# The regularity method and blow-up lemmas for sparse graphs

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July 2016

Partially supported CAPES/DAAD (430/15), CNPq (459335/2014-6, 310974/2013-5), and FAPESP (2013/07699-0, 2013/03447-6)

#### Course outline

- 1. Lecture I. Introduction to the regularity method
- 2. Lecture II. The blow-up lemma
- 3. Lecture III. The sparse case: small subgraphs
- 4. Lecture IV. The sparse case: large subgraphs

Lecture III. The sparse case (small subgraphs)

#### Motivation

- > Shall have a fairly long discussion on motivation.
- > Shall discuss rudiments of the regularity method for sparse graphs.

#### Restricted Ramsey theory

- $\triangleright$  We know  $K^6 \rightarrow K^3$ .
- ightharpoonup Are there G with  $K^6 \not\subset G$  such that  $G \to K^3$ ?
- > Yes:  $G = K^8 \setminus C^5 = C^3 \vee C^5$  (Graham 1968)
- $\triangleright$  Also  $G = (K^9 \setminus C^9) \vee K^1$  (Leader)

**Definition 1.** For positive integers r, k and ℓ, let

$$f(r,k,\ell) = min \, \big\{ n \colon \exists G = G^n \, \textit{such that} \, G \not\supset K^\ell \, \textit{and} \, G \to (K^k)_r \big\}.$$

 $\circ$  G  $\rightarrow$  (H)<sub>r</sub>: Ramsey property for r-colouring of the edges

## Restricted Ramsey theory

**Definition 2.** For positive integers r, k and ℓ, let

$$f(r,k,\ell) = min \left\{ n \colon \exists G = G^n \text{ such that } G \not\supset K^\ell \text{ and } G \to (K^k)_r \right\}.$$

 $\circ$  G  $\rightarrow$  (H)<sub>r</sub>: Ramsey property for r-colouring of the edges

**Question 3** (Erdős and Hajnal 1967). *Is*  $f(r, k, \ell) < \infty$  *for every positive* r, k *and*  $\ell$  *with*  $k < \ell$ ?

Answer. Yes: Folkman 1970, Nešetřil and Rödl 1976

**Numbers.** Graham: f(2,3,6) = 8. Nenov, Piwakowski, Radziszowski and Urbański: f(2,3,5) = 15. How about f(2,3,4)?

Erdős:  $f(2,3,4) \le 10^{10}$ ?

## Restricted Ramsey theory

Erdős:  $f(2,3,4) \le 10^{10}$ ?

- (i) Lu 2008:  $f(2,3,4) \le 9697$
- (ii) Dudek and Rödl 2008:  $f(2,3,4) \le 941$

Let us go back to the proof of  $f(r, k, \ell) < \infty$ .

Natural approach today. Investigate whether can use random graphs.

## The random graphs G(n, p)

ightharpoonup G(n,p): each element of  $\binom{[n]}{2}$  is present with probability p, independently of all others

Always interested in  $n \to \infty$ . Use the terms 'almost surely', 'almost every', 'almost always', etc to mean 'with probability  $\to 1$  as  $n \to \infty$ '.

- $\triangleright$  We are interested in the existence of mono $\chi$  subgraphs in **coloured** random graphs; i.e., properties of the form  $G(n,p) \to (H)_r$ .
- Also interested in existence of monochromatic subgraphs in 'dense' subgraphs of random graphs; i.e., statements of the form  $G(n, p) \to_{\eta} H$  (any η-fraction of the edges of G(n, p) contains a copy of H).

#### The results for $K^3$

**Theorem 4.** There is a large enough constant C such that if  $p \ge C/\sqrt{n}$ , then a.e. G(n,p) satisfies

$$G(n,p) \rightarrow (K^3)_2$$
.

**Theorem 5.** For any  $\eta > 0$ , there C such that if  $p \ge C/\sqrt{n}$  then a.e. G(n,p) satisfies

$$G(n,p) \rightarrow_{1/2+\eta} K^3$$
.

$$f(2,3,4) < \infty$$

- (i) Basic remark:  $G(n, C/\sqrt{n})$  a.s. contains  $O(p^6n^4) = O(n)$  copies of  $K^4$ .
- (ii) Two approaches:
  - Show that if we delete any O(n) edges from G(n,p) still have a Ramsey graph for  $K^3$ .
  - Show that the number of mono $\chi$  K<sup>3</sup> is large (it is actually  $(1/4 + o(1))p^3\binom{n}{3}$  a.s.) and that an edge does not belong to too many K<sup>3</sup>s.
- (iii) It follows that  $f(2,3,4) < \infty$ .

