

Approximation Algorithms for Circle Packing

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São Paulo, SP

July, 2016

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Colaborators and references

Main references

- F. K. Miyazawa, L. L. C. Pedrosa, R. C. S. Schouery, M. Sviridenko, Y. Wakabayashi. Polynomial-Time Approximation Schemes for Circle and Other Packing Problems. *Algorithmica*. To appear.
- P. H. Hokama, F. K. Miyazawa and R. C. S. Schouery. A bounded space algorithm for online circle packing. *Information Processing Letters*, 116, p. 337-342, 2016.

This work also obtained collaboration from [Lehilton L. C. Pedrosa](#), [Maxim Sviridenko](#), [Rafael C. S. Schouery](#), [Pedro H. Hokama](#) and [Yoshiko Wakabayashi](#).

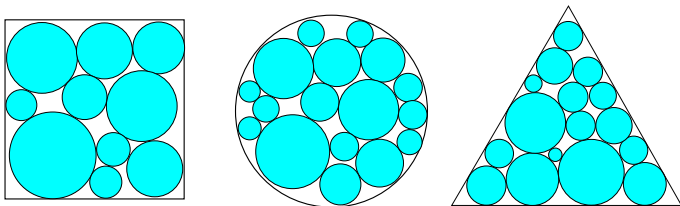
My thanks to [Lehilton Pedrosa](#) and [Rafael Schouery](#) that also contributed to part of these slides. This work obtained support from [CNPq](#), [FAPESP](#), [UNICAMP](#), [USP](#).

CIRCLE PACKING PROBLEMS

Packings

Given:

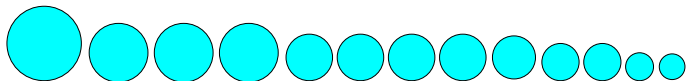
- ▶ A list of geometrical items L and bins \mathcal{B}
- ▶ Obtain a *good* packing of items in L into bin $B \in \mathcal{B}$
- ▶ The inner region of two packed items cannot overlap
- ▶ Each packed item must be totally contained in the bin



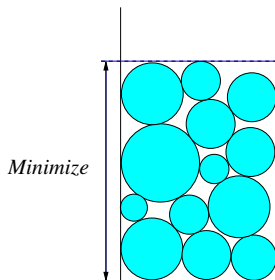
Problems

Circle Strip Packing

- ▶ **Input:** List of circles $L = (c_1, \dots, c_n)$



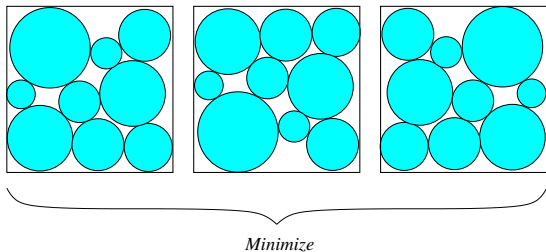
- ▶ **Output:** Packing of L into a rectangle of width 1 and minimum height.



Problems

Circle Bin Packing

- ▶ **Input:** List of circles $L = (c_1, \dots, c_n)$
- ▶ **Output:** Packing of L into the minimum number of unit bins.



Some Applications

- ▶ Cutting and Packing of circular items
- ▶ Transportation of tubes, cylinders,...
- ▶ Cable assembly/allocations
- ▶ Tree plantation
- ▶ Origami design
- ▶ Marketing
- ▶ Cylinder pallet assembly
- ▶ ...

Marketing



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Computational Complexity

Demaine, Fekete, Lang'10: To decide if a set of circles can be packed into a square is NP-hard.

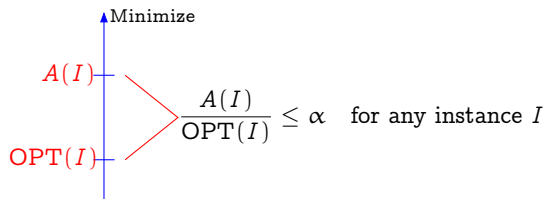
Approximation Algorithms:

- ▶ Efficient Algorithms (polynomial time)
- ▶ Analysis: How far from the optimum solution value ?
- ▶ Compromise:

Computational Time \times Solution Quality

Approximation Algorithms

- ▶ $A(I)$ Value of the solution produced by A for instance I
- ▶ $\text{OPT}(I)$ Value of an optimum solution of I
- ▶ A has approximation factor α if



- ▶ A has asymptotic approximation factor α if

$$A(I) \leq \alpha \text{OPT}(I) + \beta \quad \text{for any instance } I,$$

for some constant β

Approximation Algorithms

For Minimization Problems

PTAS: Polynomial Time Approximation Scheme

- ▶ A family of polynomial time algorithms A_ε , $\varepsilon > 0$, is a **polynomial time approximation scheme** if

$$A_\varepsilon(I) \leq (1 + \varepsilon) \text{OPT}(I), \quad \text{for any instance } I$$

APTAS: Asymptotic Polynomial Time Approximation Scheme

- ▶ A family of algorithms A_ε , $\varepsilon > 0$ is an **asymptotic polynomial time approximation scheme** if

$$A_\varepsilon(I) \leq (1 + \varepsilon) \text{OPT}(I) + \beta_\varepsilon, \quad \text{for any instance } I$$

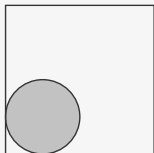
where β_ε is a constant that depends only on ε

Online Packing Algorithms

- ▶ Incoming items appears one after the other, sequentially
- ▶ An incoming item must be packed when it arrives, without the knowledge of further items
- ▶ Once an item is packed, it cannot be repacked again.

