The regularity method and blow-up lemmas for sparse graphs

Y. Kohayakawa (São Paulo)

SPSAS Algorithms, Combinatorics and Optimization

University of São Paulo

July 2016

Partially supported CAPES/DAAD (430/15), CNPq (459335/2014-6, 310974/2013-5), and FAPESP (2013/07699-0, 2013/03447-6)

Course outline

1. **Lecture I.** Introduction to the regularity method
2. **Lecture II.** The blow-up lemma
3. **Lecture III.** The sparse case: small subgraphs
4. **Lecture IV.** The sparse case: large subgraphs

Lecture IV. The sparse case (large subgraphs)

Aim of talk

- ▷ Shall only give some idea (shan't be ambitious)
- ▷ **Joint work with**
  - ○ P. Allen (London),
  - ○ J. Böttcher (London),
  - ○ H. Hàn (Santiago),
  - ○ Y. Person (Frankfurt).
- ▷ Preprint available on the school's webpage

Outline of talk

1. (Lecture I & II) The regularity method; the blow-up lemma
2. (Lecture III) The regularity method in the sparse setting
3. Applications of the blow-up lemmas in the sparse setting
4. Inheritance of regularity
5. Statement of the sparse blow-up lemmas

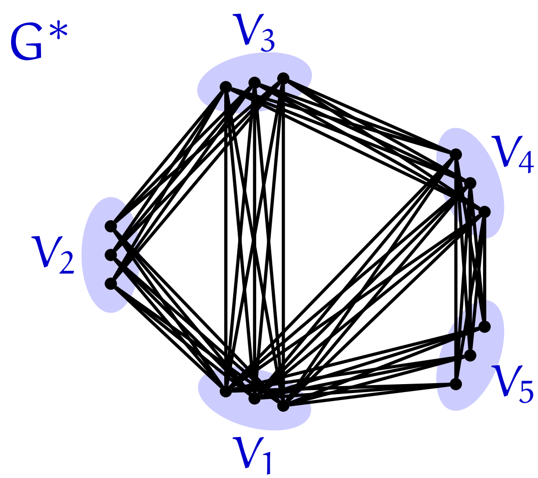
Outline of talk

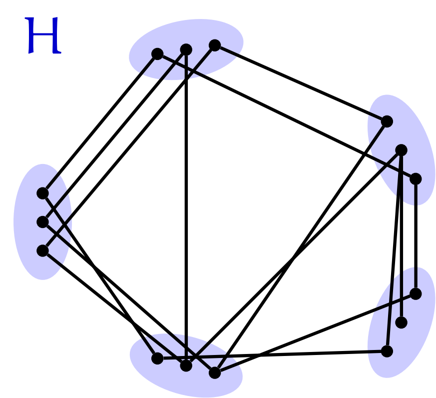
1. (Lecture I & II) The regularity method; the blow-up lemma
2. (Lecture III) The regularity method in the sparse setting
3. Applications of the blow-up lemmas in the sparse setting
4. Inheritance of regularity
5. Statement of the sparse blow-up lemmas

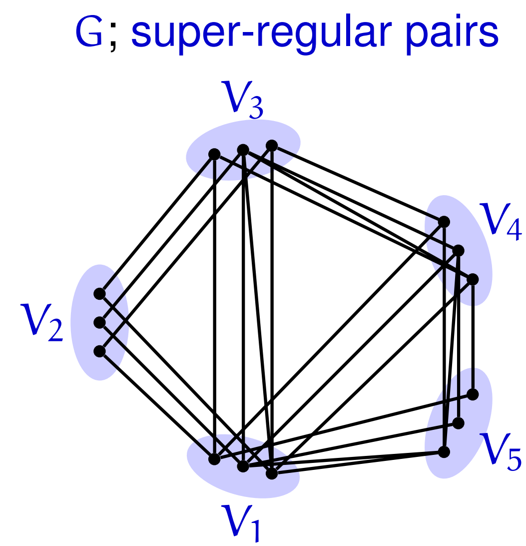
The Blow-up Lemma

Theorem 1 (The Blow-up Lemma (Komlós, Sárközy & Szemerédi '97))**.** *For every* $\delta > 0$*,* Δ*,* $r \in \mathbb{N}$ *there is* $\varepsilon > 0$ *such that the following holds. Let* $G^* = (V_1, \ldots, V_r; E^*)$ *and* $G = (V_1, \ldots, V_r; E)$ *be two graphs and let* $R \subset \binom{[r]}{2}$ *be such that* (V_i, V_j) *is a complete bipartite graph in* G^* *and* (V_i, V_j) *is an* (ε, δ)*-super-regular graph in* G *whenever* $ij \in R$*. If* H *with* $\Delta(H) \leq \Delta$ *can be embedded into* G^* *then it can also be embedded into* G*.*

