llcorner Y)(A) = \phi(A \cap Y)$, for every $A \subset X$.

Recall that if $f : X \to Y$ is a map, then a subset S of X is called f-*saturated* if $f^{-1}\big[f[S]\big] = S$. Equivalently, S is f-saturated if $S = f^{-1}[T]$, for some subset T of Y.

Proposition 13. *Let $f : X \to Y$ be a map and ϕ be an exterior measure on X. Given a subset A of Y, if $f^{-1}[A]$ is ϕ-measurable then A is $f_*\phi$-measurable. The converse holds if f is injective.*

Proof. Note that $A \subset Y$ is $f_*\phi$-measurable if and only if

$$\phi(S) \geq \phi\big(S \cap f^{-1}[A]\big) + \phi\big(S \setminus f^{-1}[A]\big),$$

for every f-saturated subset S of X. If f is injective, then every subset of X is f-saturated. □

Corollary 14. *Let X be a set, Y be a subset of X and ϕ be an exterior measure on Y. Consider the inclusion map $i : Y \to X$ and the exterior measure $i_*\phi$ on X. For every subset A of X, we have that A is $i_*\phi$-measurable if and only if $A \cap Y$ is ϕ-measurable.* □

Corollary 15. *Let X be a set, Y be a subset of X and ϕ be an exterior measure on X. For every subset A of X, we have that A is $(\phi \llcorner Y)$-measurable if and only if $A \cap Y$ is $\phi|_{\wp(Y)}$-measurable.*

Proof. Note that $\phi \llcorner Y = i_*(\phi|_{\wp(Y)})$. □

If $f : X \to Y$ is a map, ϕ is an exterior measure on X and T is a subset of X, note that $f_*(\phi \llcorner T)$ is the same as $(f|_T)_*(\phi|_{\wp(T)})$.

Proposition 16. *Let $f : X \to Y$ be a map, ϕ be an exterior measure on X and A be a subset of Y. We have that $f^{-1}[A]$ is ϕ-measurable if and only if for every $T \subset X$ it holds that A is $f_*(\phi \llcorner T)$-measurable.*

Proof. Note that for a given $T \subset X$, we have that A is $f_*(\phi \llcorner T)$-measurable if and only if the inequality

$$\phi(S) \geq \phi\big(S \cap f^{-1}[A]\big) + \phi\big(S \setminus f^{-1}[A]\big)$$

holds for every $S \subset X$ which is the intersection with T of an f-saturated subset of X. Finally, note that for every $S \subset X$ we have $S = f^{-1}\big[f[S]\big] \cap T$ with $T = S$. □

Proposition 17. *Let $f : X \to Y$ be a map, ϕ be an exterior measure on Y and A be a subset of Y. If A is ϕ-measurable then $f^{-1}[A]$ is $f^*\phi$-measurable. The converse holds if A is contained in the image of f and the image of f is ϕ-measurable.*

Proof. We have that

$$f\big[S \cap f^{-1}[A]\big] = f[S] \cap A \quad \text{and} \quad f\big[S \setminus f^{-1}[A]\big] = f[S] \setminus A,$$

for every subset S of X. It follows that $f^{-1}[A]$ is $f^*\phi$-measurable if and only if

$$\phi(T) \geq \phi(T \cap A) + \phi(T \setminus A), \tag{5}$$

for every subset T of the image of f. It remains to check that if A is contained in $f[X]$, $f[X]$ is ϕ-measurable and if (5) holds for every subset T of $f[X]$ then (5) holds for every subset T of Y. Let T be an arbitrary subset of Y. The ϕ-measurability of $f[X]$ implies

$$\phi(T) \geq \phi\big(T \cap f[X]\big) + \phi\big(T \setminus f[X]\big). \tag{6}$$

Using (5) with T replaced with $T \cap f[X]$, we obtain:

$$\phi\big(T \cap f[X]\big) \geq \phi(T \cap A) + \phi\big(T \cap f[X] \setminus A\big). \tag{7}$$

