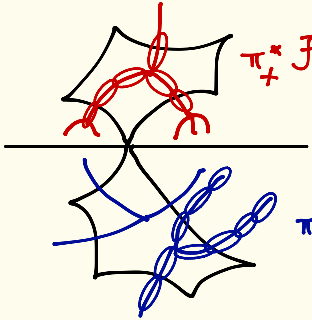
Summary:

9/11/15

$f : S \to S$ pA: $\mathcal{F}^+, \mathcal{F}^-$ measured foliations, (quad. diff.), $f^*\mathcal{F}^+ = \lambda \mathcal{F}^+$, $f^*\mathcal{F}^- = \frac{1}{\lambda}\mathcal{F}^-$

$\mathcal{F}^+ = |\mathrm{Re}\sqrt{q}|$, $\mathcal{F}^- = |\mathrm{Im}\sqrt{q}|$.

$S = \mathbb{H}^2/\Gamma$ fuchsian gp.



$\pi_+^*\mathcal{F}^+$ $\quad \pi_+ : \mathbb{H}^+ \to S$

$\pi_-^*\mathcal{F}^-$ $\quad \pi_- : \mathbb{H}^- \to S$, consider:

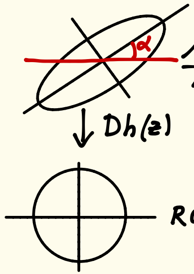
$$\mu_t = \begin{cases} t\,\dfrac{\bar{q}}{|q|} & \text{in } \mathbb{H}^+ \\ t\,\dfrac{\overline{-q}}{|-q|} & \text{in } \mathbb{H}^- \end{cases}$$

$h_t : \mathbb{P}^1 \to \mathbb{P}^1$, $\dfrac{\partial h_t}{\partial \bar{z}} = \mu_t \dfrac{\partial h_t}{\partial z}$

$$\boxed{\Gamma_t = h_t \circ \Gamma \circ h_t^{-1}}$$

Question: does $\lim_{t \to 1} \Gamma_t$ converge, at least up to conjugation?

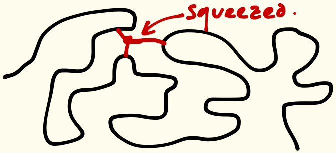
$\underset{\sim\sim}{R_k}$: $t\frac{\bar{q}}{|q|}$ is a Beltrami form: $\frac{\partial h}{\partial \bar{z}} = \mu \frac{\partial h}{\partial z}$, $\|\mu\|_\infty < 1$: μ encodes family of ellipses



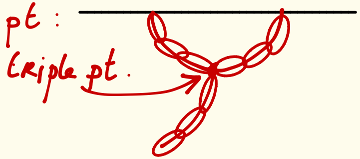
$\frac{1+|\mu|}{1-|\mu|}$ is ratio: $\frac{\text{big axis}}{\text{small axis}}$, $\alpha = \frac{1}{2}\arg \mu$.

Round circle. If $\mu = 0$: Cauchy-Riemann equ.

We are lining up the ellipses with foliations: squeeze these foliations:



coming from triple pt:



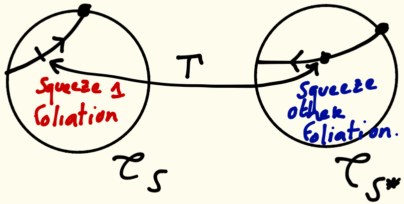
Moore's thm:

$$\begin{array}{c} S^2 \\ \downarrow \\ X \end{array}$$

X Hausdorff
X not a pt.

the fibers do not disconnect $S^2 \Rightarrow X \underset{\text{homeo.}}{\simeq} S^2$, π can be approx. by homeo.

Application: here the h_t approx. the quotient.

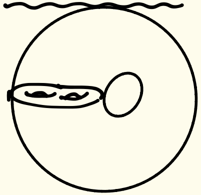
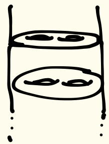


The space of qf-deformations of T is isom. to $\mathcal{T}_S \times \mathcal{T}_{S^*}$

Varying t means: moving along curves.

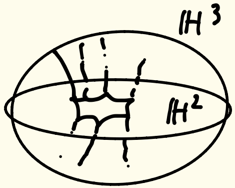
Recall: we want fixed pt for the action of f.

Moore's thm: is here to show that such a limit (after squeezing) can actually exist.

 with cover  with symmetry: moving up means applying f.

this should be hard: f is p.A, far from analytic,

but will be represented by an isometry.