## The case p = 1 (skip/exercise)

**Theorem 6** (Goodman). The number of monochromatic triangles in any 2-colouring of  $K^n$  is  $\geq \binom{n}{3} - \frac{1}{8}n(n-1)^2 = (\frac{1}{4} + o(1))\binom{n}{3}$ .

*Proof.* Count the number N of 2-coloured cherries. Say have b(x) blue edges and r(x) red edges at vertex x. Have  $N = \sum_{x} b(x)r(x) \le n(n-1)^2/4$  (use b(x) + r(x) = n - 1). The number of non-monochromatic triangles is N/2.

Clearly, there are 2-colourings with  $\leq \frac{1}{4} \binom{n}{3}$  monochromatic triangles!

## The case p = 1 (skip/exercise)

**Theorem 7.** Goodman implies Mantel:  $ex(n, K^3) \le n^2/4$ .

*Proof.* Suppose  $K^3 \not\subset G^n$  and  $e(G^n) > n^2/4$ , so that  $e(G^n) \ge n^2/4 + 3/4$ . Count the number N of pairs (e,T) where  $e \in E(G^n)$ ,  $T \in \binom{V(G^n)}{3}$  and  $e \subset T$ . Say there are  $t_i$  triples containing i edges of G. Then, by Theorem 6,

$$(n^2/4 + 3/4)(n-2) \le e(G^n)(n-2)$$
  
=  $N \le t_1 + 2t_2 \le 2(t_1 + t_2) \le n(n-1)^2/4$ ,

which is a contradiction for  $n \geq 4$ .

#### Theorems 4 and 5 from Goodman's counting (skip/exercise)

**Theorem 8.** For every  $\varepsilon > 0$  there is C such that if  $\mathfrak{p} \geq C/\sqrt{\mathfrak{n}}$ , then a.e. G(n,p) is such that any 2-colouring of its edges contains at least  $(1/4 - \varepsilon)p^3\binom{n}{3}$  monochromatic  $K^3$ .

*Proof.* Exercise!

Exercise 9. Derive Theorem 5 from Theorem 8.

**Exercise 10.** Show that the threshold for Theorem 4 is indeed  $1/\sqrt{n}$ .

**Exercise 11.** Show that the threshold for Theorem 5 is indeed  $1/\sqrt{n}$ .

#### How about $f(r, k, \ell)$ ?

We are interested in the *threshold* for the Ramsey property

$$G(n,p) \to (K^k)_r. \tag{1}$$

**Threshold:** the 'smallest' order of growth for p = p(n) such that (1) is guaranteed to hold a.s.

This was actually a very natural problem in the theory of random graphs, and was solved by Rödl and Ruciński (1995).

#### The Rödl-Ruciński theorem

**Definition 12** (2-density and  $m_2(H)$ ). The 2-density  $d_2(H)$  of a graph H with |V(H)| > 2 is

$$\frac{|E(H)| - 1}{|V(H)| - 2}.$$
 (2)

For  $H = K^1$  and  $2K^1$  let  $d_2(H) = 0$ ; set  $d_2(K^2) = 1/2$ . Let

$$m_2(H) = \max\{d_2(J): J \subset H, |V(J)| > 0\}.$$
 (3)

**Exercise 13.** Consider, say,  $H = K^h$ . Show that if  $p \ll n^{-1/m_2(H)}$ , then a.s.  $\#\{H \hookrightarrow G(n,p)\} \ll e(G(n,p))$ . On the other hand, if  $p \gg n^{-1/m_2(H)}$ , then a.s.  $\#\{H \hookrightarrow G(n,p)\} \gg e(G(n,p))$ .

#### The Rödl–Ruciński theorem

**Theorem 14.** Let H be a graph containing at least a cycle and let  $r \ge 2$  be an integer. Then there exist constants c and C such that

$$\lim_{n\to\infty} \mathbb{P}(G(n,p)\to (H)_r) = \begin{cases} 0 & \text{if } p \leq cn^{-1/m_2(H)} \\ 1 & \text{if } p \geq Cn^{-1/m_2(H)}. \end{cases} \tag{4}$$

o In particular,  $p_0 = p_0(n) = n^{-1/m_2(H)}$  is the threshold for the property  $G(n,p) \to (H)_r$ .

