# PARTIAL LIST OF BUGS ON FEDERER'S BUG ON GEOMETRIC MEASURE THEORY

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## Paragraph 2.2.17

There is nothing actually wrong in that paragraph, it's just that the result stated there contains an unnecessary assumption that is easy to remove. If $X$ and $Y$ are locally compact Hausdorff spaces, $f : X \to Y$ is a continuous proper map (with "proper" meaning that the inverse image of compact subsets is compact) and $\phi$ is a Radon measure on $X$, then $f_{\#}\phi$ is a Radon measure on $Y$. The assumption that $X$ be a countable union of compact sets is unnecessary. See [1, Section 35] for details.

## Some remarks on Paragraphs 2.5.1 and 2.5.2

The book uses a somewhat unusual definition for a lattice of functions on a set $X$. Given a set $X$, then the set $\mathbb{R}^X$ of all maps $f : X \to \mathbb{R}$ has a natural structure of a partially ordered vector space (with both the vector space operations and the partial order being defined pointwise). Moreover, this partially ordered vector space is a vector lattice, meaning that every pair of maps $f, g \in \mathbb{R}^X$ admits an infimum

$$(f \wedge g)(x) = \min\{f(x), g(x)\}, \quad x \in X$$

and a supremum

$$(f \vee g)(x) = \max\{f(x), g(x)\}, \quad x \in X.$$

A *vector sublattice* of $\mathbb{R}^X$ is then (using standard terminology) a vector subspace of $\mathbb{R}^X$ that is closed under the lattice operations $\wedge$ and $\vee$. If $L$ is a nonempty lattice of functions as defined in 2.5.1, then $L$ is not in general a vector subspace of $\mathbb{R}^X$, but

$$L - L = \big\{f - g : f, g \in L\big\}$$

is a vector sublattice of $\mathbb{R}^X$ and $L^+ = (L - L)^+$, i.e., the set of nonnegative elements of $L$ coincides with the set of nonnegative elements of $L - L$. If $\lambda : L \to \mathbb{R}$ is a map satisfying the conditions

$$\lambda(f + g) = \lambda(f) + \lambda(g) \quad \text{and} \quad \lambda(cf) = c\lambda(f), \tag{1}$$

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for all $f, g \in L$ and all $c \geq 0$, then $\lambda$ admits a unique linear extension to $L - L$. Moreover, if $\lambda$ satisfies all the assumptions of Theorem 2.5.2, then so does this extension.

These observations imply that one looses nothing by redefining a lattice of functions on $X$ to be a vector sublattice $L$ of $\mathbb{R}^X$ satisfying the additional assumption that $f \wedge c \in L$, for all $f \in L$ and for every nonnegative constant function $c$. For Theorem 2.5.2 one then replaces assumptions (1) by the assumption that $\lambda$ be linear. I think that changing the definitions and theorems in Paragraphs 2.5.1 and 2.5.2 in this way one obtains an exposition that would be easier to follow and is more in tune with standard terminology.

## Paragraph 2.5.3

There is nothing wrong there, but some missing crucial observations would make lots of proofs simpler. Given a lattice of functions $L$ on a set $X$, the book defines collections $F_0$, $F_1$ and $F_2$ of subsets of $X$ as follows. The collection $F_0$ consists of all sets of the form

$$[f > t] = \{x \in X : f(x) > t\}$$

with $f \in L$ and $t > 0$ (nothing changes if one considers only functions $f \in L$ with $f \geq 0$, since replacing $f$ with its positive part $f^+$ does not change the set $[f > t]$). The collection $F_1$ is defined as the set of unions of increasing sequences of elements of $F_0$ and, for a given measure $\phi$ on $X$, the collection $F_2$ is defined as the set of intersections of decreasing sequences of elements of $F_1$ having finite measure.