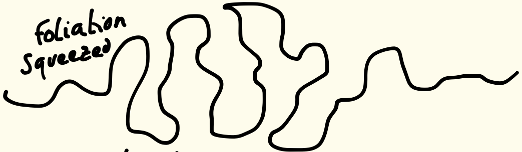
= how to build structure on $S^2 \times \mathbb{R}$: unfortunately without symmetry.

Bers: "do squeezing on top, leave bottom" $\Rightarrow$ lim. exists.

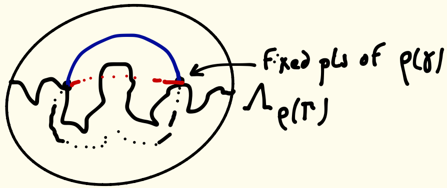
Theorem: compactness of the Bers slice.

> The image of $\mathcal{T}_S \times K$ with K compact in $\mathcal{T}_{S^*}$ in $[\mathrm{Rep}](\Gamma, PSL_2\mathbb{C})$ has compact closure.
>
> $\uparrow$ conjug. classes of representations.

Proof: $\rho : \Gamma \to PSL_2\mathbb{C}$



Analogy: 2 neighb. country: as long as <u>price of bread</u> stays reasonable in one neighbour, price will stay OK.

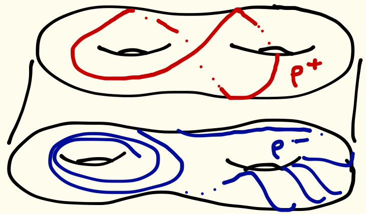


price of bread here $=$ $\mathrm{tr}\ \rho(\gamma)$ $=$ length in hyp. space.

$\exists$ geodesic joigning them in $\mathbb{H}^3$

" " " upper-half plane.

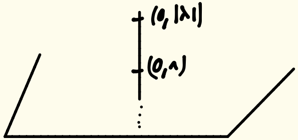
" " " lower " "



Bers: $\frac{1}{\ell^+} + \frac{1}{\ell^-} \leq \frac{2}{\ell}$.

$$\frac{\ell^+ + \ell^-}{\ell^+\ell^-} \leq \frac{2}{\ell}, \quad \ell \leq 2\,\frac{\ell^+\ell^-}{\ell^+ + \ell^-} = 2\ell^+\,\frac{\ell^-}{\ell^+ + \ell^-} \leq 2\ell^+ \text{ hence } \ell \leq 2\inf(\ell^+, \ell^-).$$

Recall: $z \mapsto \lambda z$, in matrix form: $\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}(z) = \lambda z$. Trace is $\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}$.



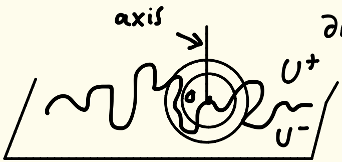
$$\ell = \int_1^{|\lambda|} \frac{dt}{t} = \log|\lambda|.$$

thus tr same as $\log|\lambda|$.

Hence tr of all elements stay bounded, hence conj. classes stay in a compact.

Hence we need to show: $\frac{1}{\ell^+} + \frac{1}{\ell^-} \leq \frac{2}{\ell}$.

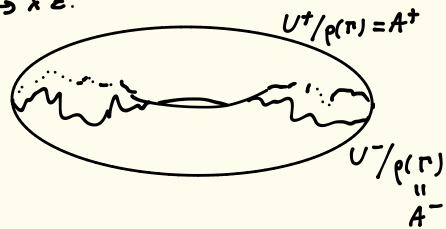
Idea: comes from Grötzsch:



draw circle, and its image by $z \mapsto \lambda z$.

$\mathbb{C}^*/\lambda^{\mathbb{Z}}$ is a torus

containing image of limit set



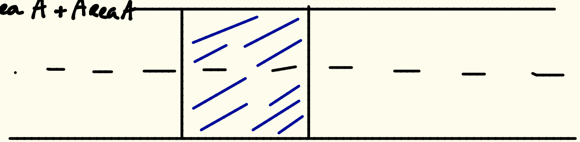
But: can't put very fat annuli in a fixed torus.