#### The Rödl-Ruciński theorem

**Exercise 15.** Prove that  $f(r, k, \ell) < \infty$  whenever  $k < \ell$ .

**Exercise 16.** Given a graph G, let  $\mathcal{H}_3(G)$  be the 3-uniform hypergraph whose hypervertices are the edges of G and the hyperedges are the edge sets of the triangles in G. Show that, for any integers  $\ell$  and r, there is a graph G satisfying  $G \to (K^3)_r$  such that  $\mathcal{H}_3(G)$  has girth  $\geq \ell$ .

#### Generalized Turán number:

$$ex(G, H) = max \{ |E(G')| : H \not\subset G' \subset G \}.$$
 (5)

$$\triangleright \operatorname{ex}(n, H) = \operatorname{ex}(K^n, H)$$

$$\triangleright \operatorname{ex}(Q^{\operatorname{d}}, C^4) = ?$$

$$\triangleright$$
 ex(G, K<sup>h</sup>) = ? for (n, d,  $\lambda$ )-graphs G

**Exercise 17.** Show that, for any G and H, we have

$$ex(G,H) \ge \left(1 - \frac{1}{\chi(H) - 1}\right)e(G). \tag{6}$$

 $\triangleright$  Interested in knowing when this is sharp for G = G(n, p) (up to o(e(G)))

**Theorem 18** (K., Rödl and Schacht 2004). Let H be a graph with degeneracy d and suppose  $np^d \gg 1$ . Then, almost surely,

$$ex(G(n,p),H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)p\binom{n}{2}.$$
 (7)

**Conjecture 19.** For any graph H, if  $np^{m_2(H)} \to \infty$  then (7) holds almost surely.

Example: if  $H = K^k$ , have  $m_2(H) = (k+1)/2$ , but Theorem 18 supposes  $np^{1/(k-1)} \gg 1$ .

Conjecture 19 was proved in 2009/2010:

**Theorem 20** (Schacht 2016; Conlon and Gowers 2016). *Conjecture 19 holds.* 

- Quite different approaches:
  - Schacht: combinatorial, multi-round exposure method
  - Conlon and Gowers: analytic approach
  - General transference principles

## General transference principles

- Schacht and Conlon & Gowers obtained general transference results: results that allow one to transfer classical density results ("full environment") to random sparse subenvironments
- Applications: results in additive combinatorics; graph and hypergraph theorems

# The regularity method for sparse random graphs

- - Schacht and Conlon and Gowers bypassed the regularity approach.
- ▶ In short: the regularity method of Szemerédi works in sparse random environments.

#### Recall

- > Shall have a fairly long discussion on motivation.
- > Shall discuss rudiments of the regularity method for sparse graphs.

## Szemerédi's regularity lemma

- 1. Tool for identifying the quasirandom structure of deterministic graphs.
- 2. Works very well for large, dense graphs: n-vertex graphs with  $\geq cn^2$  edges,  $n \to \infty$ .
- 3. Variant for sparse graphs exists (sparse = with  $o(n^2)$  edges).
- 4. With the recent embedding lemmas, the method works nicely/easily when dealing with small subgraphs.

#### ε-regularity

**Definition 21** ( $\varepsilon$ -regular pair). G = (V, E) a graph;  $U, W \subset V$  non-empty and disjoint. Say (U, W) is  $\varepsilon$ -regular (in G) if

ho for all  $U' \subset U$ ,  $W' \subset W$  with  $|U'| \geq \varepsilon |U|$  and  $|W'| \geq \varepsilon |W|$ , we have

$$\left|\frac{|\mathsf{E}(\mathsf{U}',\mathsf{W}')|}{|\mathsf{U}'||\mathsf{W}'|} - \frac{|\mathsf{E}(\mathsf{U},\mathsf{W})|}{|\mathsf{U}||\mathsf{W}|}\right| \le \varepsilon. \tag{8}$$

#### ε-regularity

The pair (U, W) is  $\varepsilon$ -regular if

$$\left|\frac{|\mathsf{E}(\mathsf{U}',\mathsf{W}')|}{|\mathsf{U}'||\mathsf{W}'|} - \frac{|\mathsf{E}(\mathsf{U},\mathsf{W})|}{|\mathsf{U}||\mathsf{W}|}\right| \le \varepsilon. \tag{9}$$

Equivalently,

$$|\mathsf{E}(\mathsf{U}',\mathsf{W}')| = |\mathsf{U}'||\mathsf{W}'| \left(\frac{|\mathsf{E}(\mathsf{U},\mathsf{W})|}{|\mathsf{U}||\mathsf{W}|} \pm \varepsilon\right). \tag{10}$$

Clearly, not meaningful if

$$\frac{|\mathsf{E}(\mathsf{U},\mathsf{W})|}{|\mathsf{U}||\mathsf{W}|} \to 0 \tag{11}$$

and  $\varepsilon$  is fixed. (We think of G = (V, E) with  $n = |V| \to \infty$ .)