A crucial observation that makes many proofs easier is the observation that the class $F_0$ is closed under finite unions and intersections. Namely, since $L$ is closed under the operation of multiplication by a positive constant, it follows that the collection $F_0$ coincides with the collection of all sets of the form

$$[f > 1]$$

with $f \in L$. Moreover

$$[f > 1] \cup [g > 1] = [(f \vee g) > 1]$$

and:

$$[f > 1] \cap [g > 1] = [(f \wedge g) > 1].$$

The fact that $F_0$ is closed under finite unions implies that $F_1$ coincides with the collection of all countable unions of elements of $F_0$. Thus $F_1$ is closed under countable unions and finite intersections. The fact that $F_1$ is closed under finite intersections implies that $F_2$ equals the collection of all countable intersections of elements of $F_1$ with finite measure (or countable intersections of elements of $F_1$ in which at least one of them has finite measure). Thus $F_2$ is closed under countable intersections and finite unions.

Using the observations above, one sees easily that the condition the book calls $L$-regularity for a measure $\phi$ can be equivalently defined by requiring that[1]

$$\phi(A) = \inf \big\{\phi(B) : B \in F_1,\ B \supset A\big\},$$

for every subset $A$ of $X$. I'm assuming here a measure $\phi$ for which every function in $L$ is $\phi$-measurable, so that every element in the classes $F_0$, $F_1$ and $F_2$ is also $\phi$-measurable.

Let $\lambda$ be a monotone Daniell integral on $L$ (i.e., $\lambda$ satisfies the assumptions of Theorem 2.5.2). I will say that a measure $\phi$ over $X$ *represents* $\lambda$ if every $f \in L$ is ($\phi$-measurable and) $\phi$-integrable and the equality $\lambda(f) = \int_X f \,\mathrm{d}\phi$ holds. It is easy to see that the measure $\phi$ constructed in the proof of Theorem 2.5.2 (i.e., the unique $L$-regular measure representing $\lambda$) is the largest measure representing $\lambda$. The book claims that two measures representing $\lambda$ coincide over the class $F_2$. This is indeed true, but a much better result can be obtained: let $\mathcal{R}_L$ denote the $\sigma$-ring generated by $F_0$. Since $F_0$ is closed under finite intersections and every measure that represents $\lambda$ is finite on elements of $F_0$, it follows from [2, Lema 5.2.12] that *any two measures representing $\lambda$ agree on $\mathcal{R}_L$*. If

$$\mathcal{R}_L' = \big\{X \setminus A : A \in \mathcal{R}_L\big\}$$

denotes the set of all complements in $X$ of elements of $\mathcal{R}_L$ then the union $\mathcal{A}_L = \mathcal{R}_L \cup \mathcal{R}_L'$ is the $\sigma$-algebra genetated by $F_0$. It coincides with the smallest $\sigma$-algebra for which every element of $L$ is a measurable function. If $X$ is in $F_1$, then $\mathcal{A}_L = \mathcal{R}_L$; otherwise, $\mathcal{R}_L$ and $\mathcal{R}_L'$ are disjoint and the measure $\phi$ constructed in the proof of Theorem 2.5.2 gives the value $+\infty$ for every element of $\mathcal{R}_L'$ (since it gives the value $+\infty$ for every subset of $X$ that is not contained in an element of $F_1$). If $X$ is a locally compact Hausdorff topological space and $L$ is the lattice of continuous functions with compact support on $X$, then the class $F_0$ consists of all open subsets of $X$ that are both $\sigma$-compact and relatively compact and the class $F_1$ consists of all $\sigma$-compact open subsets of $X$ (a subset is called *$\sigma$-compact* if it is a countable union of compact subsets). If $\lambda$ is a bounded (in the supremum norm) positive linear functional on $\mathcal{K}(X)$, then the Radon measure representing $\lambda$ is finite, while the measure $\phi$ constructed in the proof of Theorem 2.5.2 gives the value $+\infty$ for every element of $\mathcal{R}_L'$ in case $X$ is not $\sigma$-compact. Hence, it is not in general true that two measures representing $\lambda$ agree on the whole $\sigma$-algebra $\mathcal{A}_L$.

