COUNTEREXAMPLE TO L'HOSPITAL'S RULE

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We will construct two positive functions $F : I \to \mathbb{R}$ and $G : I \to \mathbb{R}$ of class C^1, with $I \subset \mathbb{R}$ an interval that is unbounded above, in such a way that

$$\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} G(x) = 0, \quad \lim_{x\to+\infty} \frac{F(x)}{G(x)} = 2$$

and

$$\lim_{x\to+\infty} \frac{F'(x)}{G'(x)} = 1,$$

thus obtaining a "counterexample" to L'Hospital's rule. The catch is that there are arbitrarily large values of x for which $G'(x) = 0$, contradicting one (often forgotten) assumption of the rule. Our construction is easy to be modified in order to make the functions F and G of class C^∞. In [1] it is given a family of counterexamples to L'Hospital's rule involving elementary functions given by simple formulas which are more suited to be presented in introductory Differential Calculus courses, but in those examples the limit $\lim_{x\to+\infty} \frac{F(x)}{G(x)}$ does not exist. Our example has the disadvantage of requiring a more involved construction, but it is more eye catching in the sense that both limits $\lim_{x\to+\infty} \frac{F(x)}{G(x)}$ and $\lim_{x\to+\infty} \frac{F'(x)}{G'(x)}$ exist but are different.

Let $\phi : \big[0, \frac{1}{2}\big] \to \mathbb{R}$ be a fixed continuous function such that:

- $\phi(0) = \phi\big(\frac{1}{2}\big) = 0$;
- ϕ is positive in the open interval $\big]0, \frac{1}{2}\big[$;
- $\int_0^{\frac{1}{2}} \phi(t)\,\mathrm{d}t = 1$.

We will define continuous functions $f : [1, +\infty[\to \mathbb{R}$ and $g : [1, +\infty[\to \mathbb{R}$ such that, for every positive integer n, the restriction of f and g to $\big[n, n+\frac{1}{2}\big]$ is a positive scalar multiple of a translation of ϕ and the restriction of f and g to $\big[n+\frac{1}{2}, n+1\big]$ is a negative scalar multiple of a translation of ϕ. More precisely, we will construct sequences

$$(a_n)_{n\ge1}, \quad (\bar{a}_n)_{n\ge1}, \quad (b_n)_{n\ge1}, \quad (\bar{b}_n)_{n\ge1} \tag{1}$$

of positive real numbers and with those sequences in hand we set

$$f(x) = a_n\phi(x-n), \quad g(x) = b_n\phi(x-n),$$

for every $x \in \big[n, n+\frac{1}{2}\big]$ and every positive integer n and

$$f(x) = -\bar{a}_n\phi\big(x-n-\tfrac{1}{2}\big), \quad g(x) = -\bar{b}_n\phi\big(x-n-\tfrac{1}{2}\big),$$

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for every $x \in \left[n + \frac{1}{2}, n+1\right]$ and every positive integer n. The sequences (1) will be chosen with

$$\sum_{n=1}^{\infty} a_n < +\infty, \quad \sum_{n=1}^{\infty} \bar{a}_n < +\infty, \quad \sum_{n=1}^{\infty} b_n < +\infty, \quad \sum_{n=1}^{\infty} \bar{b}_n < +\infty$$

which implies that the improper integrals of f and g over $[1, +\infty[$ are absolutely convergent. The functions $F : [1, +\infty[\to \mathbb{R}$ and $G : [1, +\infty[\to \mathbb{R}$ defined by

$$F(x) = \int_x^{+\infty} f(t)\,\mathrm{d}t, \quad G(x) = \int_x^{+\infty} g(t)\,\mathrm{d}t,$$

for all $x \geq 1$ are thus of class C^1 and satisfy:

$$\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} G(x) = 0.$$

Moreover, we have $F' = -f$ and $G' = -g$. Let us now construct the sequences (1). First, we set

$$x_n = \frac{1}{\sqrt{n}} + \frac{3}{n}, \qquad \bar{x}_n = \frac{1}{\sqrt{n}} + \frac{1}{n},$$
$$y_n = \frac{1}{\sqrt{n}} + \frac{1}{n}, \qquad \bar{y}_n = \frac{1}{\sqrt{n}},$$

and then we define (1) by setting

$$a_n = x_n - x_{n+1}, \qquad \bar{a}_n = \bar{x}_n - \bar{x}_{n+1}$$
$$b_n = y_n - y_{n+1}, \qquad \bar{b}_n = \bar{y}_n - \bar{y}_{n+1},$$