Online Circle Bin Packing

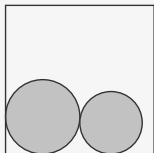
Example



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Online Circle Bin Packing

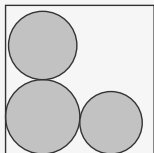
Example



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Online Circle Bin Packing

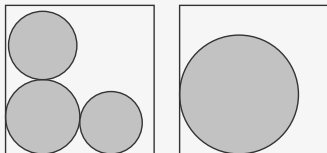
Example



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Online Circle Bin Packing

Example



4

Preliminaries

Some Notation

- ▶ If $f : D \rightarrow \mathbb{R}$ is a numerical function, we may write
 - ▶ f_e and $f(e)$, indistinctly
 - ▶ $f(S)$ as the value $\sum_{e \in D} f(e)$, there is no explicit definition

- ▶ If c is a circle and $L = (c_1, \dots, c_n)$ a list of circles, then
 - ▶ c is also used to denote its **radius**
 - ▶ \hat{c} is the **square** with side lengths $2c$
 - ▶ \hat{L} is the list $\hat{L} = (\hat{c}_1, \dots, \hat{c}_n)$
 - ▶ $\max(L)$ is the **maximum radius** of a circle in L
 - ▶ $\text{Area}(L)$ is the **total area** of the circles in L
 - ▶ \bar{L} is the list with $|L|$ equal circles with radius $\max(L)$
 - ▶ \mathcal{C} is the set containing **all lists of circles** for the input problem

Preliminaries

If \mathcal{E} is a packing or other geometrical composition, \mathcal{E} may be considered as the solid structure

- ▶ $\text{width}(\mathcal{E})$ is the width of \mathcal{E}
- ▶ $\text{height}(\mathcal{E})$ is the height of \mathcal{E}

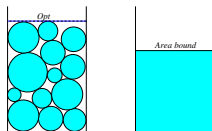
First considerations

- ▶ We consider a more general computational model
- ▶ Possible to operate over polynomial solutions

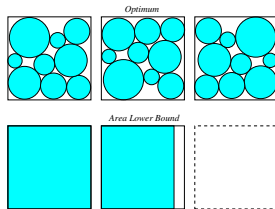
BASIC ALGORITHMS

Area Lower Bound

- ▶ Circle Strip Packing with bin width 1



- ▶ Circle Bin Packing with unit square bins



Area based algorithms

Let
 \mathcal{C} the set containing all lists of circles, and
 \mathcal{Q} the set containing all lists of squares
 next algorithm is a circle version $\mathbb{C}\mathcal{A}$ from square packing algorithm \mathcal{A}

$\mathbb{C}\mathcal{A}(L)$

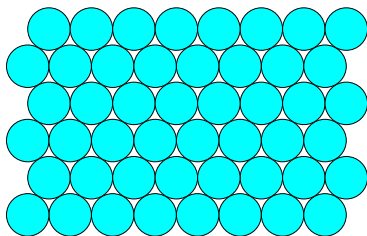
1. Let $\hat{\mathcal{P}} \leftarrow \mathcal{A}(\hat{L})$.
2. Let \mathcal{P} the packing $\hat{\mathcal{P}}$ replacing \hat{c}_i by c_i .
3. Return \mathcal{P} .

Lemma. If \mathcal{A} is a square packing algorithm and α, β are constants, st. $\mathcal{A}(S) \leq \alpha \text{Area}(S) + \beta$, for any $S \in \mathcal{Q}$

$$\mathbb{C}\mathcal{A}(L) \leq \alpha \frac{4}{\pi} \text{Area}(L) + \beta, \text{ for any } L \in \mathcal{C}$$

Area based algorithms

Best possible density



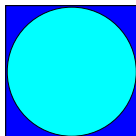
Hexagonal Packing

- Density $\frac{\pi}{\sqrt{12}} \approx 0.9069$

Lemma. There is no algorithm with approximation factor, based only on area arguments, better than $\frac{\sqrt{12}}{\pi} \approx 1.10266$

Using square packing algorithms

- ▶ Round each circle c_i to a square \hat{c}_i :

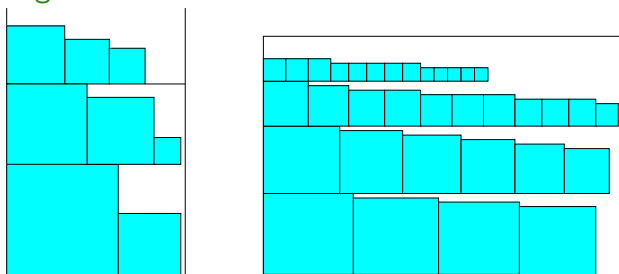


- ▶ Use square packing algorithms
- ▶ Area increasing: $\frac{\text{Area}(\hat{c}_i)}{\text{Area}(c_i)} = \frac{4}{\pi} \approx 1.27324$
- ▶ Let $\hat{L} = (\hat{c}_1, \dots, \hat{c}_n)$ the list L rounding each circle to a square
- ▶ **Bounding the optimum with the area:**

$$\text{Area}(\hat{L}) = \frac{4}{\pi} \text{Area}(L) \leq \frac{4}{\pi} \text{OPT}(L)$$

Using square packing algorithms

Shelf Packing:

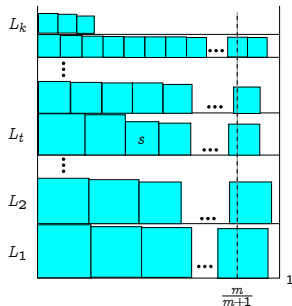


- ▶ Items are packed over shelves (of zero thickness)
- ▶ side by side in a leftmost way
- ▶ Items in a same shelf are packed at the same height.
- ▶ Item s can be packed in a shelf S if $\text{width}(s) + \text{width}(S) \leq 1$

Using square packing algorithms

$\text{NFDH}^s(L)$ # for Strip Packing

1. Sort $L = (s_1, \dots, s_n)$ st. $s_1 \geq \dots \geq s_n$
2. For $i \leftarrow 1$ to n :
3. Pack s_i into the last shelf, **if possible**
4. **otherwise**, pack s_i in a new shelf on top of the previous shelf or bin's bottom (in case there is no previous shelf)



Lemma. If L has only squares with side lengths at most $1/m$

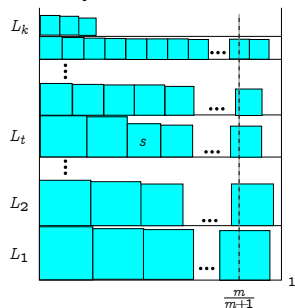
$$\text{NFDH}^s(L) \leq \frac{m+1}{m} \text{Area}(L) + \frac{1}{m}$$

Using square packing algorithms

Sketch.

Each of the first $k-1$ shelves have width filled by at least $\frac{m}{m+1}$

Let L_t the first shelf having square with side $\leq 1/(m+1)$



- L_1, \dots, L_{t-1} : m squares of side $> \frac{1}{m+1}$, each.
- L_t, \dots, L_k : width filled $> 1 - \frac{1}{m+1} = \frac{m}{m+1}$, otherwise receive another item.