## Paragraph 2.5.8

It is claimed that if $\phi$ measures $X$ (using more standard terminology, this means that $\phi$ is an exterior measure on $X$) and $\mu$ is a Daniell integral on $L_\infty(\phi)$, then $\mu$ is represented by a function $k \in L_1(\phi)$, i.e., $\mu(f) = \int_X fk \,\mathrm{d}\phi$, for every $f \in L_\infty(\phi)$. It should be noted that the spaces $L_p(\phi)$ are spaces of functions here, not of equivalence classes of functions that are equal almost everywhere (also the elements of $L_\infty(\phi)$ are assumed by definition to have

[1]The infimum of the empty set is $+\infty$.

$\sigma$-finite support — otherwise the result would not hold). So here is the minor bug: there is nothing that ensures that if $f$ and $g$ in $L_\infty(\phi)$ are equal almost everywhere then $\mu(f) = \mu(g)$. Adding this assumption, it will indeed follow that $\mu$ is bounded and thus represented by some $k \in L_1(\phi)$. Without this assumption, there are simple counterexamples: consider, for instance, the Daniell integral $\mu(f) = f(x)$ given by evaluation at some fixed point $x \in X$ with $\phi\big(\{x\}\big) = 0$.

## Paragraph 2.5.19

Let $X$ be a locally compact Hausdorff topological space and denote by $\mathcal{K}(X)$ the space of real-valued continuous functions on $X$ with compact support. Given a map $M : \mathcal{K}(X) \to [0, +\infty[$, it is claimed that the set

$$\begin{aligned}\big\{\mu \in \mathcal{K}(X)^* : \mu \text{ is a Daniell integral and } \mu^+(f) + \mu^-(f) \le M(f),\\ \text{for all } f \in \mathcal{K}(X)\big\}\end{aligned} \tag{2}$$

is compact in the weak topology (which using more standard terminology would be called the weak* topology). This claim is true if we assume that $M(f) = M(|f|)$, for all $f \in \mathcal{K}(X)$. Equivalently, we could replace "for all $f \in \mathcal{K}(X)$" in (2) with "for all nonnegative $f \in \mathcal{K}(X)$". Here is a counterexample to the claim stated in the book: set $X = [0, 1]$ and let $f_0 : X \to \mathbb{R}$ be a continuous function such that $f_0(x) = 1$ for $x \in \big[0, \frac{1}{3}\big]$ and $f_0(x) = -1$ for $x \in \big[\frac{2}{3}, 1\big]$. Define $M$ by setting $M(f) = 4\|f\|_{\sup}$, for $f \ne f_0$ and $M(f_0) = 1$. Let $(x_n)_{n\ge1}$ be a convergent sequence in $\big[\frac{2}{3}, 1\big]$ with limit $x$ such that $x_n \ne x$ for all $n$ and let $y \in \big[0, \frac{1}{3}\big]$ be given. Consider the sequence $(\lambda_n)_{n\ge1}$ of Daniell integrals on $\mathcal{K}(X)$ given by $\lambda_n = \delta_{x_n} - \delta_x + 2\delta_y$, for all $n \ge 1$, where $\delta_t$ denotes evaluation at $t$. We have that $(\lambda_n)_{n\ge1}$ converges to $\lambda = 2\delta_y$. Moreover, $\lambda_n^+ = \delta_{x_n} + 2\delta_y$, $\lambda_n^- = \delta_x$, $\lambda^+ = \lambda$ and $\lambda^- = 0$. It follows that $\lambda_n$ is in (2) for all $n$ but $\lambda$ isn't in (2), since $\lambda(f_0) = 2 > M(f_0)$.

## 1. Paragraph 2.6.4

Let $X$ and $Y$ be locally compact Hausdorff spaces and let $\lambda$ and $\mu$ be monotone Daniell integrals on $\mathcal{K}(X)$ and $\mathcal{K}(Y)$, respectively. Let $\nu$ be the Daniell integral on $\mathcal{K}(X \times Y)$ that extends $\lambda \otimes \mu$. It is claimed that if $\alpha$ is the $\mathcal{K}(X)$-regular measure on $X$ that represents $\lambda$ and $\beta$ is the $\mathcal{K}(Y)$-regular measure that represents $\mu$, then the product measure $\alpha \times \beta$ is the $\mathcal{K}(X \times Y)$-regular measure that represents $\nu$. It is true that $\alpha \times \beta$ represents $\nu$, but it is not in general true that $\alpha \times \beta$ is $\mathcal{K}(X \times Y)$-regular. For instance, if $X$ is not a countable union of compact sets and if there exists a nonempty $\beta$-measurable subset $S$ of $Y$ with $\beta(S) = 0$, then $(\alpha \times \beta)(X \times S) = \alpha(X)\beta(S) = 0$. On the other hand, if $\gamma$ is a $\mathcal{K}(X \times Y)$-regular measure on $X \times Y$ then $\gamma(X \times S) = +\infty$, since $X \times S$ is not contained in a countable union of compact sets. The claim on the book is true if $X$ and $Y$ are both assumed to be countable unions of compact sets.