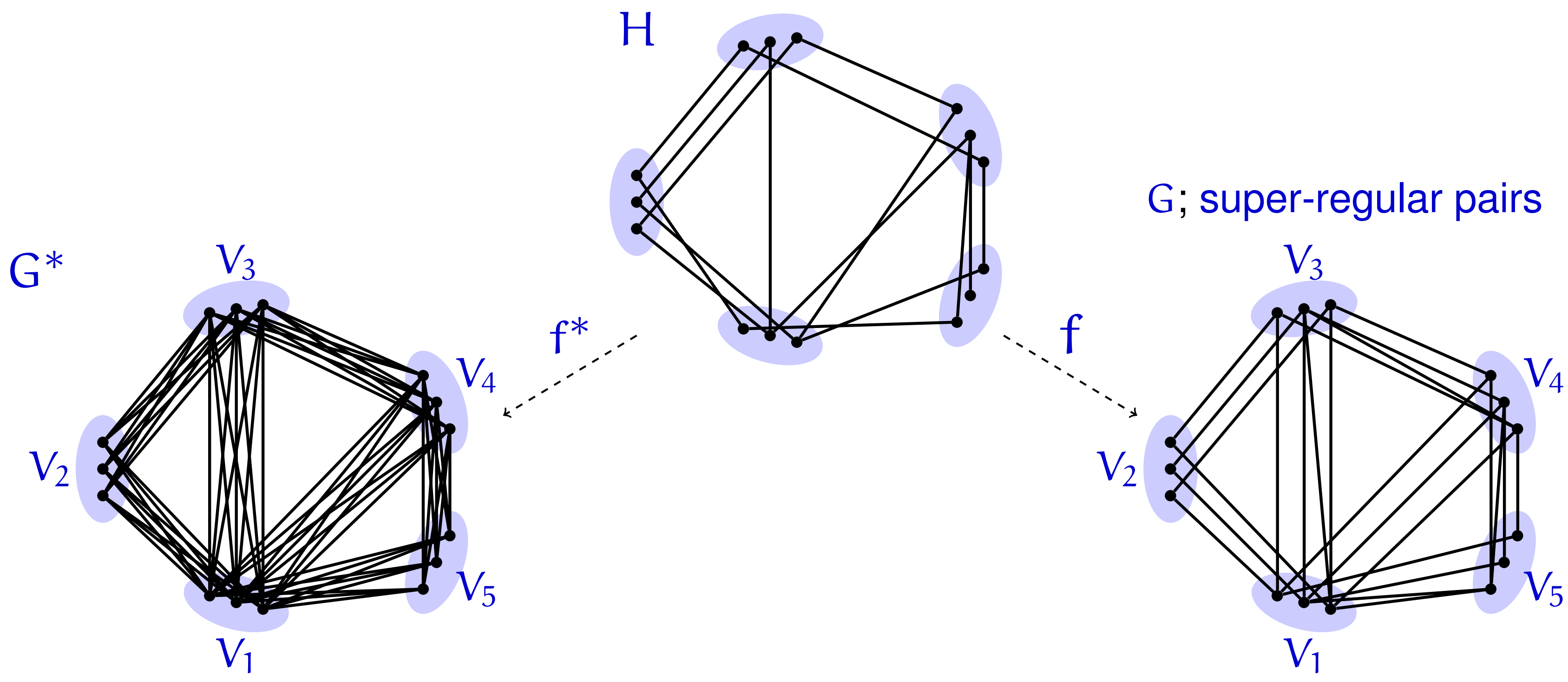
The Blow-up Lemma







The Blow-up Lemma

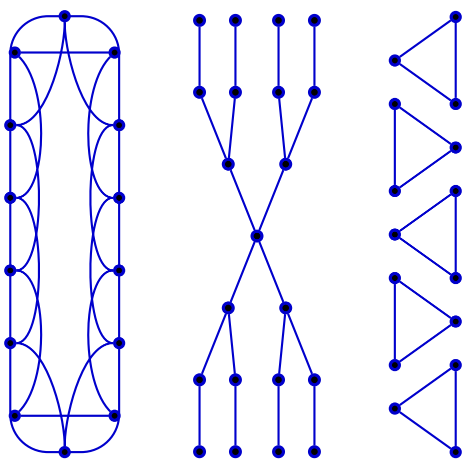


If f^* exists, the f exists.

Applications of the Blow-up Lemma

Do graphs with sufficiently high minimum degree contain ...

- ▷ Pósa & Seymour '74: r-th powers of Hamiltonian cycles?
- ▷ Alon & Yuster '96: F-factors (for F fixed)?



Answer: Yes!