The subadditivity of ϕ gives

$$\phi\big(T \setminus f[X]\big) + \phi\big(T \cap f[X] \setminus A\big) \geq \phi(T \setminus A) \tag{8}$$

and then we obtain from (6), (7) and (8) that (5) holds. □

Corollary 18. *Let X be a set, ϕ be an exterior measure on X and Y be a subset of X. We have that $A \cap Y$ is $\phi|_{\wp(Y)}$-measurable for every ϕ-measurable subset A of X. If Y is ϕ-measurable, then a subset of Y is $\phi|_{\wp(Y)}$-measurable if and only if it is ϕ-measurable.*

Proof. Use Proposition 17 with the inclusion map $i : Y \to X$. $\square$

Corollary 19. *Let X be a set, ϕ be an exterior measure on X and Y be a subset of X. Every ϕ-measurable subset of X is also $(\phi \llcorner Y)$-measurable. If Y is ϕ-measurable then a subset A of X is $(\phi \llcorner Y)$-measurable if and only if $A \cap Y$ is ϕ-measurable.*

Proof. Follows from Corollaries 15 and 18. $\square$

Lemma 20. *Let ϕ be a regular exterior measure on a set X. Given $A \subset X$ with $\phi(A) < +\infty$, if there exists a ϕ-measurable set $A' \subset X$ contained in A with $\phi(A') = \phi(A)$ then A is ϕ-measurable.*

Proof. By regularity, there exists a ϕ-measurable set A'' containing A with $\phi(A'') = \phi(A)$. Then

$$\phi(A \setminus A') \leq \phi(A'' \setminus A') = \phi(A'') - \phi(A') = 0,$$

which implies that $A \setminus A'$ and hence $A = (A \setminus A') \cup A'$ are ϕ-measurable. $\square$

Corollary 21. *Let ϕ be a regular exterior measure on a set X and let $A, B \subset X$ be such that $A \cup B$ is ϕ-measurable and:*

$$\phi(A \cup B) = \phi(A) + \phi(B) < +\infty.$$

Then both A and B are ϕ-measurable.

Proof. Let $B' \subset X$ be a ϕ-measurable set containing B with $\phi(B') = \phi(B)$. By replacing B' with $B' \cap (A \cup B)$ we can assume that $B' \subset A \cup B$. Note that $A \setminus B' = (A \cup B) \setminus B'$ is ϕ-measurable so that, by Lemma 20, to prove that A is ϕ-measurable it is sufficient to check that $\phi(A) = \phi(A \setminus B')$. Since both $A \cup B$ and B' are ϕ-measurable and $B' \subset A \cup B$, we have:

$$\phi(A \setminus B') = \phi\big((A \cup B) \setminus B'\big) = \phi(A \cup B) - \phi(B') = \phi(A) + \phi(B) - \phi(B) = \phi(A).$$

The proof that B is ϕ-measurable is analogous. $\square$

Corollary 22. *If ϕ is a finite regular exterior measure on a set X, then a subset A of X is ϕ-measurable if and only if $\phi(X) = \phi(A) + \phi(X \setminus A)$.*

Proof. Use Corollary 21 with $B = X \setminus A$. $\square$

Corollary 23. *Let $f : X \to Y$ be a map, ϕ be a finite regular exterior measure on X and A be a subset of Y. Then A is $f_*\phi$-measurable if and only if $f^{-1}[A]$ is ϕ-measurable.*

Proof. We already know that if $f^{-1}[A]$ is ϕ-measurable then A is $f_*\phi$-measurable (Proposition 13). Now note that if A is $f_*\phi$-measurable then

$$(f_*\phi)(Y) = (f_*\phi)(A) + (f_*\phi)(Y \setminus A),$$

which yields

$$\phi(X) = \phi\big(f^{-1}[A]\big) + \phi\big(X \setminus f^{-1}[A]\big)$$

and implies that $f^{-1}[A]$ is ϕ-measurable by Corollary 22. $\square$

Additive families

Definition 24. Let ϕ be an exterior measure on a set X. A family $(A_i)_{i\in I}$ of subsets of X is called *ϕ-additive* if

$$\phi(S) \ge \sum_{i\in I} \phi(S\cap A_i), \tag{9}$$

for every $S \subset X$.