## 2. Paragraph 2.6.5

It is not true that the product of measures is associative (this is true for $\sigma$-finite measures, as follows from Lemma 1 below). If $\mathfrak{m}$ denotes the Lebesgue exterior measure on $\mathbb{R}$, $\mathfrak{c}$ denotes the counting measure on $\mathbb{R}$ and $S = \Delta \times \mathbb{R}$, with $\Delta = \big\{(x,x) : x \in \mathbb{R}\big\}$, then

$$\big((\mathfrak{m} \times \mathfrak{m}) \times \mathfrak{c}\big)(S) = (\mathfrak{m} \times \mathfrak{m})(\Delta)\mathfrak{c}(\mathbb{R}) = 0 \cdot (+\infty) = 0.$$

On the other hand, we will show below (Corollary 3) that $\big(\mathfrak{m} \times (\mathfrak{m} \times \mathfrak{c})\big)(S)$ is infinite. Given a measure $\phi$ on a set $X$, we denote by $\mathfrak{M}_\phi$ the set of $\phi$-measurable sets (we note that what is being called a "measure" here is more usually called an exterior measure).

**Lemma 1.** *Let $\alpha$, $\beta$ and $\gamma$ be measures on sets $X$, $Y$ and $Z$, respectively. If $\alpha$ is $\sigma$-finite (i.e., $X$ is a countable union of $\alpha$-measurable sets of finite measure) then $\big(\alpha \times (\beta \times \gamma)\big)(U)$ equals the infimum of the set*

$$\Big\{\sum_{i=1}^{\infty} \alpha(A_i)\beta(B_i)\gamma(C_i) : U \subset \bigcup_{i=1}^{\infty}(A_i \times B_i \times C_i),\quad A_i \in \mathfrak{M}_\alpha,\ B_i \in \mathfrak{M}_\beta \text{ and } C_i \in \mathfrak{M}_\gamma\Big\}, \tag{3}$$

*for every $U \subset X \times Y \times Z$.*

*Proof.* By definition, $\big(\alpha \times (\beta \times \gamma)\big)(U)$ equals the infimum of the set:

$$\Big\{\sum_{i=1}^{\infty} \alpha(A_i)(\beta \times \gamma)(P_i) : U \subset \bigcup_{i=1}^{\infty}(A_i \times P_i),\quad A_i \in \mathfrak{M}_\alpha \text{ and } P_i \in \mathfrak{M}_{\beta\times\gamma}\Big\}. \tag{4}$$

Every element of (3) is in (4), since we can take $P_i = B_i \times C_i$. The fact that $\alpha$ is $\sigma$-finite implies that the set (4) is unchanged if we add the condition that $\alpha(A_i) < +\infty$, for all $i$. The conclusion then follows from the observation that $(\beta \times \gamma)(P_i)$ can be approximated by a sum $\sum_{j=1}^{\infty} \beta(B_{ij})\gamma(C_{ij})$, with $B_{ij} \in \mathfrak{M}_\beta$ and $C_{ij} \in \mathfrak{M}_\gamma$ so that, since $\alpha(A_i)$ is finite, the sum in (4) is approximated by the sum $\sum_{i,j=1}^{\infty} \alpha(A_i)\beta(B_{ij})\gamma(C_{ij})$. □