- ▷ Invented by Komlós, Sárközy & Szemerédi

Spanning subgraphs with constant maximum degree

Outline of talk

1. (Lecture I & II) The regularity method; the blow-up lemma
2. (Lecture III) The regularity method in the sparse setting
3. Applications of the blow-up lemmas in the sparse setting
4. Inheritance of regularity
5. Statement of the sparse blow-up lemmas

The sparse setting

- ▷ So far: we have been concerned with dense graphs, i.e., graphs $G = G^n$ with $\Omega(n^2)$ edges.
- ▷ *Sparse*: graphs with $o(n^2)$ edges.
- ▷ Densities $d(U, U') = e(U, U')/|U||U'|$ in such sparse graphs vanish as $n \to \infty$. Therefore, the regularity condition

$$\forall U' \subset U,\ W' \subset W \text{ with } |U'|/|U|,\ |W'|/|W| \geq \varepsilon$$
$$\text{we have } |d(U', W') - d(U, W)| \leq \varepsilon$$

  becomes trivial.
- ▷ Easy fix: normalize the density by dividing by $p = p(n) = e(G)\binom{n}{2}^{-1}$.

Szemerédi's regularity lemma, sparse version

Definition 2. $B = (U, W; E)$ *is* $(\varepsilon, d; p)$*-regular if for all* $U' \subset U$ *and* $W' \subset W$ *with* $|U'|/|U|$, $|W'|/|W| \geq \varepsilon$, *we have* $|d_p(U', W') - d| \leq \varepsilon$, *where*

$$d_p(U', W') = \frac{e(U', W')}{p|U'||W'|}.$$

Typical applications: B is a subgraph of the random graph $G(n, p)$ or of a pseudorandom graph of some density p, with $p \to 0$.

Ambient graph: we shall usually have a *sparse*, *ambient graph* Γ.

Szemerédi's regularity lemma, sparse version

Any graph with no 'dense patches' admits a Szemerédi partition with the above notion of ε-regularity.

Definition 3. *Let us say* $G = (V, E)$ *is locally* (η, b)*-bounded if for all* $U \subset V$ *with* $|U| \geq \eta|V|$*, we have*

$$\#\{\textit{edges within } U\} \leq b|E|\binom{|U|}{2}\binom{|V|}{2}^{-1}.$$

Szemerédi's regularity lemma, sparse version

Theorem 4 (The regularity lemma)**.** *For any* $\varepsilon > 0$, $t_0 \geq 1$, *and* b, *there exist* $\eta > 0$ *and* T_0 *such that any locally* (η, b)*-bounded graph* $G = G^n$ *admits a partition* $V = V_1 \cup \cdots \cup V_t$ *such that*

- ▷ $t_0 \leq t \leq T_0$,
- ▷ $|V_1| \leq \cdots \leq |V_t| \leq |V_1| + 1$,
- ▷ (V_i, V_j) *is* $(\varepsilon; p)$*-regular for at least* $(1 - \varepsilon)\binom{t}{2}$ *pairs* $ij \in \binom{t}{2}$, *where*

$$p = e(G)\binom{n}{2}^{-1}.$$

- ○ A structure theorem for locally bounded sparse graphs

Embedding lemma; sparse setting

- Thanks to the work of several researchers, we now know how to use the regularity lemma in the sparse setting.

  Contributors: Balogh, Conlon, Fox, Gerke, Gowers, K., Łuczak, Morris, Rödl, Samotij, Saxton, Schacht, Steger, Thomason, Zhao, among others.

- Typical applications: when $G \subset \Gamma$, and $\Gamma = G(n, p)$ or Γ is a strongly pseudorandom graph.

- More precisely: we know suitable embedding lemmas for graphs $H = H^h$ with $h = O(1)$.

- How about *large* H? E.g., $h = n$?

Sparse blow-up lemmas

- ▷ **Main result:** suitable blow-up lemmas exist in the sparse setting.
- ▷ Statements are technical: we'll discuss some applications first.

Outline of talk

1. (Lecture I & II) The regularity method; the blow-up lemma
2. (Lecture III) The regularity method in the sparse setting
3. Applications of the blow-up lemmas in the sparse setting
4. Inheritance of regularity
5. Statement of the sparse blow-up lemmas

Sparse blow-up applications

1. Universality results for random and pseudorandom graphs
2. Partition-universality results for random and pseudorandom graphs
3. Resilience and robustness of low-bandwidth graphs
4. Maker-Breaker games

Universality

- $\triangleright$ $\mathcal{H}$: a set of graphs
- $\triangleright$ A graph G is $\mathcal{H}$-universal if $H \subset G$ for each $H \in \mathcal{H}$.

