# SUBGROUPS OF VECTOR SPACES

DANIEL V. TAUSK

In what follows, finite dimensional real vector spaces are always endowed with their usual topology, i.e., the topology induced from an arbitrary norm (or, equivalently, the topology induced from the Euclidean topology of $\mathbb{R}^n$ by an arbitrary choice of basis). Also, vector spaces are always endowed with their (abelian) additive group structure.

**Proposition 1.** *Let $V$ be a finite dimensional real vector space. If $G$ is a discrete subgroup of $V$ then $G$ is spanned (as a group) by a subset of $V$ which is linearly independent (over $\mathbb{R}$). In particular, $G$ is a free abelian finitely generated group whose rank is less than or equal to the dimension of $V$.*

*Proof.* By replacing $V$ with the vector subspace of $V$ spanned by $G$, we can assume without loss of generality that $G$ spans $V$ as a vector space. Then there exists a basis of $V$ contained in $G$. By replacing $V$ with $\mathbb{R}^n$ and $G$ with its image under the isomorphism from $V$ to $\mathbb{R}^n$ that maps such a basis to the canonical basis of $\mathbb{R}^n$, we can also assume without loss of generality that $V = \mathbb{R}^n$ and that $G$ is a discrete subgroup of $\mathbb{R}^n$ containing the canonical basis. Then $G$ contains $\mathbb{Z}^n$. Consider the quotient map $\mathfrak{q} : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$, where $\mathbb{R}^n/\mathbb{Z}^n$ is endowed with the quotient topology. The map $\mathfrak{q}$ is open and the quotient $\mathbb{R}^n/\mathbb{Z}^n$ is compact (being homeomorphic to the product of $n$ copies of the circle). The fact that $G$ is discrete and $\mathfrak{q}$-saturated[1] implies that $\mathfrak{q}(G)$ is discrete[2]. Moreover, by the definition of the quotient topology, the fact that $G$ is closed[3] and $\mathfrak{q}$-saturated implies that $\mathfrak{q}(G)$ is closed. Then $\mathfrak{q}(G) \cong G/\mathbb{Z}^n$ is compact and discrete, hence finite. Since $\mathbb{Z}^n$ and $G/\mathbb{Z}^n$ are both finitely generated, it follows that also $G$ is finitely generated[4]. Since $V$ (and thus $G$) is torsion-free, we obtain that $G$ is free abelian[5]. The fact that $G/\mathbb{Z}^n$ is finite then implies that the rank of $G$ is equal[6] to $n$. If $B$ is a basis of $G$ as a $\mathbb{Z}$-module then $B$ also spans $\mathbb{R}^n$ as a vector space and, since $B$ has $n$ elements, this implies that $B$ is linearly independent over $\mathbb{R}$ and we are done. □

**Lemma 2.** *Let $V$ be a finite dimensional real vector space and $G$ be a closed subgroup of $V$. if $G$ is not discrete then $G$ contains a non zero vector subspace of $V$.*

*Proof.* Let $\|\cdot\|$ be a norm in $V$. Since $G$ is not discrete, the origin is not isolated[7] in $G$. Let $(g_k)_{k\geq 1}$ be a sequence in $G\setminus\{0\}$ converging to zero. Since

*Date*: October 2011.

the unit sphere of $V$ is (sequentially) compact, by passing to a subsequence, we can assume that:

$$\lim_{k\to+\infty} \frac{g_k}{\|g_k\|} = v,$$

for some $v \in V$ with $\|v\| = 1$. Let us prove that $tv \in G$ for all $t \in \mathbb{R}$. Since this is trivial for $t = 0$, and since $tv \in G$ implies $-tv \in G$, we can assume that $t > 0$. Set:

$$\alpha_k = \frac{t}{\|g_k\|},$$

so that $\lim_{k\to+\infty} \alpha_k = +\infty$ and $\lim_{k\to+\infty} \alpha_k g_k = tv$. Set $n_k = \lfloor \alpha_k \rfloor$, i.e., $n_k$ is the only integer such that $n_k \leq \alpha_k < n_k + 1$. For $k$ sufficiently large, we have $n_k > 0$, so that:

$$1 \leq \frac{\alpha_k}{n_k} < 1 + \frac{1}{n_k}.$$

Since $\lim_{k\to+\infty} n_k = +\infty$, we obtain $\lim_{k\to+\infty} \frac{\alpha_k}{n_k} = 1$. This yields:

$$\lim_{k\to+\infty} n_k g_k = \lim_{k\to+\infty} \Big(\frac{n_k}{\alpha_k}\, \alpha_k g_k\Big) = tv.$$