#### ε-regularity; multiplicative error version

Replace

$$|E(U', W')| = |U'||W'| \left(\frac{|E(U, W)|}{|U||W|} \pm \varepsilon\right)$$
 (12)

by

$$|E(U', W')| = (1 \pm \varepsilon)|E(U, W)| \frac{|U'||W'|}{|U||W|}.$$
 (13)

Altered condition becomes

 $\triangleright$  for all  $U' \subset U$ ,  $W' \subset W$  with  $|U'| \ge \varepsilon |U|$  and  $|W'| \ge \varepsilon |W|$ , we have

$$\left| |\mathsf{E}(\mathsf{U}', \mathsf{W}')| - |\mathsf{E}(\mathsf{U}, \mathsf{W})| \frac{|\mathsf{U}'||\mathsf{W}'|}{|\mathsf{U}||\mathsf{W}|} \right| \le \varepsilon |\mathsf{E}(\mathsf{U}, \mathsf{W})| \frac{|\mathsf{U}'||\mathsf{W}'|}{|\mathsf{U}||\mathsf{W}|}. \tag{14}$$

OK even if  $|E(U, W)|/|U||W| \rightarrow 0$ .

#### Szemerédi's regularity lemma, sparse version

Any graph with no 'dense patches' admits a Szemerédi partition with the new notion of  $\varepsilon$ -regularity.

**Definition 22** ( $(\eta, b)$ -bounded). Say G = (V, E) is  $(\eta, b)$ -bounded if for all  $U \subset V$  with  $|U| \ge \eta |V|$ , we have

$$\#\{edges \ within \ U\} \le b|E| {|U| \choose 2} {|V| \choose 2}^{-1}. \tag{15}$$

# Szemerédi's regularity lemma, sparse version

**Theorem 23** (The regularity lemma). For any  $\epsilon > 0$ ,  $t_0 \ge 1$ , and b, there exist  $\eta > 0$  and  $T_0$  such that any  $(\eta, b)$ -bounded graph G admits a partition  $V = V_1 \cup \cdots \cup V_t$  such that

- 1.  $|V_1| \leq \cdots \leq |V_t| \leq |V_1| + 1$
- 2.  $t_0 \le t \le T_0$
- 3. at least  $(1 \epsilon) {t \choose 2}$  pairs  $(V_i, V_j)$  (i < j) are  $\epsilon$ -regular.

Proof. Just follow Szemerédi's original proof.

# A counting lemma (simplest version)

**Setup.**  $G = (V_1, V_2, V_3; E)$  tripartite with

- 1.  $|V_i| = m$  for all i
- 2.  $(V_i, V_j)$   $\varepsilon$ -regular for all i < j
- 3.  $|E(V_i, V_j)| = \rho m^2$  for all i < j

**Notation:**  $G = G_3^{(\epsilon)}(m, \rho)$  [G is an  $\epsilon$ -regular triple with density  $\rho$ ]

 $\triangleright$  Wish to embed  $K^3$  with  $V(K^3) = \{x_1, x_2, x_3\}$  such that  $x_i$  is placed in  $V_i$ .

#### A counting lemma (simplest version)

#### Just like random:

**Lemma 24** (Counting Lemma; Embedding lemma).  $\forall \rho > 0, \ \delta > 0 \ \exists \epsilon > 0, \ m_0$ : if  $m \ge m_0$ , then

$$\left| \# \{ \mathsf{K}^3 \hookrightarrow \mathsf{G}_3^{(\varepsilon)}(\mathsf{m}, \rho) \} - \rho^3 \mathsf{m}^3 \right| \le \delta \mathsf{m}^3. \tag{16}$$

## Tough: counting Lemma is false if $\rho \to 0$

Fact 25. 
$$\forall \varepsilon > 0 \; \exists \rho > 0, \; m_0 \; \forall m \geq m_0 \; \exists G_3^{(\varepsilon)}(m, \rho) \; \textit{with}$$

$$K^3 \not\subset G_3^{(\varepsilon)}(m, \rho). \tag{17}$$

[cf. Lemma 24]

Exercise 26. Prove Fact 25.