A universality result (random graphs)

$\mathcal{H}(n, d, \Delta)$: class of graphs $H = H^n$ with $\Delta(H) = \Delta$ and degeneracy d

Theorem 5. *For every* d *and* $\Delta \in \mathbb{N}$ *there is* C *such that if*

$$p \geq C \left(\frac{\log n}{n}\right)^{1/(2d+1)},$$

then the r.g. $G(n, p)$ *is a.a.s.* $\mathcal{H}(n, d, \Delta)$*-universal.*

Theorem 6. *For every* d*,* $\Delta \in \mathbb{N}$ *and* $\gamma > 0$ *there is* C *such that if*

$$p \geq C \left(\frac{\log n}{n}\right)^{1/2d},$$

then the r.g. $G((1+\gamma)n, p)$ *is a.a.s.* $\mathcal{H}(n, d, \Delta)$*-universal.*

Remarks

- ▷ Strengthen results of Dellamonica, K., Rödl and Ruciński (2015) and Kim and Lee (2013+), Ferber, Nenadov and Peter (2013+). [For $d = 1$, Montgomery can do $p = \widetilde{O}(1/n)$.]
- ▷ One of the forms of the blow-up lemma is tailored to handle graphs with low degeneracy.

A universality result (pseudorandom graphs)

Pseudorandom concept we shall use: G is (p, β)-*bi-jumbled* if, for all U, $W \subset V(G)$, we have

$$\Big| e(U, W) - p|U||W| \Big| \leq \beta\sqrt{|U||W|}.$$

Thomason (1987): (p, β)-jumbled graphs

Remarks:

(*i*) graph discrepancy results imply that if $G = G^n$ is (p, β)-bi-jumbled, then $\beta \geq \sqrt{np(1-p)/80}$ as long as $p(1-p) \geq 1/n$.

(*ii*) (n, d, λ)-graphs are $(d/n, \lambda)$-bi-jumbled.

(*iii*) Shall just say *jumbled* for *bi-jumbled*.

A universality result (pseudorandom graphs)

$\mathcal{H}(n, \Delta)$: class of graphs $H = H^n$ with $\Delta(H) = \Delta$

Theorem 7. *For every* $\Delta \geq 2$ *there is* $c > 0$ *such that, for any* $p > 0$*, if* $\beta \leq cp^{3\Delta/2+1/2}n$*, then any* n*-vertex* (p, β)*-jumbled graph* G *with* $\delta(G) \geq \frac{1}{2}pn$ *is* $\mathcal{H}(n, \Delta)$*-universal.*

A partition-universality result (random graphs)

G is *r-partition universal for $\mathcal{H}$* if in any r-colouring of $E(G)$ there is a colour class which is $\mathcal{H}$-universal.

$\mathcal{H}(n, d, \Delta)$: class of graphs $H = H^n$ with $\Delta(H) = \Delta$ and degeneracy d

Theorem 8. *For every r, d, and $\Delta \in \mathbb{N}$, there is C such that if*

$$p \geq C \left(\frac{\log n}{n}\right)^{1/2d},$$

then a.a.s. $G(Cn, p)$ is r-partition universal for $\mathcal{H}(n, d, \Delta)$.

- ▷ Strengthens a result of K., Rödl, Schacht and Szemerédi (2011).

A partition-universality result (pseudorandom graphs)

$\mathcal{H}(n, \Delta)$: class of graphs $H = H^n$ with $\Delta(H) = \Delta$

Theorem 9. *For every r and $\Delta \in \mathbb{N}$ there is $c > 0$ such that if $p > 0$ and G is an n/c-vertex graph that is $\left(p, p^{3\Delta/2+1/2}n\right)$-jumbled, then G is r-partition universal for $\mathcal{H}(n, \Delta)$.*

▷ Answers a question of K., Rödl, Schacht and Szemerédi (2011).

The bandwidth of a graph

$H = H^n$ has bandwidth $\mathrm{bw}(H)$ at most b if there is an ordering $x_1, \ldots, x_n$ of the vertices of H such that every edge $x_i x_j$ of H is such that $|j - i| \le b$.

Naturally,

$$\mathrm{bw}(H) = \min b,$$

with b as above.

The bandwidth theorem

Theorem 10 (Böttcher, Schacht and Taraz (2009))**.** *For every* $\gamma > 0$ *and* Δ *there is* $\beta > 0$ *such that for all sufficiently large* n *the following holds. If* $H = H^n$ *has* $\Delta(H) \leq \Delta$ *and* $\mathrm{bw}(H) \leq \beta n$*, and* $G = G^n$ *is such that*

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)} + \gamma\right) n,$$

then $H \subset G$*.*

- Conjectured by Bollobás and Komlós.
- Generalizes, up to a small error term, several results dealing with spanning graphs in extremal graph theory.

Resilience of low-bandwidth graphs (approximate form)

Theorem 11. *For every* $\gamma > 0$ *and* Δ *there is* $\beta > 0$ *and* C *such that if*

$$p \geq C\left(\frac{\log n}{n}\right)^{1/\Delta},$$

then a.a.s. $\Gamma = G(n, p)$ *has the following property. If* H *is any* $(1-\gamma)n$*-vertex graph with* $\Delta(H) \leq \Delta$ *and* $\mathrm{bw}(H) \leq \beta n$*, and* $G = G^n \subset \Gamma$ *is such that*

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)} + \gamma\right)pn,$$

then $H \subset G$*.*

Remarks

- ▷ Forthcoming result of Allen, Böttcher, Ehrenmüller and Taraz: the case in which $|V(H)| = n - Cp^{-2}$ if $p \leq 1/\log n$ (which is optimal) and even $H = H^n$ for some H.
- ▷ Previous results by Böttcher, K. and Taraz (2013) and Huang, Lee and Sudakov (2012).