Sliding up squares in L_i can cover all rectangular region of shelf L_{i+1} , up to width $\frac{m}{m+1}$.

$$(\text{NFDH}^S(L) - \text{height}(L_1)) \frac{m}{m+1} \leq \text{Area}(L)$$

So,

$$\text{NFDH}^S(L) \leq \frac{m+1}{m} \text{Area}(L) + \text{height}(L_1) \leq \frac{m+1}{m} \text{Area}(L) + \frac{1}{m}$$

Using square packing algorithms

Let

\mathcal{C}_m the set of lists with small circles (diam. $\leq 1/m$, m integer)

Corollary. If $L \in \mathcal{C}_m$, then

$$\text{CNFDH}^s(L) \leq \frac{m+1}{m} \frac{4}{\pi} \text{OPT}(L) + \frac{1}{m} \quad \forall L$$

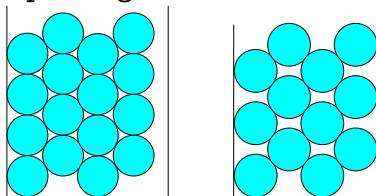
Corollary. If L is a list of circles then

$$\text{CNFDH}^s(L) \leq 2.548 \text{OPT}(L) + 1$$

Rounding circles to circles

Algorithm EqualCircles(L) # all circles in L have a same size

1. Let \mathcal{P}' and \mathcal{P}'' packings of L as below



2. Return packing $\mathcal{P} \in \{\mathcal{P}', \mathcal{P}''\}$ with minimum height.

Lemma. If all circles of L have radius r , then

$$\text{EqualCircles}(L) \leq 1.654 \text{Area}(L) + 2r.$$

Rounding circles to circles

Idea: Circles with close radius are rounded up to the same radius
 Given $L \in \mathcal{C}$, \bar{L} is the list with $|L|$ circles with radius $\max(L)$

Algorithm $\mathcal{A}_\varepsilon(L)$

1. $\delta \leftarrow \frac{\varepsilon}{6}$.
2. For $i \geq 0$ do
3. $L_i \leftarrow \{r \in L : \frac{1/2}{(1+\delta)^{i+1}} < r \leq \frac{1/2}{(1+\delta)^i}\}$.
4. $\mathcal{P}_i \leftarrow \text{EqualCircles}(\bar{L}_i)$.
5. $\mathcal{P} \leftarrow \mathcal{P}_0 \parallel \mathcal{P}_1 \parallel \mathcal{P}_2 \parallel \dots$ # concatenation of packings
6. Return \mathcal{P} .

Theorem. Given $\varepsilon > 0$, we have

$$\mathcal{A}_\varepsilon(L) \leq (1.654 + \varepsilon) \text{Area}(L) + C_\varepsilon, \text{ for any } L \in \mathcal{C}$$

Online Circle Strip Packing

Online Packing

- ▶ Incoming items appears one after the other, sequentially
- ▶ An incoming item must be packed when it arrives, without the knowledge of further items
- ▶ Once an item is packed, it cannot be repacked again.

Baker, Schwarz'83: For $0 < p < 1$, there exists online algorithm CNFS_p s.t.,

$$\text{CNFS}_p(L) \leq \frac{2.548}{p} \text{OPT}(L) + \frac{1}{p(1-p)}, \text{ for any } L \in \mathcal{C}$$

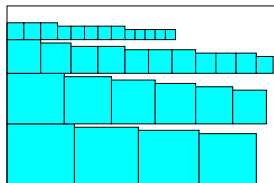
CIRCLE BIN PACKING

Rounding to squares

Adaptation of the strip packing version NFDH^s.

NFDH^b(L) # For the bin packing version

1. Sort $L = (s_1, \dots, s_n)$ st. $s_1 \geq \dots \geq s_n$
2. For $i \leftarrow 1$ to n :
3. Pack s_i in the last shelf (of the last bin), **if possible**
4. **otherwise**, pack s_i in a new shelf at the top of the previous shelf, **if possible**
5. **otherwise**, pack s_i in a new shelf of a new bin.



Rounding to squares

Meir, Moser'68. If all squares of L have side lengths at most $\frac{1}{m}$

$$\text{NFDH}^b(L) \leq \left(\frac{m+1}{m}\right)^2 \text{Area}(L) + \frac{m+2}{m}$$

Proof: Exercise (analogous to the proof of NFDH^s)

Corollary. For any list $L \in \mathcal{C}$ with diameters at most $\frac{1}{m}$

$$\text{CNFDH}^b(L) \leq \frac{4}{\pi} \left(\frac{m+1}{m}\right)^2 \text{Area}(L) + \frac{m+2}{m}$$

Corollary. For any list $L \in \mathcal{C}$

$$\text{CNFDH}^b(L) \leq 5.1\text{OPT}(L) + 3$$

Corollary. As radius of circles decrease, the density of the packing is improved and CNFDH^b goes to $4/\pi \approx 1.27324$.

BOUNDED SPACE ONLINE BIN PACKING

Bounded Space Online Bin Packing

- ▶ Algorithms must be online
- ▶ At any moment, bins are classified as *open* or *closed*
- ▶ Only open bins can receive new items
- ▶ A bin starts open and once it became closed, it cannot be open again.
- ▶ The number of open bins is bounded by a constant

Bounded Space Online Bin Packing

Related results with asymptotic approximation:

- ▶ **Lee and Lee:** Algorithm with factor 1.69103 for 1-dimensional items and showed that no algorithm can have better performance
- ▶ **Epstein, van Stee'07:** Algorithms with factors 2.3722 for packing squares and 3.0672 for packing cubes.

Bounded Space Online Bin Packing

We will see

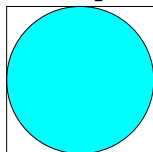
- ▶ Algorithm with asymptotic approximation factor 2.44
- ▶ Lower bound of 2.29

Techniques

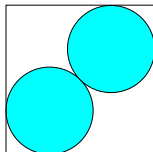
- ▶ Weighting system to obtain approximation factors
- ▶ Specific algorithms to deal with big and small circles
- ▶ Grouping circles to consider as equal circles
- ▶ Geometric Partition to combine items of the same type

Packing equal circles

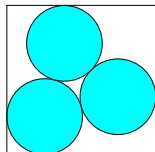
Find the largest ρ^* st. k circles of radius ρ^* can be packed in a unit square



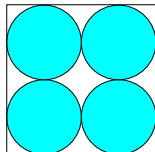
$$\rho_1^* = 0.5$$



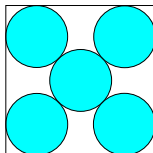
$$\rho_2^* = 0.2928$$



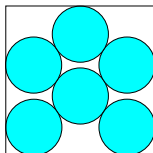
$$\rho_3^* = 0.2543$$



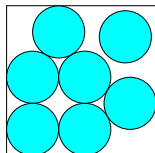
$$\rho_4^* = 0.1963$$



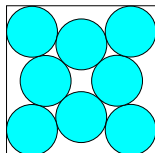
$$\rho_5^* = 0.2071$$



$$\rho_6^* = 0.1876$$



$$\rho_7^* = 0.1744$$



$$\rho_8^* = 0.1705$$

Previous results: It is known the exact values of ρ_n^* , for $n \leq 30$ and good lower bounds for many.