Clearly, a subset A of X is ϕ-measurable if and only if the family consisting of A and $X\setminus A$ is ϕ-additive.

Lemma 25. *Let ϕ be an exterior measure on a set X and $(A_i)_{i\in I}$ be a family of subsets of X. The following conditions are equivalent:*

(i) $(A_i)_{i\in I}$ *is ϕ-additive;*
(ii) (9) *holds for every subset S of $\bigcup_{i\in I} A_i$;*
(iii) $\phi\big(S\cap\bigcup_{i\in I} A_i\big) \ge \sum_{i\in I}\phi(S\cap A_i)$, *for every subset S of X;*
(iv) $\phi\big(S\cap\bigcup_{i\in I} A_i\big) \ge \sum_{i\in I}\phi(S\cap A_i)$, *for every subset S of $\bigcup_{i\in I} A_i$.*

If I is countable, then (i)—(iv) are also equivalent to:

(v) $\phi\big(S\cap\bigcup_{i\in I} A_i\big) = \sum_{i\in I}\phi(S\cap A_i)$, *for every subset S of X;*
(vi) $\phi\big(S\cap\bigcup_{i\in I} A_i\big) = \sum_{i\in I}\phi(S\cap A_i)$, *for every subset S of $\bigcup_{i\in I} A_i$.*

Proof. The equivalence between (i), (ii), (iii) and (iv) is obtained simply by noting that one can use (9) with S replaced by $S\cap\bigcup_{i\in I} A_i$. The equivalence with (v) and (vi) follows from the countable subadditivity of ϕ. □

The proof of the following result is straightforward.

Lemma 26. *Let ϕ be an exterior measure on a set X.*

(i) *if $(A_i)_{i\in I}$ is a ϕ-additive family and if $A_i'\subset A_i$ for every $i\in I$ then $(A_i')_{i\in I}$ is a ϕ-additive family;*
(ii) *a subfamily of a ϕ-additive family is ϕ-additive;*
(iii) *if every finite subfamily of $(A_i)_{i\in I}$ is ϕ-additive then $(A_i)_{i\in I}$ is ϕ-additive.* □

Lemma 27. *Let ϕ be an exterior measure on a set X. Let $(A_i)_{i\in I}$ be a family of subsets of X and for each $i\in I$ let $(A_{ij})_{j\in J_i}$ be a family of subsets of X with $A_i=\bigcup_{j\in J_i} A_{ij}$. Set $\Lambda=\big\{(i,j): i\in I,\ j\in J_i\big\}$ and consider the family $(A_\lambda)_{\lambda\in\Lambda}$ given by $A_\lambda = A_{ij}$, for all $\lambda=(i,j)\in\Lambda$.*

(i) *If $(A_i)_{i\in I}$ is ϕ-additive and $(A_{ij})_{j\in J_i}$ is ϕ-additive for all $i\in I$, then $(A_\lambda)_{\lambda\in\Lambda}$ is ϕ-additive.*
(ii) *If $(A_\lambda)_{\lambda\in\Lambda}$ is ϕ-additive and J_i is countable for all $i\in I$ then $(A_i)_{i\in I}$ is ϕ-additive.*

Proof. To prove (i), simply note that for every $S\subset X$ we have:

$$\phi(S) \ge \sum_{i\in I}\phi(S\cap A_i) \ge \sum_{i\in I}\sum_{j\in J_i}\phi(S\cap A_{ij}) = \sum_{\lambda\in\Lambda}\phi(S\cap A_\lambda).$$

To prove (ii) note that for every $S \subset X$ the ϕ-additivity of $(A_\lambda)_{\lambda\in\Lambda}$ yields

$$\phi(S) \geq \sum_{i\in I} \sum_{j\in J_i} \phi(S \cap A_{ij})$$

and the fact that J_i is countable implies $\sum_{j\in J_i} \phi(S \cap A_{ij}) \geq \phi(S \cap A_i)$. □

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