Robustness of the Bandwidth Theorem

▷ **Robustness:** proposed by Krivelevich, Lee and Sudakov (2014), who proved that Pósa's result on the Hamiltonicity of $G(n, p)$ with $p = C(\log n)/n$ is very *robust*, in the sense that random subgraphs $G_p = G_p^n$ of *Dirac graphs* are a.a.s. Hamiltonian for $p = C'(\log n)/n$. [Dirac graph: $G = G^n$ with $\delta(G) \geq n/2$].

Robustness of the Bandwidth Theorem

If G is a graph and $0 \le p \le 1$, then G_p is random spanning subgraph of G in which each $e \in E(G)$ is included in G_p with probability p, independently of all other edges.

Theorem 12. *For every $\gamma > 0$ and $\Delta \ge 2$ there are $\beta > 0$ and C such that if $p \ge C\left(\frac{\log n}{n}\right)^{1/\Delta}$, then the following holds. If $H = H^n$ is a graph with $\Delta(H) \le \Delta$ and $\mathrm{bw}(H) \le \beta n$, and G is any n-vertex graph with*

$$\delta(G) \ge \left(1 - \frac{1}{\chi(H)} + \gamma\right) n,$$

then a.a.s. $H \subset G_p$.

Make-Breaker games

Maker-Breaker H-game on K^n with bias b: Maker and Breaker take turns to colour the edges of K^n red and blue. In each turn, Maker colours one edge, while Breaker colours b edges (and edges may not be recoloured). Maker's aim: create a red copy of H; Breaker's aim: prevent Maker from doing so.

$\mathcal{H}$-universality game: Maker's graph contains simultaneously each $H \in \mathcal{H}$.

Make-Breaker games

Theorem 13. *For every* $d, \Delta \in \mathbb{N}$ *and* $\delta > 0$ *there is* $c > 0$ *with the following properties.*

(*i*) *If* $b \leq c\left(\frac{n}{\log n}\right)^{1/\Delta}$ *then Maker wins the* $\mathcal{H}(n, \Delta)$*-universality game on* $K^{(1+\delta)n}$ *with bias* b*.*

(*ii*) *If* $b \leq c\left(\frac{n}{\log n}\right)^{1/2d}$ *then Maker wins the* $\mathcal{H}(n, d, \Delta)$*-universality game on* $K^{(1+\delta)n}$ *with bias* b*.*

Outline of talk

1. (Lecture I & II) The regularity method; the blow-up lemma
2. (Lecture III) The regularity method in the sparse setting
3. Applications of the blow-up lemmas in the sparse setting
4. Inheritance of regularity
5. Statement of the sparse blow-up lemmas

An ingredient: inheritance of density and regularity

- ▷ Immediate inheritance
- ▷ Inheritance by very small sets

Immediate inheritance; dense case

▷ Suppose have an ε-regular triple $G = (X, U, W; E)$, with all pairs with density d.

Fact 14. *Suppose $d \gg \varepsilon$. Then typical vertices $x \in X$ are such that*

$$(N_G(x) \cap U, N_G(x) \cap W) \tag{1}$$

has density $d(U, W) \pm \varepsilon$. Moreover, the pair is, say, $2\varepsilon/d$-regular.

In particular, (1) induces an edge, and hence $K^3 \subset G$. Even have many such edges, and hence many such K^3. The regularity of pairs of the form (1) is useful in inductive arguments.

Inheritance

Question 15. *Suppose* $B = (U, W; E)$ *is an* ε*-regular pair of density* $d = |E|/|U||W|$*. Suppose* $W' \subset W$ *and* $U' \subset U$*. What can we say about the density and the regularity of* $B[U', W']$*?*

- **Inheritance:** positive results in this direction.
- Immediate if can take

$$\varepsilon \ll |U'|/|U|,\ |W'|/|W|. \tag{2}$$

- In several applications, can take (2).
- What if $|U'| \ll \varepsilon|U|$ or $|W'| \ll \varepsilon|W|$?

▷ Often, U' and W' are neighbourhoods (or intersections of neighbourhoods). If the graphs are **sparse**, then certainly (2) fails.

Inheritance results

Even in the case $|U'|/|U|$, $|W'|/|W| \ll \varepsilon$,

- $\triangleright$ there are inheritance results for subgraphs of r.gs: $B = (U, W; E) \subset \Gamma = G(n, p)$, and
- $\triangleright$ there are inheritance results for subgraphs of (p, β)-jumbled graphs Γ: $B = (U, W; E) \subset \Gamma$,

as long as one considers U' and W' that are neighbourhoods or joint neighbourhoods of vertices in Γ.