Packing big circles

Round up big circles to the nearest value of ρ
(to bound the number of different circles)

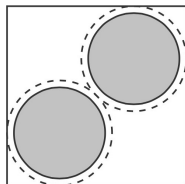
- ▶ A circle is **big** if its radius is larger than $1/M$
- ▶ Let ρ_i be the value of ρ_i^* , when it is known, otherwise, the best known lower bound.
- ▶ Let K be such that $\rho_{K+1} \leq 1/M < \rho_K$

A circle r is of **type i** if:

- ▶ $\rho_{i+1} < r \leq \rho_i$ (for $1 \leq i < K$)
- ▶ $1/M < r \leq \rho_K$ (for $i = K$)

Packing big circles

- ▶ For $1 \leq i \leq K$, a *c-bin* of type i is a circular bin of radius ρ_i
- ▶ Circles of type i are packed in a *c-bin* of type i
- ▶ Packing in a *c-bins* of type 2



Algorithm - Part 1

To pack a big circle c of type i :

if there is no empty c -bin of type i

 close the current bin of type i (if any)

 open a new bin of type i containing i c -bins of type i

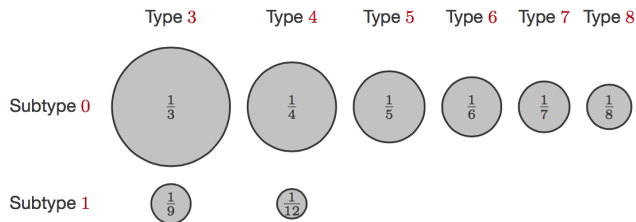
Pack c into a empty c -bin of type i

Small circles

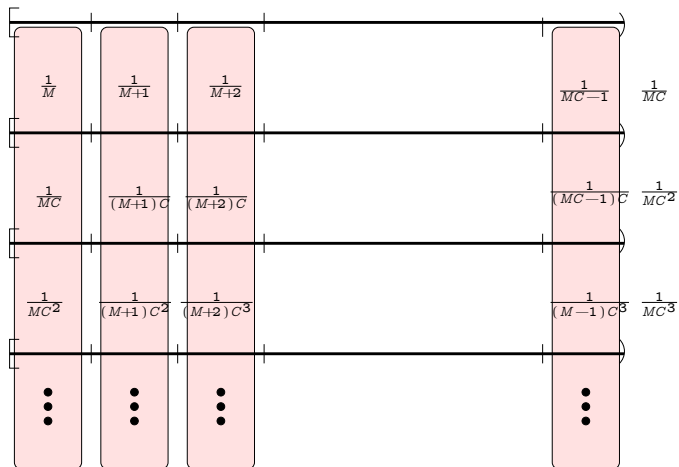
Let $C > 0$ be an integer multiple of 3

A small circle of radius r is of type i , subtype k if

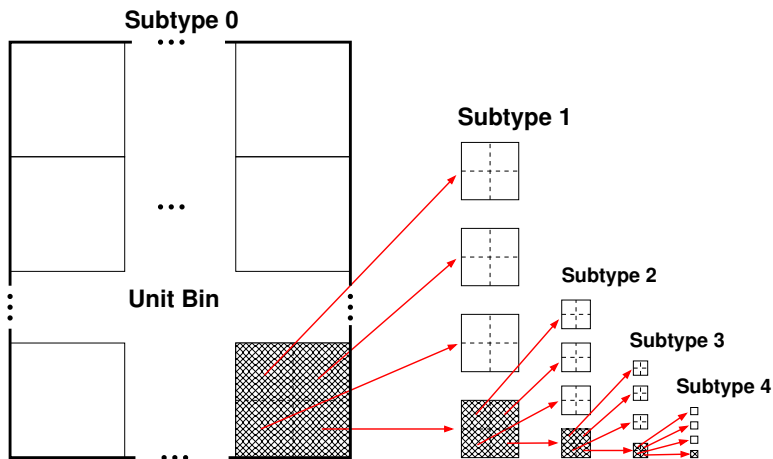
- ▶ $1/(i+1) < C^k r \leq 1/i$
- ▶ where k is the largest integer such that $C^k r \leq 1/M$
- ▶ and the circle is said to be of type (i, k)



Small circles



Idea: Geometric Subdivision



Subdivisions within a same type: Example with squares

Sub-bins (h-bins and t-bins)

Idea: Round/Pack small circles into hexagonal bins

h-bin of type (i, k) :

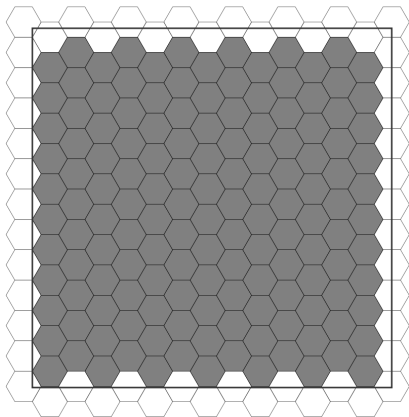
- ▶ hexagonal bin of side length $2/(\sqrt{3} C^k i)$
- ▶ Receives a small circle of type (i, k)

t-bin of type (i, k) :

- ▶ trapezoidal bin obtained by the subdivision of a **h-bin** of type (i, k) in the center

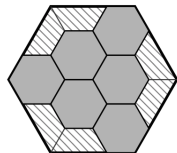
Subdividing a square into h -bins

Subdividing a square into h -Bins

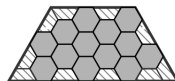
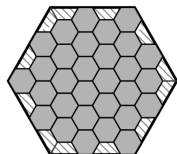


Partitioning sub-bins

For all $M \leq i < CM$ and $k \geq 0$, if C is multiple of 3 then, it is possible to partition an **h-bin** or an **t-bin** of type (i, k) into **h-bins** and **t-bins** of type $(i, k + 1)$.



$$C = 3$$



$$C = 6$$

Algorithm - Part 2

When a small circle c of type (i, k) arrives:

if there is no empty **h-bin** of type (i, k) or an empty **sub-bin** of type (i, k') with $k' < k$

close the current bin of type i (if any)

open a bin of type i subdividing into **h-bins** of type $(i, 0)$

while there is no **h-bin** of type (i, k)

let k' the largest number such that $k' < k$ and there exists an empty **h-bin** of type (i, k')

if there exists an empty **t-bin** of type (i, k')

B tal **t-bin**

else

let B an **h-bin** of type (i, k')

particionate B in **sub-bins** of type $(i, k' + 1)$

packs c into a **h-bin** of type (i, k)

Analysis by weighting function

Given algorithm \mathcal{A} and weight function $w: L \rightarrow \mathbb{R}_{\geq 0}$ st.