**Corollary 2.** *Let $\alpha$, $\beta$ and $\gamma$ be measures on sets $X$, $Y$ and $Z$, respectively. If both $\alpha$ and $\gamma$ are $\sigma$-finite, then $\alpha \times (\beta \times \gamma) = (\alpha \times \beta) \times \gamma$.* □

**Corollary 3.** *With the notation introduced above, we have:*

$$\big(\mathfrak{m} \times (\mathfrak{m} \times \mathfrak{c})\big)(S) = +\infty.$$

*Proof.* Assume $S \subset \bigcup_{i=1}^{\infty}(A_i \times B_i \times C_i)$ and let us show that

$$\mathfrak{m}(A_i)\mathfrak{m}(B_i)\mathfrak{c}(C_i) = +\infty,$$

for some $i$. Pick $y \in \mathbb{R}$ not in the union of the sets $C_i$ that are finite. Then for every $x \in \mathbb{R}$ we have $(x,x,y) \in S$ and therefore there exists $i$ such that

$x \in A_i \cap B_i$ and $C_i$ is infinite. Thus

$$\mathbb{R} = \bigcup \big\{A_i \cap B_i : C_i \text{ is infinite}\big\}$$

which proves that there exists $i$ such that $C_i$ is infinite, $\mathfrak{m}(A_i) > 0$ and $\mathfrak{m}(B_i) > 0$. □

## 3. Paragraphs 2.7.14 and 2.7.15

There is something important missing from the proofs of Theorem 2.7.14 and Corollary 2.7.15. Both results are true, however. In the proof of Theorem 2.7.14, the book considers the $\mathcal{K}(G)$-regular measure $\alpha$ representing a left $\chi$-covariant monotone Daniell integral $\lambda$. The proof proceeds to show correctly that every open subset of $G$ is $\alpha$-measurable. However, something else is needed to complete the argument, as explained below. The situation with Corollary 2.7.15 is similar.

Let $X$ be an arbitrary locally compact Hausdorff topological space and $\alpha$ be a $\mathcal{K}(X)$-regular measure representing an arbitrary monotone Daniell integral $\lambda$ on $\mathcal{K}(X)$. Assume that every open subset of $X$ is $\alpha$-measurable. The results mentioned in the proof of Theorem 2.7.14 (paragraphs 2.5.3 and 2.5.14) imply that $\alpha$ satisfies all the conditions in the definition of a Radon measure except for the fact that the equality

$$\alpha(U) = \sup\big\{\alpha(K) : K \subset U,\, K \text{ compact}\big\} \tag{5}$$

might not hold if $U$ is an open subset of $X$ with $\alpha(U) = +\infty$. Equality (5) does hold if $U$ is contained in a countable union of compact subsets. Namely, in this case we can write $U = \bigcup_{n=1}^{\infty} U_n$ as a disjoint union of Borel subsets $U_n$ with finite measure and for each $n$ we can find a compact subset $K_n$ of $U_n$ with $\alpha(U_n \setminus K_n) < \frac{1}{2^n}$. Then $\bigcup_{i=1}^{n} K_i$ is a compact subset of $U$ for all $n$ and $\lim_{n\to+\infty} \alpha\big(\bigcup_{i=1}^{n} K_i\big) = +\infty$ if $\alpha(U) = +\infty$.

In general, equality (5) might not hold. If $U$ is not contained in a countable union of compact subsets then we necessarily have $\alpha(U) = +\infty$. However, it might happen for instance that $U$ is disjoint from the support of $\lambda$, so that $\alpha(K) = 0$ for every compact subset $K$ of $U$. Even assuming that the support of $\lambda$ equals $X$ itself, equality (5) does not follow, as shown in the counterexample given below in Subsection 3.1. In the context of Theorem 2.7.14 and Corollary 2.7.15 the equality does hold, however, due to the following two lemmas.

**Lemma 4.** *Let $X$ be a locally compact Hausdorff topological space and $\lambda$ be a monotone Daniell integral on $\mathcal{K}(X)$ whose support is $X$. Let $\alpha$ be the $\mathcal{K}(X)$-regular measure that represents $\lambda$ and assume that every open subset of $X$ is $\alpha$-measurable. If $X$ is a disjoint union*[2] *of open $\sigma$-compact subsets, then $\alpha$ is a Radon measure.*

[2]A subset of $X$ is *$\sigma$-compact* if it is a countable union of compact subsets. For a locally compact Hausdorff space $X$, it holds that $X$ is a disjoint union of open $\sigma$-compact subsets if and only if $X$ is paracompact.