Random case

- **Dense/classical case:** overwhelming majority of pairs (U', W') induce ε'-regular bipartite graphs $B[U', W']$, with $\varepsilon' \to 0$ as $\varepsilon \to 0$, as long as $|U'|$, $|W'| \geq C/d$, where $C = C(\varepsilon')$.
- Statement above is true even if $d \to 0$ (as long as we replace ε'-regularity by ε'-lower-regularity) [Gerke, K., Rödl, Steger 2007]

Definition 16. $B = (U, W; E)$ *is* $(\varepsilon, d; p)$*-lower-regular if for all* $U' \subset U$ *and* $W' \subset W$ *with* $|U'|/|U|$, $|W'|/|W| \geq \varepsilon$, *we have*

$$d_p(U', W') = \frac{e(U', W')}{p|U'||W'|} \geq d - \varepsilon.$$

Inheritance for subgraphs of $G(n, p)$ **[SKIP]**

Set-up 17. $J = J^{3m}$ *is a graph with vertex classes* X, U, *and* W, *all of cardinality* m; *the graphs* $J[X, U]$, $J[X, W]$, *and* $J[U, W]$ *are* $(\varepsilon, d; p)$*-lower-regular, and* $J = J[X, U] \cup J[X, W] \cup J[U, W]$.

Definition 18. J *is* ε'*-good if* $\exists X' \subset X$ *with* $|X'| \geq (1 - \varepsilon')m$ *such that* $\forall x' \in X'$ *we have that* $J[N(x') \cap U, N(x') \cap W]$ *is* $(\varepsilon', d; p)$*-regular.*

Inheritance for subgraphs of $G(n, p)$

Theorem 19 (Gerke, K., Rödl, Steger 2007). *For all* ε' *and* $d > 0$, *there is* $\varepsilon > 0$ *such that if* $p^2 m \gg 1$ *and* $pm \gg \log n$, *then*

$$\mathbb{P}(\exists J^{3m} \subset G(n, p)\ \varepsilon'\text{-}bad) = o(1).$$

- Similar statement holds for the setting in which we are concerned with $J^{4m} = J[X, U] \cup J[U, W] \cup J[W, Y]$ and pairs $(x, y) \in X \times Y$.

Inheritance for subgraphs of (p, β)-jumbled graphs

(*i*) Recall Set-up 17.

(*ii*) Suppose $J = J^{3m} \subset \Gamma$, with Γ $(p, \gamma p^d n)$-jumbled, for some $d \geq 1$.

(*iii*) Furthermore, suppose $m = \Omega(p^f n)$ and $d \geq f + 3$.

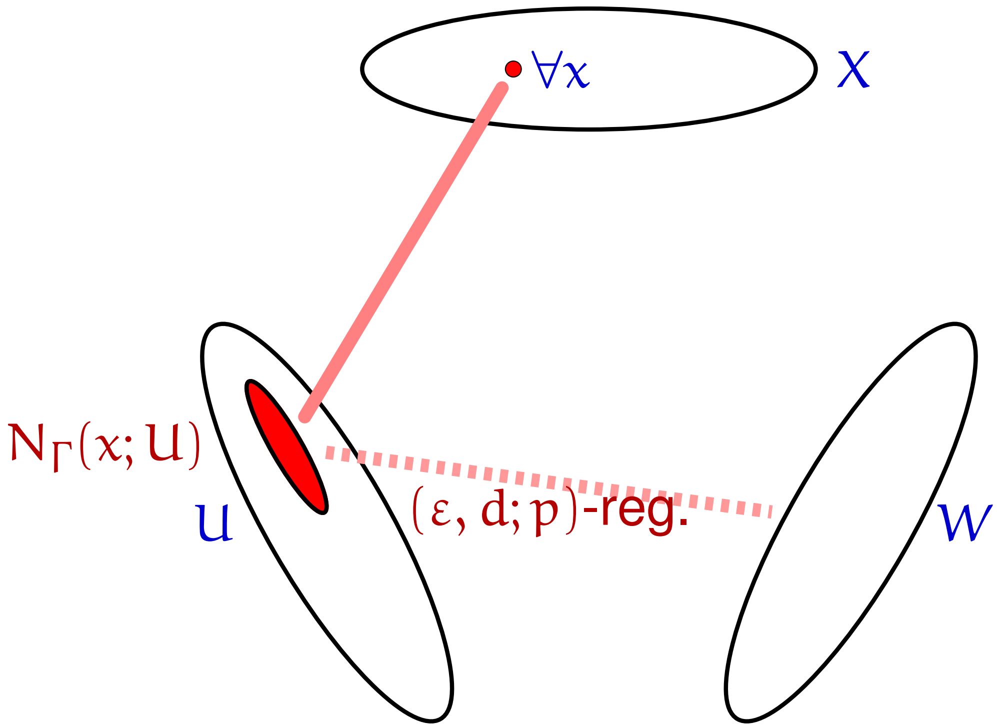
(*iv*) (Recall) J is ε'-good if $\exists X' \subset X$ with $|X'| \geq (1 - \varepsilon')m$ such that $\forall x' \in X'$ we have that $J[N(x') \cap U, N(x') \cap W]$ is $(\varepsilon', d; p)$-regular.