**Hint 27.** Consider the random tripartite graph  $G(t, t, t; \delta/\sqrt{t})$  ( $t = t(\epsilon)$  and  $\delta = \delta(\epsilon)$  appropriate constants). Remove few edges to make it  $K^3$ -free. Blow up each vertex to an independent set of size n/t.

Change of focus (just for simplicity): from counting  $K^3 \subset G_3^{(\epsilon)}(\mathfrak{m},\rho)$  to existence of  $K^3 \subset G_3^{(\epsilon)}(\mathfrak{m},\rho)$ .

# Key observation

Counterexamples to the embedding lemma in the sparse setting do exist (Fact 25), but

are extremely rare.

## An asymptotic enumeration lemma

Lemma 28. 
$$\forall \beta > 0 \ \exists \varepsilon > 0, \ C > 0, \ m_0: \ \textit{if} \ T = \rho m^2 \ge C m^{3/2}, \ \textit{then}$$
 
$$\# \{G_3^{(\varepsilon)}(m,\rho) \not\supset K^3\} \le \beta^T {m^2 \choose T}^3. \tag{18}$$

Observe that  $\rho \geq C/\sqrt{m} \rightarrow 0$ .

#### Consequence for random graphs

Easy expectation calculations imply

ightrightarrow if  $p \gg 1/\sqrt{n}$ , then almost every G(n,p) is such that

$$\left(\mathsf{K}^3\text{-free }\mathsf{G}_3^{(\varepsilon)}(\mathsf{m},\mathsf{p})\right)\not\subset\mathsf{G}(\mathsf{n},\mathsf{p}),$$
 (19)

if (\*) mp  $\gg \log n$  and  $\rho \geq \alpha p$  for some fixed  $\alpha$ .

**Conclusion.** Recovered an 'embedding lemma' in the sparse setting, *for* subgraphs of random graphs.

**Corollary 29** (EL for subgraphs of r.gs). If  $p \gg 1/\sqrt{n}$  and (\*) holds, then almost every G(n,p) is such that if  $G_3^{(\epsilon)}(m,\rho) \subset G(n,p)$ , then

$$\exists \iota \colon \mathsf{K}^3 \hookrightarrow \mathsf{G}_3^{(\varepsilon)}(\mathsf{m}, \rho) \subset \mathsf{G}(\mathsf{n}, \mathsf{p}). \tag{20}$$

# The easy calculation

Recall  $T = \rho m^2$ ,  $\rho \ge \alpha p$ , and  $mp \gg \log n$ . Therefore

$$\begin{split} \mathbb{E}(\#\{K^3\text{-free }G_3^{(\epsilon)}(m,\rho)\hookrightarrow G(n,p)\}) &\leq n^{3m}o(1)^T{m^2\choose T}^3p^{3T}\\ &\leq n^{3m}\left(o(1)\frac{em^2}{T}\right)^{3T}p^{3T} \leq n^{3m}\left(o(1)\frac{em^2p}{T}\right)^{3T}\\ &= n^{3m}\left(o(1)\frac{em^2p}{\rho m^2}\right)^{3T} \leq n^{3m}\left(o(1)\frac{e}{\alpha}\right)^{3T}\\ &\leq e^{3m\log n}\left(o(1)\frac{e}{\alpha}\right)^{3\alpha m\log n} = o(1). \end{split}$$

### Superexponential bounds

Suppose we wish to prove a statement about all subgraphs of G(n, p).

- Too many such subgraphs: about  $2^{p\binom{n}{2}}$
- G(n,p) has no edges with probability  $(1-p)^{\binom{n}{2}} \ge \exp\{-2pn^2\}$ , if, say,  $p \le 1/2$ .
- Concentration inequalities won't do  $(2^{p\binom{n}{2}})$  vs  $e^{-2pn^2}$ .
- Bounds of the form

$$o(1)^{\mathsf{T}} \binom{\binom{\mathsf{m}}{2}}{\mathsf{T}} \tag{21}$$

for the cardinality of a family of 'undesirable subgraphs' U(m, T) do the job. Use of such bounds goes back to Füredi (1994).