- ▶ \mathcal{A} produce bins with **average weight at least 1**
I.e., $w(L)/\mathcal{A}(L) \geq 1$ and therefore

$$\mathcal{A}(L) \leq w(L), \quad \text{for any instance } L$$

- ▶ Find $\alpha \geq$ *maximum bin weight*. I.e.,

$$\alpha \geq \sup\{w(S) : S \subseteq L \text{ and } \exists \text{ packing of } S \text{ in one bin}\}$$

Optimum uses at least $\frac{w(L)}{\alpha}$ bins: $w(L) \leq \alpha \text{OPT}$

Algorithm \mathcal{A} has approximation factor α :

$$\mathcal{A}(L) \leq w(L) \leq \alpha \text{OPT}(L)$$

Removing *few* bins, before average leads to asymptotic factor

Circle weights

How to obtain **average weight ≥ 1** ?

- ▶ Algorithm produce bins with weight ≥ 1

$$w(c) = \begin{cases} 1/i & \text{if } c \text{ is big and type } i \\ \text{Area}(c)/\gamma & \text{if } c \text{ is a small circle,} \end{cases}$$

where γ is area density or lower bound, for bins with small

- ▶ If B is closed type i bin (big items) then B has i circles of weight $1/i$ and $w(B) = 1$.
- ▶ If B is closed bin for small circles, then $\text{Area}(B)$ is also its density and $\text{Area}(B) \geq \gamma$. So

$$w(B) = \sum_{c \in B} w(c) = \sum_{c \in B} \frac{\text{Area}(c)}{\gamma} \geq 1$$

Circle weights

γ : lower bound for area covered in closed bins by small items

The non-covered regions are due to:

- ▶ \mathcal{L}_B : upper bound for the non-covered region due to the shape and partial intersection of hexagons with the **border** of the square bin, and is at most $5.89/M$
- ▶ \mathcal{L}_F : upper bound for to the set of non-covered hexagons when a bin is closed: $2\sqrt{3}C^2/(M^2(C^2 - 1))$
- ▶ \mathcal{L}_H : loss factor due to the *rounding* of circles into hexagons:

$$\frac{\pi}{\sqrt{12}} \frac{M^2}{(M+1)^2}$$

That is

$$\gamma = (1 - \mathcal{L}_B - \mathcal{L}_F) \mathcal{L}_H$$

Computing β

Value of β is obtained via Mixed Integer Programming:

- ▶ x_i : number of circles of type i
- ▶ y : area of small circles

$$\text{maximize } \frac{y}{\alpha} + \sum_{i=1}^K \frac{x_i}{i}$$

$$\text{subject to } y + \sum_{i=1}^K \pi \rho_{i+1}^2 x_i \leq 1$$

$$x_i \in \mathbb{Z}_+ \quad \forall 1 \leq i \leq K$$

$$y \geq 0$$

Computing β

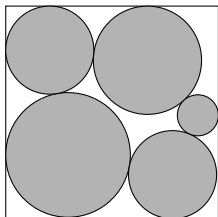
On the other hand, we do not know if a solution can indeed be packed in one bin

- ▶ Using **Constraint Programming** to verify if a solution can be packed in only one bin, with time limit
- ▶ If it is not possible, we add a constraint in the model to avoid such solution

For $M = 59$ and $K = 992$, the value of β is **2.4394**

- ▶ But we do not know if the solution can in fact be packed in only one bin

Lower bound for any competitive factor



- ▶ 1 circle of type 1
 - ▶ 1 circle of type 2
 - ▶ 2 circle of type 4
 - ▶ 1 circle of type 25
- (the area covered by the above circles: 0.77139)
- ▶ remaining space is completed with sand
(very small circles, non-necessarily equal)

Lower Bound

Consider N copies of the lower bound pattern with circles sorted by radius

An online bounded space algorithm B uses:

- ▶ at least $N - B$ bins for the circles of type 1
- ▶ at least $N/2 - 2B$ bins for the circles of type 2
- ▶ at least $2N/4 - 2B$ bins for the circles of type 4
- ▶ at least $N/25 - 2B$ bins for the circles of type 25

At least $2.04N - 7B$ for the circles

Lower bound

Let $S = (1 - 0.77139 - \varepsilon)$ the remaining area used by sand

- ▶ The best way to obtain a dense packing of equal circles is the hexagonal packing
 - ▶ with densities $\pi/\sqrt{12}$
- ▶ Even for very small circles, the algorithm cannot do better than the hexagonal packing
- ▶ The algorithm uses at least $S\sqrt{12}/\pi - 2kB$ bins
 - ▶ k is the number of different radius

The algorithm uses at least $2.2920N - \delta N - O(1)$ bins, that tends to $2.2920 - \delta$ when N goes to infinity

An offline algorithm uses at most N bins

Exercises

- ▶ Obtain bounded online approximation algorithms to pack items into bins, each one could be one of the following: equilateral triangles, squares, circles, hexagons, etc.
- ▶ For the previous exercise, consider the three-dimensional or d -dimensional case.

APPROXIMATION SCHEMES

We will see ideas to obtain

- ▶ APTAS for circle bin packing with resource augmentation
- ▶ APTAS for the circle strip packing

Techniques

- ▶ Solving packings for non small items (applicable to many shapes)
- ▶ Linear Rounding technique
- ▶ Improved approach to combine small and large items

Related works

For cubes and rectangles

- ▶ An asymptotic 1.405-approximation (Bansal and Khan)
- ▶ APTAS for d -dimensional cubes (Bansal et al.)
- ▶ APTAS for 3-dimensional strip packing (Bansal et al.)