for every positive integer n. Clearly $(x_n)_{n\geq1}$, $(\bar{x}_n)_{n\geq1}$, $(y_n)_{n\geq1}$ and $(\bar{y}_n)_{n\geq1}$ are strictly decreasing sequences of positive real numbers converging to zero and $(a_n)_{n\geq1}$, $(\bar{a}_n)_{n\geq1}$, $(b_n)_{n\geq1}$ and $(\bar{b}_n)_{n\geq1}$ are sequences of positive real numbers such that

$$\sum_{i=n}^{\infty} a_i = x_n, \qquad \sum_{i=n}^{\infty} \bar{a}_i = \bar{x}_n, \tag{2}$$

$$\sum_{i=n}^{\infty} b_i = y_n, \qquad \sum_{i=n}^{\infty} \bar{b}_i = \bar{y}_n, \tag{3}$$

for every positive integer n. A simple computation shows that

$$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n(n+1)}} + \frac{3}{n(n+1)} = \frac{1}{\sqrt{n(n+1)}\big(\sqrt{n} + \sqrt{n+1}\big)} + \frac{3}{n(n+1)},$$

$$\bar{a}_n = b_n = \frac{1}{\sqrt{n(n+1)}\big(\sqrt{n} + \sqrt{n+1}\big)} + \frac{1}{n(n+1)},$$

$$\bar{b}_n = \frac{1}{\sqrt{n(n+1)}\big(\sqrt{n} + \sqrt{n+1}\big)}$$

for every positive integer n, from which it follows easily that

$$\lim_{n\to+\infty} a_n\, n^{\frac{3}{2}} = \frac{1}{2}, \qquad \lim_{n\to+\infty} \bar{a}_n\, n^{\frac{3}{2}} = \frac{1}{2}, \tag{4}$$

$$\lim_{n\to+\infty} b_n\, n^{\frac{3}{2}} = \frac{1}{2}, \qquad \lim_{n\to+\infty} \bar{b}_n\, n^{\frac{3}{2}} = \frac{1}{2} \tag{5}$$

and:

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = 1, \qquad \lim_{n\to+\infty} \frac{\bar{a}_n}{\bar{b}_n} = 1.$$

The equality

$$\lim_{x\to+\infty} \frac{F'(x)}{G'(x)} = \lim_{x\to+\infty} \frac{f(x)}{g(x)} = 1$$

is now easily established. It remains to show that $\lim_{x\to+\infty} \frac{F(x)}{G(x)} = 2$. We start by considering integer values of x. Using (2) and (3), we see that for each positive integer n we have

$$F(n) = x_n - \bar{x}_n = \frac{2}{n}, \quad G(n) = y_n - \bar{y}_n = \frac{1}{n} \tag{6}$$

and

$$\frac{F(n)}{G(n)} = \frac{x_n - \bar{x}_n}{y_n - \bar{y}_n} = 2; \tag{7}$$

moreover, for $x \in [n, n+1]$ we have the following estimates:

$$|F(x) - F(n)| \leq \int_n^{n+1} |f(t)|\, \mathrm{d}t = a_n + \bar{a}_n, \tag{8}$$

$$|G(x) - G(n)| \leq \int_n^{n+1} |g(t)|\, \mathrm{d}t = b_n + \bar{b}_n. \tag{9}$$

If for each $x \geq 1$ we denote by $n(x)$ the unique positive integer such that $x \in [n(x), n(x)+1[$, then (4), (5), (6), (8) and (9) yield:

$$\lim_{x\to+\infty} \frac{F(x) - F\big(n(x)\big)}{F\big(n(x)\big)} = 0, \quad \lim_{x\to+\infty} \frac{G(x) - G\big(n(x)\big)}{G\big(n(x)\big)} = 0.$$

Thus

$$\lim_{x\to+\infty} \frac{F(x)}{F\big(n(x)\big)} = 1, \quad \lim_{x\to+\infty} \frac{G(x)}{G\big(n(x)\big)} = 1 \tag{10}$$

and using (7) we get:

$$\lim_{x\to+\infty} \frac{F(x)}{G(x)} = \lim_{x\to+\infty} \frac{F\big(n(x)\big)}{G\big(n(x)\big)} = 2.$$

Note that (10) also implies that the quotients

$$\frac{F(x)}{F\big(n(x)\big)}, \quad \frac{G(x)}{G\big(n(x)\big)}$$

are positive for sufficiently large x and thus by (6) we conclude that $F(x)$ and $G(x)$ are also both positive for sufficiently large x.

References

[1] R. P. Boas, *Counterexamples to L'Hôpital's rule*, American Mathematical Monthly, 1986, https://www.maa.org/programs/faculty-and-departments/classroom-capsules-and-notes/counterexamples-to-lh-pitals-rule.

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