One way to see torus: $\mathbb{C}/_{2\pi i\mathbb{Z}+\log\lambda\mathbb{Z}} = \mathbb{C}^*/_{\lambda^{\mathbb{Z}}}$, since $\mathbb{C}^* = \mathbb{C}/_{2\pi i\mathbb{Z}}$ by $\begin{array}{ccc}\mathbb{C} & \to & \mathbb{C}^* \\ z & \mapsto & e^z\end{array}$

Grötzsch

Give torus its euclidean metric: $2\pi|\ln\lambda| = \text{Area } T \geqslant \text{Area } A^+ + \text{Area } A^-$

$\mathbb{B} = \{z \,/\, |\mathrm{Im}\, z| < \frac{\pi}{2}\}$ with metric $\frac{|dz|}{\cos y}$ "band model".

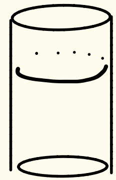


$A^+ = \mathbb{B}/_{\ell^+\mathbb{Z}}$ $A^- = \mathbb{B}/_{\ell^-\mathbb{Z}}$

$A^+ \xleftarrow{\varphi^+} C^+$

$$\int_{C^+} |(\varphi^+)'(z)|^2 dx\,dy + \int_{C^-} |(\varphi^-)'(z)|^2 dx\,dy$$

$$= \frac{1}{\ell^+}\int_{-\pi/2}^{\pi/2}\left(\int_0^{\ell^+} |\varphi'(z)|^2 dx\right)\left(\int_0^{\ell^+} 1^2 dx\right) dy + \frac{1}{\ell^-}\,(\text{corresponding quantity}).$$



$$\geqslant \frac{1}{\ell^+}\int_{-\frac{\pi}{2}}^{\pi/2}\Big(\underbrace{\int_0^{\ell^+} |\varphi'(z)|\,dx}_{\text{a length}}\Big)^2 dy + \text{corresponding qty.}$$

Cauchy-Schwarz

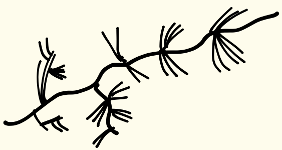
$$\geqslant \frac{1}{\ell^+}\left(\int_{-\pi/2}^{\pi/2} |2\pi p + q\log\lambda|^2 dy\right) + \cdots$$

$$\geqslant \pi\,|2\pi p + q\log\lambda|^2 \left(\frac{1}{\ell^+} + \frac{1}{\ell^-}\right) \geqslant \cancel{\pi}\,\ell^2\left(\frac{1}{\ell^+} + \frac{1}{\ell^-}\right)$$ but $2\cancel{\pi}\ell = 2\cancel{\pi}|\ln\lambda| = \text{Area } T.$ $\square$

Conclusion: $\boxed{\frac{2}{\ell} \geqslant \frac{1}{\ell^+} + \frac{1}{\ell^-}}$

Now: "if $\lim_{t \to 1} T_t$ does not exist, then hyperbolic space degenerates into an $\mathbb{R}$-tree."

Def: an $\mathbb{R}$-tree is a metric space T s.t $\forall x, y \in T$, the intersection of the connected subsets containing x and y is isometric to an interval.



A Γ-tree is an $\mathbb{R}$-tree on which Γ acts by isometries.

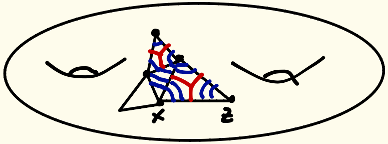
The action is minimal if there is no invariant proper subtree.

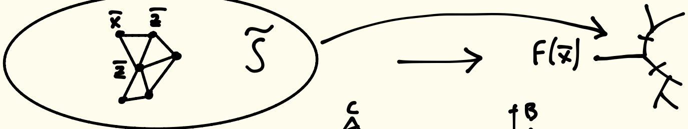
Principal exemple: is the set of leaves of $\tilde{\mathcal{F}}$, where $\mathcal{F}$ is a measured foliation on a surface S, $\pi : \tilde{S} \to S$ is a universal cov. map and $\tilde{\mathcal{F}} = \pi^* \mathcal{F}$.

Hatcher: if $\Gamma = \pi_1(S, s)$ and if T is Γ-tree, then there exists a measured foliation on S and a Γ-equiv. map $\mathcal{L}(\tilde{\mathcal{F}}) \to T$.

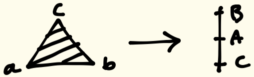
Skora: if ~~f~~ all edges have small stabilizers, the map above is an isometry.
$\{1\}$ or $\mathbb{Z}$

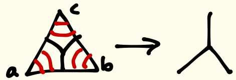
Hatcher:





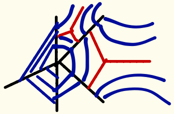
Tree





tripod

Problem: could have



End of lecture Monday 9/11/2015 $\square$