Common approach to obtain PTAS in packing problems

Main idea

- ▶ Separate **small** and **large** items
- ▶ use a **PTAS for large** items
- ▶ use simple algorithm pack **small items in free space**

Area occupation for **small items**

- ▶ **square packing** with side lengths are at most ε
NFDH can cover $1 + O(\varepsilon)$ of the bin area
- ▶ **equal circle packing** with radii at most ε
at most a factor of the bin area

- ▶ More complex way to divide and define items as:
small, **medium** and **large**

Subroutines for special cases:

- ▶ Optimum algorithm when $|L|$ is **constant**
- ▶ Optimum algorithm when $\min(L) \geq \varepsilon$ and **diff(L) is constant**,
where **diff(L)** is the number of different items in L .
- ▶ PTAS to obtain a solution for **relatively large items**

Optimum packing when $|L|$ is constant

One Bin Problem: Given list of circles $Q = (r_1, \dots, r_n)$, obtain a packing of L into a unit square.

Formulation:

Obtain positions (x_i, y_i) for circle r_i , for $i = 1, \dots, n$, respecting packing constraints:

$$\begin{aligned}
 (P) \quad & (x_i - x_j)^2 + (y_i - y_j)^2 \geq (r_i + r_j)^2 && \text{for } 1 \leq i < j \leq n, \\
 & r_i \leq x_i \leq 1 - r_i && \text{for } 1 \leq i \leq n, \text{ and} \\
 & r_i \leq y_i \leq 1 - r_i && \text{for } 1 \leq i \leq n.
 \end{aligned}$$

$$(\exists x_1)(\exists y_1) \dots (\exists x_n)(\exists y_n) \bigwedge_{i=0}^s f_i(x_1, y_1, \dots, x_n, y_n) \geq 0.$$

Optimum packing when $|L|$ is constant

One Bin Problem

Resolution of system of polynomials can be solved by standard algebraic quantifier elimination (Grigore'v and Vorobjov Jr.)

Concerning algebraic quantifier elimination algorithms:

- ▶ System resolution may obtain **irrational points**

Computational model

- ▶ Use a **stronger model** of computation
- ▶ Obtain **rational solutions**, within an **error**
circles may intersect or
stay outside the bin within an small error

For rational solutions within an error

- ▶ obtained solution may be invalid packing
- ▶ use *resource augmentation*: Modify solution within an error to a packing, but into bin of dimensions $(1, 1 + \gamma)$.
- ▶ Idea: shift items by small steps possibly extending the solution to the augmented border

Optimum packing when $|L|$ is constant

Multiples Bins Problem

Given list of circles $Q = (r_1, \dots, r_n)$, obtain a packing of L into the minimum number of unit squares.

Obtain an optimum packing

- ▶ enumerate all packable partitions of L into one bin
- ▶ choose one with small number of parts.

Packing without small circles and constant different radius

Let

- ▶ δ be a lower bound for the radius of each circle in L , a constant
- ▶ k the number of different radius in L

Idea

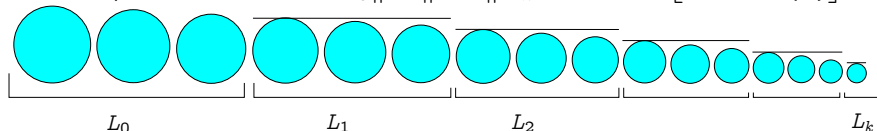
1. Number of circles in a bin is bounded by $1/\text{Area}(\delta)$
2. There are k type possibilities for each circle
3. Obtain all possible packings into one bin (patterns):
 P_1, \dots, P_t
4. Enumerate all possible packings for L .
Each pattern is used at most n times: $O(n^t)$

Packing without small circles

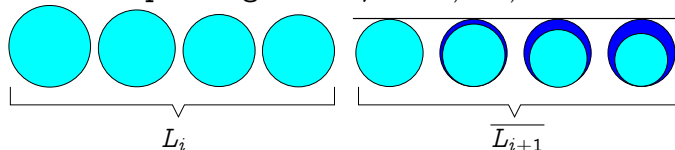
Lemma. \exists PTAS when $\min(L) \geq \delta$, for constant $\delta > 0$.

Linear Rounding:

- Sort L in non-increasing order of radius and divide in $k + 1$ sublists, such that $L = L_0 || L_1 || \dots || L_k$ and $k = \lfloor n \varepsilon \text{Area}(\delta) \rfloor$

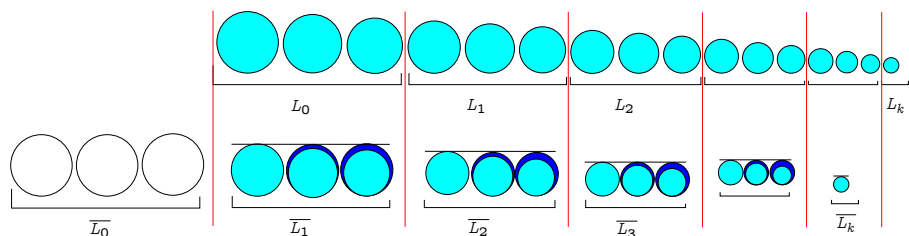


- Let \overline{L}_i the sublist L_i rounding up each circle to the largest in the corresponding sublist, $i = 1, \dots, k$.



Packing without small circles

Packing of L_1, \dots, L_k :



$$\overline{L_{i+1}} \preceq L_i$$

4. We have: $\overline{L_1} \parallel \overline{L_2} \parallel \dots \parallel \overline{L_k} \preceq L_0 \parallel L_1 \parallel \dots \parallel L_{k-1} \preceq L$

So, $\text{OPT}(\overline{L_1} \parallel \overline{L_2} \parallel \dots \parallel \overline{L_k}) \leq \text{OPT}(L_0 \parallel L_1 \parallel \dots \parallel L_{k-1}) \leq \text{OPT}(L)$

where $k = \frac{n}{\lfloor n \varepsilon \text{Area}(\delta) \rfloor} + 1 = O(1/(\varepsilon \text{Area}(\delta)))$

Packing without small circles

Packing of L_0 :

1. Build packing \mathcal{P}_0 placing each circle in a new bin.

$$\begin{aligned}
 |\mathcal{P}_0| &= \lfloor n\varepsilon' \text{Area}(\delta) \rfloor \leq \varepsilon'(n \text{Area}(\delta)) \\
 &\leq \varepsilon' \text{Area}(L) \\
 &\leq \varepsilon' \text{OPT}(L)
 \end{aligned}$$