Since $n_k \in \mathbb{Z}$, we have $n_k g_k \in G$ for all $k$ and, since $G$ is closed, we obtain that $tv \in G$. □

**Lemma 3.** *If $V$ is a vector space and $G$ is a subgroup of $V$ then there exists a largest vector subspace of $V$ contained in $G$, i.e., there exists a vector subspace $S$ of $V$ contained in $G$ which contains every vector subspace of $V$ contained in $G$.*

*Proof.* Simply let $S$ be the sum of all vector subspaces of $V$ contained in $G$. Since $G$ is closed under addition, we have $S \subset G$. □

**Proposition 4.** *Let $V$ be a finite dimensional real vector space and $G$ be a closed subgroup of $V$. If $S$ is the largest vector subspace of $V$ contained in $G$ then $G/S$ is a discrete subgroup of $V/S$.*

*Proof.* Since the quotient map $\mathfrak{q} : V \to V/S$ is a quotient map in the topological sense[8] and since $G$ is a closed $\mathfrak{q}$-saturated subset of $V$, it follows that $\mathfrak{q}(G) = G/S$ is closed in $V/S$. The subgroup $G/S$ of $V/S$ does not contain a non zero subspace of $V/S$. Namely, a subspace of $V/S$ contained in $G/S$ is of the form $W/S$, with $W$ a subspace of $V$ such that $S \subset W \subset G$. Then $W = S$ and $W/S$ is trivial. It follows from Lemma 2 that $G/S$ is discrete. □

**Corollary 5.** *Let $V$ be a finite dimensional real vector space and $G$ be a closed subgroup of $V$. If $S$ is the largest vector subspace of $V$ contained in $G$ then $S$ is open in $G$.*

*Proof.* Since the quotient map $\mathfrak{q} : G \to G/S$ is continuous and the origin $\{0\}$ is open in $G/S$, then $S = \mathfrak{q}^{-1}(0)$ is open in $G$. □

**Corollary 6.** *Let $V$ be a finite dimensional real vector space and $G$ be a closed subgroup of $V$. If $S$ is the largest vector subspace of $V$ contained in $G$ then $S$ is the connected component of $G$ containing the origin.*

*Proof.* It follows from the fact that $S$ is connected and that $S$ is open and closed in $G$. □

**Proposition 7.** *Let $V$ be a finite dimensional real vector space and $G$ be a subgroup of $V$. Then $G$ is not dense in $V$ if and only if there exists a non zero linear functional $\alpha : V \to \mathbb{R}$ such that $\alpha(G) \subset \mathbb{Z}$.*

*Proof.* If $\alpha$ is a non zero linear functional with $\alpha(G) \subset \mathbb{Z}$ then $\alpha^{-1}\big(\,]0,1[\,\big)$ is a non empty open subset of $V$ that does not intersect $G$, so that $G$ is not dense in $V$. Now assume $\overline{G} \neq V$. Let $S$ be the largest vector subspace of $V$ contained in $\overline{G}$. Since $\overline{G}$ is a closed subgroup of $V$, Proposition 4 implies that $\overline{G}/S$ is a discrete subgroup of $V/S$. Since $\overline{G} \neq V$, we have $S \neq V$ and thus $V/S$ is non zero. By Proposition 1, $\overline{G}/S$ is spanned as a group by some linearly independent subset $B$ of $V/S$. The fact that $V/S$ is non zero implies then that there exists a non zero linear functional $\bar{\alpha} : V/S \to \mathbb{R}$ such that $\bar{\alpha}(B) \subset \mathbb{Z}$. Then $\bar{\alpha}(\overline{G}/S) \subset \mathbb{Z}$. The linear functional $\alpha$ is then obtained by taking the composition of $\bar{\alpha}$ with the quotient map $V \to V/S$. □

**Corollary 8.** *Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ then the group $G = \mathbb{Z}x + \mathbb{Z}^n$ is dense in $\mathbb{R}^n$ if and only if the sequence $(1, x_1, \ldots, x_n)$ is linearly independent over $\mathbb{Z}$ (or, equivalently, if it is linearly independent over $\mathbb{Q}$).*