*Proof.* It remains to check that the righthand side of (5) is infinite if $U$ is an open subset of $X$ that is not contained in a $\sigma$-compact subset. Write $X = \bigcup_{i\in I} X_i$ as a disjoint union of open $\sigma$-compact subsets $X_i$. Since $U$ is not contained in a $\sigma$-compact subset, it follows that $U$ intersects $X_i$ for uncountably many $i$. Since the support of $\lambda$ is $X$, we have that the measure of a nonempty open set is positive. Thus, $\alpha(U \cap X_i)$ is positive for uncountably many $i$. It follows that there exists $\varepsilon > 0$ and an infinite set $J \subset I$ such that $\alpha(U \cap X_i) > \varepsilon$ for all $i \in J$. For each $i \in J$ let $K_i$ be a compact subset of $U \cap X_i$ such that $\alpha(K_i) > \varepsilon$. The unions $\bigcup_{i\in F} K_i$ with $F \subset J$ finite are compact subsets of $U$ and their measure can be made as large as desired. □

If $G$ acts transitively on $X$ and $\lambda$ is nonzero and $\chi$-covariant, then the support of $\lambda$ is $G$-invariant and nonempty and thus it is equal to $X$. To conclude the argument, we need the following result.

**Lemma 5.** *If $G$ is a locally compact Hausdorff topological group, then $G$ admits an open $\sigma$-compact subgroup $G_0$. If $G$ acts on $X$ and the maps*

$$G \ni g \longmapsto g \cdot x \in X$$

*are continuous and open for all $x \in X$ then the orbits of the action of $G_0$ on $X$ yield a partition of $X$ into open $\sigma$-compact subsets.*

*Proof.* Simply take $G_0$ to be the subgroup generated by a compact neighborhood of 1. The second statement is immediate. □

3.1. **Counterexample.** Let $\omega$ denote the set of natural numbers and let $(A_i)_{i\in I}$ be an uncountable almost disjoint family of subsets of $\omega$ (i.e., each $A_i$ is an infinite subset of $\omega$ and $A_i \cap A_j$ is finite, if $i \neq j$). Topologize the disjoint union $X = \omega \cup I$ so that each point of $\omega$ is isolated and the fundamental neighborhoods of each $i \in I$ are unions of $\{i\}$ with cofinite subsets of $A_i$. Then $X$ is a locally compact Hausdorff topological space and $\omega$ is a countable open dense subset of $X$. Since $I$ is closed in $X$ and discrete in the relative topology, it follows that the intersection with $I$ of every compact subset of $X$ is finite. Thus, the $\sigma$-compact subsets of $X$ are precisely the countable subsets of $X$. Let $(p_n)_{n\in\omega}$ be a sequence of positive real numbers with $\sum_{n\in\omega} p_n < +\infty$ and let $\lambda$ be the monotone Daniell integral on $\mathcal{K}(X)$ defined by $\lambda(f) = \sum_{n\in\omega} p_n f(n)$, for all $f \in \mathcal{K}(X)$. The support of $\lambda$ is $X$ itself, because $\omega$ is dense in $X$. The $\mathcal{K}(X)$-regular measure $\alpha$ representing $\lambda$ is given as follows: if $A$ is an uncountable subset of $X$, then $\alpha(A) = +\infty$. If $A \subset X$ is countable, then $\alpha(A) = \sum_{n\in A\cap\omega} p_n$. Every subset of $X$ is $\alpha$-measurable. Equality (5) does not hold with $U = X$, since $\alpha(X) = +\infty$ and the righthand side of (5) equals the sum $\sum_{n\in\omega} p_n$.

## References

[1] D. V. Tausk, http://www.ime.usp.br/~tausk/texts/goodlem.pdf.
[2] D. V. Tausk, http://www.ime.usp.br/~tausk/texts/NotasMedida.pdf.

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