Packing without small circles

Algorithm \mathcal{A}_ε

1. $\varepsilon' \leftarrow \varepsilon/2$
2. If L uses at most k/ε' bins, pack optimally.
3. Otherwise:
4. $\mathcal{P}_0 \leftarrow$ pack L_0 using at most $\varepsilon' \text{OPT}(L)$ bins.
5. $\mathcal{P}_{1k} \leftarrow$ pack $\overline{L_1} \parallel \overline{L_2} \parallel \dots \parallel \overline{L_k}$ with up to $(1 + \varepsilon') \text{OPT}(L)$ bins

\mathcal{A}_ε is PTAS:

$$\begin{aligned}
 \mathcal{A}_\varepsilon(L) &= |\mathcal{P}_0| + |\mathcal{P}_{1k}| \\
 &= \varepsilon' \text{OPT}(L) + (1 + \varepsilon') \text{OPT}(L) \\
 &= (1 + \varepsilon) \text{OPT}(L)
 \end{aligned}$$

Application: APTAS for bin packing

Bin Packing: Given list $L = (s_1, \dots, s_n)$, where $0 < s_i \leq 1$, obtain a packing of L into the minimum number of bins of size 1.

Changes in

- ▶ resolution of the problem with constant sizes
- ▶ replace $\text{Area}(\cdot)$ by the function $s(\cdot)$.

PTAS for bin packing:

1. Let $B = \{i \in L : s_i \geq \varepsilon\}$ # **Big items**
2. Let S the remaining set # **Small items**
3. $\mathcal{P}_B \leftarrow$ PTAS for B
4. Pack items in S :
5. Greedily pack items of S in the remaining space of \mathcal{P}_B
6. **If needed**, use new bins and continue

Application: APTAS for bin packing

PTAS for bin packing:

Two cases considering “If needed” new bins to pack S :

- ▶ **No new bins:** continue as PTAS

$$\mathcal{P}_B \leq (1 + \varepsilon)\text{OPT}(B) \leq (1 + \varepsilon)\text{OPT}(L)$$

- ▶ **Used new bins:** Bins are almost full before obtain new bin
Proof follows by **area arguments**

Final packing \mathcal{P} such that $|\mathcal{P}| \leq (1 + \varepsilon)\text{OPT}(L) + 1$

PTAS for circle bin packing

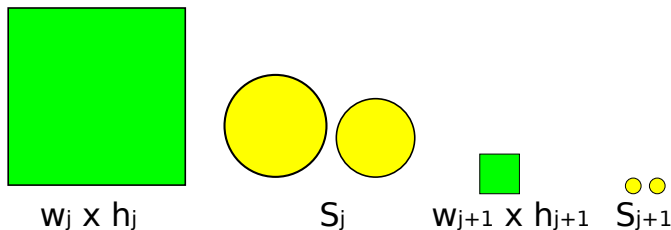
Main steps:

- ▶ Get sets from input list: S_1, S_2, \dots
- ▶ Sizes in S_{i+1} are much smaller than in S_i
- ▶ Use of **medium** items to obtain gap between sets S_i and S_{i+1}
- ▶ Use a separate algorithm to pack medium items
- ▶ Apply the method recursively to pack items of S_{i+1} in the remaining space (**do not use a simple algorithm**)

(3) Packing iteratively

Setting

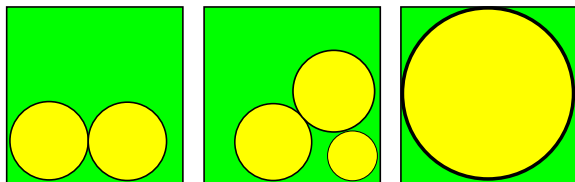
- ▶ Items of S_j are packed in bins $w_j \times (1 + \gamma)h_j$:
- ▶ There are no items of intermediate sizes:
- ▶ Bins $w_{j+1} \times (1 + \gamma)h_{j+1}$ are much smaller than circles in S_j
- ▶ Circles in S_{j+1} are much smaller than bins $w_{j+1} \times (1 + \gamma)h_{j+1}$



Improved approach

Packing recursively

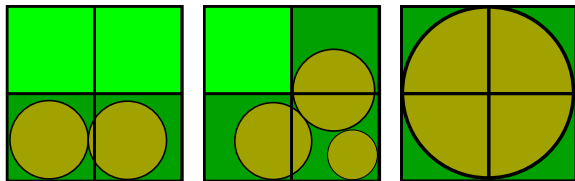
- ▶ Split the free space left by large items in small **sub-bins**
- ▶ Pack the remaining items **recursively**



Improved approach

Packing recursively

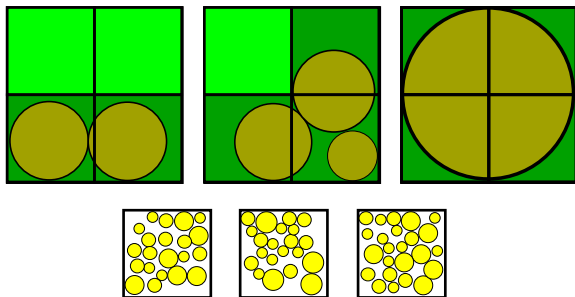
- ▶ Split the free space left by large items in small **sub-bins**
- ▶ Pack the remaining items **recursively**



Improved approach

Packing recursively

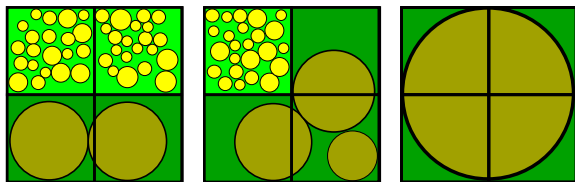
- ▶ Split the free space left by large items in small **sub-bins**
- ▶ Pack the remaining items **recursively**



Improved approach

Packing recursively

- ▶ Split the free space left by large items in small **sub-bins**
- ▶ Pack the remaining items **recursively**



Large, small, and medium items

- ▶ Make groups where radii between consecutive **decrease** $\times \varepsilon^2$
- ▶ Let $H_j = G_j \cup G_{j+r} \cup G_{j+2r} \cup \dots$, for $j \leq r-1$ and $r = \lceil 1/\varepsilon \rceil + 1$
- ▶ Choose H_t s.t. $\text{Area}(H_t) \leq \varepsilon \text{Area}(L)$ and remove H_t
- ▶ Let S_0, S_1, \dots union of consecutive groups and pack iteratively

	H_{t+1}	H_0	H_{t-1}	H_t
S_0 :		G_0, \dots	G_{t-1}	G_t
S_1 :	$G_{t+1},$	$G_{t+2},$	$\dots G_{t+r-1}$	G_{t+r}
S_2 :	$G_{t+r+1},$	$G_{t+r+2},$	$\dots G_{t+2r-1}$	G_{t+2r}
S_3 :	$G_{t+2r+1},$	$G_{t+2r+2},$	$\dots G_{t+3r-1}$	G_{t+3r}
		\vdots		\vdots

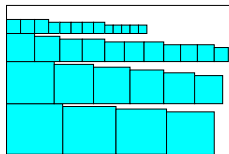
H_t are the **medium** items and are packed separately!