*Proof.* We have that $G$ is not dense in $\mathbb{R}^n$ if and only if there exists a linear functional $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(G) \subset \mathbb{Z}$. Such a linear functional exists if and only if there exists a non zero sequence of integers $(\alpha_i)_{i=1}^n$ such that $\sum_{i=1}^n \alpha_i x_i \in \mathbb{Z}$, i.e., if and only if $(1, x_1, \ldots, x_n)$ is linearly dependent over $\mathbb{Z}$. □

Observe that if $X$, $Y$ are topological spaces and $f : X \to Y$ is a continuous, open surjective map then, given a subset $D$ of $X$, we have that $f(D)$ is dense in $Y$ if and only if the $f$-saturation $f^{-1}\big(f(D)\big)$ is dense in $X$. Applying this observation to the quotient map $\mathfrak{q} : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ and to the cyclic subgroup $D = \mathbb{Z}x$ generated by some $x \in \mathbb{R}^n$ then $\mathfrak{q}^{-1}\big(\mathfrak{q}(D)\big) = \mathbb{Z}x + \mathbb{Z}^n$ and thus Corollary 8 says that the cyclic subgroup of $\mathbb{R}^n/\mathbb{Z}^n$ spanned by $\mathfrak{q}(x)$ is dense in $\mathbb{R}^n/\mathbb{Z}^n$ if and only if $(1, x_1, \ldots, x_n)$ is linearly independent over $\mathbb{Z}$ (or, equivalently, linearly independent over $\mathbb{Q}$).

## Notes

[1]This means that $G = \mathfrak{q}^{-1}\big(\mathfrak{q}(G)\big)$. Since $\mathfrak{q}$ is a group homomorphism and $G$ is a subgroup, the statement that $G$ is $\mathfrak{q}$-saturated is equivalent to the statement that $G$ contains $\mathrm{Ker}(\mathfrak{q}) = \mathbb{Z}^n$.

[2]If $X$, $Y$ are topological spaces, $f : X \to Y$ is an open mapping and $D$ is a discrete $f$-saturated subspace of $X$ then $f(D)$ is a discrete subspace of $Y$. Namely, given $x \in D$ then $\{x\} = U \cap D$ for some open subset $U$ of $X$ and, since $D$ is $f$-saturated, we have $\{f(x)\} = f(U) \cap f(D)$, with $f(U)$ open in $Y$.

[3]If $G$ were not closed, there would exist an element $g \in \overline{G} \setminus G$ and a sequence $(g_k)_{k\geq 1}$ in $G$ of pairwise distinct elements converging to $g$. Then $(g_k - g_{k+1})_{k\geq 1}$ is a sequence in $G \setminus \{0\}$ converging to zero, contradicting the fact that the origin is isolated in $G$.

[4]If $G$ is an abelian group, $H$ is a subgroup of $G$ and both $H$ and $G/H$ are finitely generated then also $G$ is finitely generated. Namely, if $A$ is a finite subset of $G$ whose image under the quotient map $G \to G/H$ spans $G/H$ and if $B$ is a finite subset of $H$ that spans $H$ then $A \cup B$ spans $G$.

[5]A finitely generated abelian group is isomorphic to a finite direct sum of cyclic groups. If it is torsion-free, then it is isomorphic to a finite direct sum of copies of $\mathbb{Z}$ and it is therefore free abelian.

[6]If $G$ is a free abelian group of finite rank $m$ and $H$ is a subgroup of $G$ then $H$ is free abelian of rank $n \leq m$. Moreover, there exists a basis $(g_i)_{i=1}^m$ of $G$ and positive integers $\alpha_1, \ldots, \alpha_n$ such that $(\alpha_i g_i)_{i=1}^n$ is a basis of $H$. Then $G/H$ is isomorphic to $\mathbb{Z}^{m-n} \oplus \bigoplus_{i=1}^n \mathbb{Z}/(\alpha_i \mathbb{Z})$. In particular, if $G/H$ is finite then $m = n$.

[7]If the origin were isolated in $G$, i.e., if $\{0\}$ were open in $G$ then $G$ would be discrete. Namely, given $g \in G$ then the translation $x \mapsto x + g$ is a homeomorphism of $G$ that sends the open set $\{0\}$ to the open set $\{g\}$.

[8]If $V$, $W$ are finite dimensional real vector spaces and $T : V \to W$ is a surjective linear map then $T$ is continuous, open and surjective and therefore a quotient map in the topological sense, i.e., a subset of $W$ is open if and only if its inverse image by $T$ is open in $V$. The fact that $T$ is open follows from the observation that $T$ is represented by a projection with respect to suitable choices of bases of $V$ and $W$.