Theorem 20 (K., Rödl, Schacht & Skokan (2010)). *For all ε' and $d > 0$, there are ε and $\gamma > 0$ such that if (ii) and (iii) hold, then J is ε'-good.*

One-sided inheritance in Γ **[RESUME]**

Definition 21. *Let* (X, U, W) *be a triple of pairwise disjoint vertex sets in* $G \subset \Gamma$*. We say that* (X, U, W) *has one-sided* $(\varepsilon, d; p)$*-inheritance if for each* $x \in X$ *the pair* $(N_\Gamma(x; U), W)$ *is* $(\varepsilon, d; p)$*-regular in* G*.*

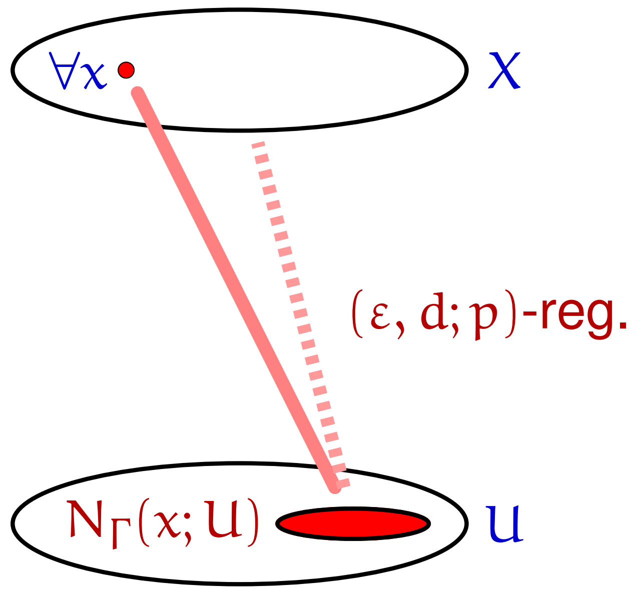
One-sided inheritance in Γ



One-sided inheritance in Γ

Definition 22. *Let (X, U) be a pair of vertex sets in $G \subset \Gamma$, with X and U disjoint. We say that (X, U) has one-sided $(\varepsilon, d; p)$-inheritance if for each $x \in X$ the pair $(N_\Gamma(x; U), X)$ is $(\varepsilon, d; p)$-regular in G.*

One-sided inheritance in Γ

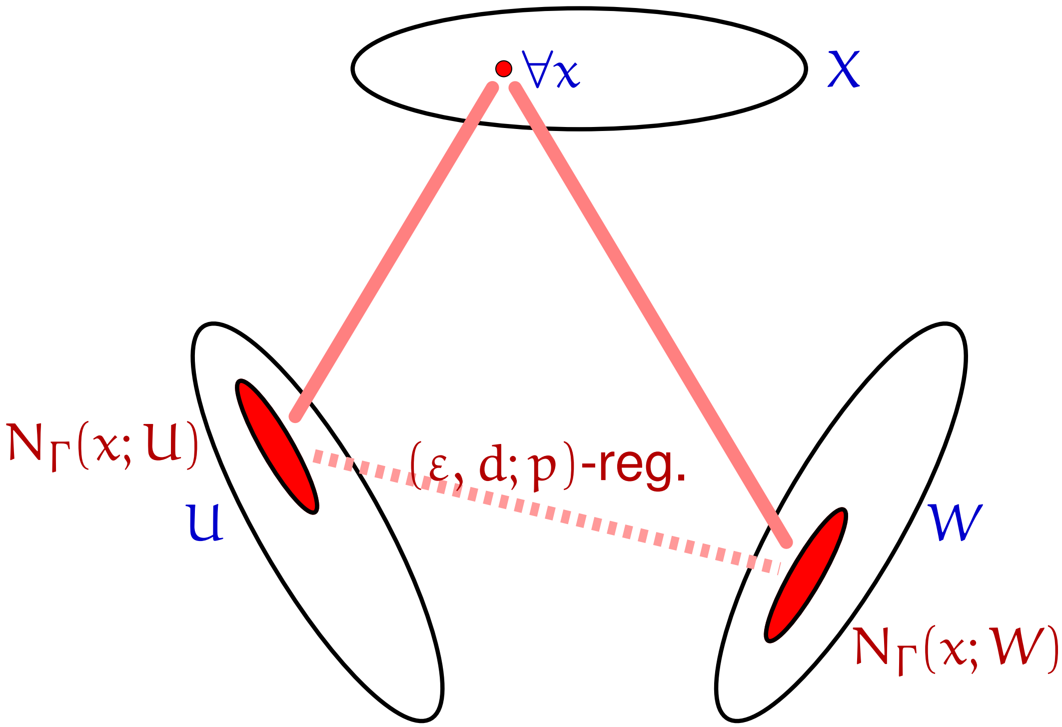


- ▷ To talk about one-sided inheritance for (X, U, W) and (X, U) at the same time, we allow the triple (X, U, X).