Iter. #0: **large circles:** S_0 **small circles:** S_1, S_2, S_3, \dots

(1) Packing medium items

Use algorithm NFDH^b

- ▶ Wrap medium circles in their bounding box
- ▶ Use NFDH to pack boxes in unit squares



Analysis

- ▶ $\text{Area}(H_t) \leq \varepsilon \text{Area}(L)$
- ▶ NFDH^b guarantees occupation of $\frac{\pi}{4} \frac{1}{4} \geq \frac{1}{12}$

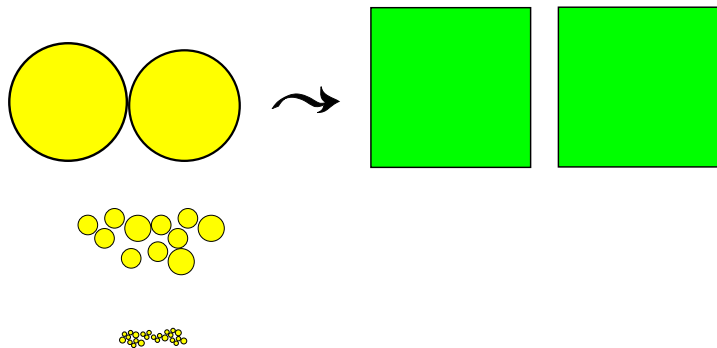
$$\text{NFDH}^b(H_t) \leq 12\varepsilon \text{Area}(L) + 1$$

(2) Packing sets S_j 's

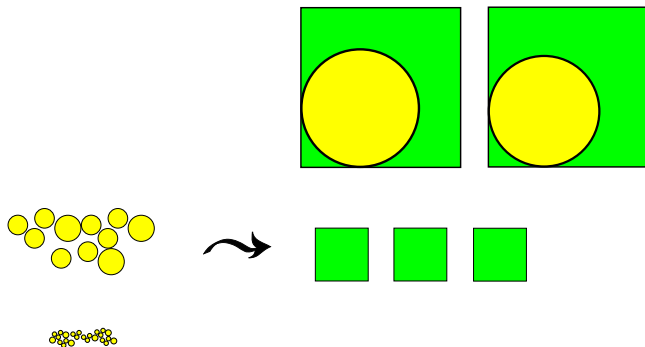
Use PTAS (when items are at least a constant) to pack each S_j

- ▶ Item sizes decrease exponentially but also their corresponding bins
- ▶ Circles radii in any S_j are at least some δ , relative to its bin
- ▶ For each level j , we can maintain constant number of empty bins

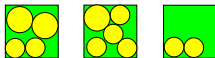
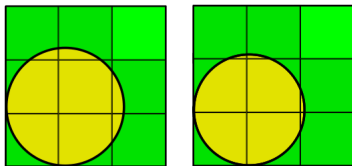
Running example



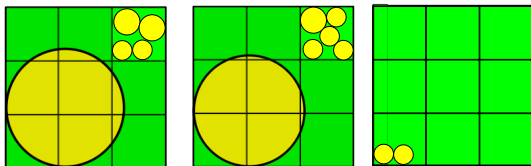
Running example



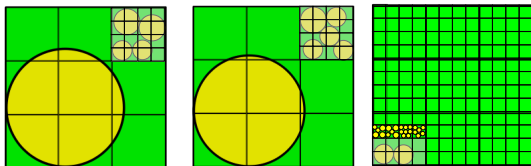
Running example



Running example



Running example



Comparing to the optimum

The generated solution is “almost” optimal, except that it is constrained in two ways.

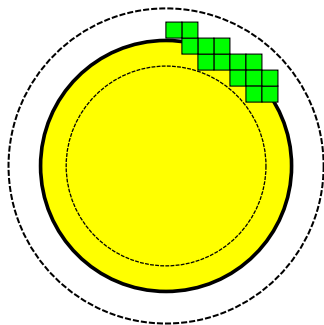
Constraints

- A. Sub-bins that partially intersect a larger circle is not used
- B. Circles of S_j do not intersect the grid of sides $w_j \times (1 + \gamma)h_j$

To calculate the cost of the generated solution, we **modify an optimal solution**, and bound the wasted space.

(A) Bins that intersect larger circles

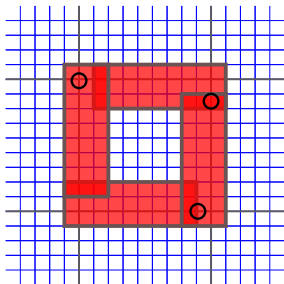
- ▶ Bins are **much smaller** than the circles with partial intersection



$$\text{wasted space} \leq O(\epsilon)\text{Area}(L) \leq O(\epsilon)\text{OPT}$$

(B) Circles that intersect grid lines

- ▶ circles with intersection not respect bins are much smaller than the bins
- ▶ we repack the circles in free space respecting the grid



We do not use a small fraction of bins of an optimal solution.

$$\text{additional space} \leq O(\epsilon)\text{OPT}$$

Summing everything

An APTAS for circle bin packing (with resource augmentation)

$$\begin{aligned}
 \text{SOL} &\leq [\text{cost of medium items}] + [\text{cost of modified optimal solution}] \\
 &\leq \lceil O(\varepsilon)\text{OPT} \rceil + \lceil \text{OPT} + O(\varepsilon)\text{OPT} + O(\varepsilon)\text{OPT} \rceil \\
 &\leq (1 + O(\varepsilon))\text{OPT} + 2
 \end{aligned}$$

Corollary

- ▶ An APTAS for circle strip packing.

Summary

Result

- ▶ First polynomial-time approximation (scheme) for circle bin packing and circle strip packing

Extensions

- ▶ May be generalized to other kinds of items: spheres, ellipsoids, etc.
- ▶ Bins of different shapes may also be considered

Open problem

- ▶ Is it possible to always obtain a rational solution from the packing decision problem? (*and thus avoid using resource augmentation*)