Two-sided inheritance in Γ

Definition 23. *Let* (X, U, W) *be a triple of pairwise disjoint vertex sets in* $G \subset \Gamma$*. We say that* (X, U, W) *has two-sided* $(\varepsilon, d; p)$*-inheritance if for each* $x \in X$ *the pair* $(N_\Gamma(x; U), N_\Gamma(x; W))$ *is* $(\varepsilon, d; p)$*-regular in* G*.*

Two-sided inheritance in Γ



Outline of talk

1. (Lecture I & II) The regularity method; the blow-up lemma
2. (Lecture III) The regularity method in the sparse setting
3. Applications of the blow-up lemmas in the sparse setting
4. Inheritance of regularity
5. Statement of the sparse blow-up lemmas

Sparse super-regularity

Recall: $B = (U, W; E)$ is $(\varepsilon, d; p)$-lower-regular if for all $U' \subset U$ and $W' \subset W$ with $|U'|/|U|$, $|W'|/|W| \geq \varepsilon$, we have $d_p(U', W') \geq d - \varepsilon$, where $d_p(U', W') = e(U', W')/p|U'||W'|$.

Definition 24 (sparse super-regularity in random graphs)**.** *A pair (U, W) in $G \subset \Gamma$ is called $(\varepsilon, d; p)$-super-regular in G if it is $(\varepsilon, d; p)$-lower-regular and, for every $u \in U$ and $w \in W$, we have*

$$\deg_G(u; W) > (d - \varepsilon) \max\{p|W|, \deg_\Gamma(u; W)/2\}.$$
$$\deg_G(w; U) > (d - \varepsilon) \max\{p|U|, \deg_\Gamma(w; U)/2\}.$$

A simplified blow-up lemma for random graphs

The objects:

- $\Gamma = G(n, p)$, where $p \geq C(\log n/n)^{1/\Delta}$,
- $G = G^n \subset \Gamma$ and $H = H^n$ with $\Delta(H) \leq \Delta$,
- H is r-partite, with r-partition $V(H) = X_1 \cup \cdots \cup X_r$. Moreover, $V(G) = V_1 \cup \cdots \cup V_r$ is an equitable partition with $|V_i| \geq |X_i|$ for all $i \in [r]$.

A simplified blow-up lemma for random graphs

Furthermore, we suppose that the following conditions hold:

(*i*) (V_i, V_j) is $(\varepsilon, d; p)$-super-regular in G for each i, $j \in [r]$ with $i \neq j$.

(*ii*) (V_i, V_j, V_k) has one-sided $(\varepsilon, d; p)$-inheritance for each i, j, $k \in [r]$ with $i \neq j$ and $j \neq k$.

(*iii*) (V_i, V_j, V_k) has two-sided $(\varepsilon, d; p)$-inheritance for each i, j, $k \in [r]$ with i, j and k all distinct.

Conclusion: Then H is a subgraph of G.

Theorem 25. *For all Δ, $r \in \mathbb{N}$ and $d > 0$ there exist $\varepsilon > 0$ and C such that if $p \geq C(\log n/n)^{1/\Delta}$, then the random graph $\Gamma = G(n, p)$ asymptotically almost surely satisfies the above.*

Full blow-up lemma for random graphs

Features include:

(*i*) Partition of $V(G)$ is allowed to be just approximately balanced.

(*ii*) Have two 'reduced graphs' R and R' to encode regular and super-regular pairs ($R' \subset R$).

(*iii*) Require two-sided regularity inheritance only where triangles of H need to be embedded.

(*iv*) The required regularity, i.e., the value of ε, does not depend on r (number of parts in the partitions of H and G), but only on $\Delta(R')$. (New for $p = 1$ also)

(*v*) Image restrictions are permitted.

Full blow-up lemma for random graphs, cont.

We have a specific version for embedding *degenerate* graphs H:

- E.g., for embedding $H = H^n$ with $\Delta(H) = \Delta$ and degeneracy D into suitably compatible $G = G^n \subset G(n, p)$ with

$$p \geq C_\Delta \left(\frac{\log n}{n}\right)^{2D+1}.$$

Full blow-up lemma for pseudorandom graphs

Rough statement:

- ▷ We are able to embed $H = H^n$ with $\Delta(H) = \Delta$ into suitably compatible $G^n \subset \Gamma$, when $\Gamma = \Gamma^n$ is a $(p, o(p^{(3\Delta+1)/2}n))$-jumbled graph.