# An embedding lemma for K<sup>3</sup> (sparse setting)

#### Scheme:

 $[T = \rho m^2]$ 

1. Proved an asymptotic enumeration lemma:

$$\#\{G_3^{(\varepsilon)}(m,\rho) \not\supset K^3\} = o(1)^T {m^2 \choose T}^3.$$
 (22)

- 2. Observed that this implies a.e.  $G(n, \omega/\sqrt{n})$  contains no  $K^3$ -free  $G_3^{(\epsilon)}(m, \rho)$  [any  $\omega = \omega(n) \to \infty$  as  $n \to \infty$ ].
- 3. Obtained a K³-embedding lemma for subgraphs of G(n,p), even when  $p = \omega/\sqrt{n}$ : for all  $G_3^{(\epsilon)}(m,\rho) \subset G(n,p)$ , have  $K^3 \subset G_3^{(\epsilon)}(m,\rho)$ .

#### General graphs H?

Suppose |V(H)| > 2. Recall

$$d_2(H) = \frac{|E(H)| - 1}{|V(H)| - 2}$$
 (23)

and

$$m_2(H) = \max\{d_2(J) \colon J \subset H\}. \tag{24}$$

**Theorem 30** (Balogh, Morris & Samotij 2015 and Saxton & Thomason 2015).  $\forall H, \beta > 0 \exists \epsilon > 0, C > 0, m_0$ : if  $T = \rho m^2 \geq C m^{2-1/m_2(H)}$ , then

$$\#\{G_{H}^{(\varepsilon)}(m,\rho) \not\supset H\} \le \beta^{\mathsf{T}} {m^2 \choose \mathsf{T}}^{e(\mathsf{H})}. \tag{25}$$

□ Can't miss: container lectures by Rob Morris next week!

#### Consequences of Theorem 30

- 1. The Rödl–Ruciński theorem on the threshold for Ramsey properties of random graphs (1-statement) and the Turán counterpart, with the best possible edge probability.
- 2. Łuczak (2000): structural and enumerative consequences for H-free graphs on n vertices and M edges

#### Consequences of Theorem 30

**Theorem 31** (Łuczak 2000). Suppose  $\chi(H) = h \ge 3$ . Then for every  $\delta > 0$  there exists  $C = C(\delta, H)$  such that, almost surely, a graph chosen uniformly at random from the family of all H-free labelled graphs on  $\mathfrak n$  vertices and  $M \ge C\mathfrak n^{2-1/m_2(H)}$  edges can be made (h-1)-partite by removing  $\le \delta M$  edges.

#### Consequences of Theorem 30

**Theorem 32** (Łuczak 2000). Suppose  $\chi(H) = h \ge 3$ . Then for every  $\varepsilon > 0$  there exist  $C = C(\varepsilon, H)$  and  $n_0 = n_0(\varepsilon, H)$  such that, for  $n \ge n_0$  and  $Cn^{2-1/d_2(H)} \le M \le n^2/C$ , we have

$$\left(\frac{h-2}{h-1}-\varepsilon\right)^{M} \leq \mathbb{P}\left(G(n,M) \not\supset H\right) \leq \left(\frac{h-2}{h-1}+\varepsilon\right)^{M}. \tag{26}$$

 $\triangleright G(n, M)$ : random graph of order n with M edges (uniform distribution)

# The hereditary nature of regularity

**Setup.** B = (U, W; E) an  $\varepsilon$ -regular bipartite graph with |U| = |W| = m and  $|E| = \rho m^2$ ,  $\rho > 0$  constant, and an integer d. Sample N  $\subset$  U and N'  $\subset$  W with |N| = |N'| = d uniformly at random.

**Theorem 33** (Duke and Rödl 1985). For any  $\beta > 0$ ,  $\rho > 0$ , and  $\epsilon' > 0$ , if  $\epsilon \leq \epsilon_0(\beta, \rho, \epsilon')$ ,  $d \geq d_0(\beta, \rho, \epsilon')$ , and  $m \geq m_0(\beta, \rho, \epsilon')$ , then

$$\mathbb{P}\big((\mathsf{N},\mathsf{N}')\;\mathit{bad}\big) \leq \beta^{\mathrm{d}},\tag{27}$$

where (N, N') is bad if  $\left| |E(N, N')| d^{-2} - \rho \right| > \epsilon'$  or else (N, N') is not  $\epsilon'$ -regular.

### The hereditary nature of regularity

**Exercise 34.** For any k and  $\delta > 0$ , there is C such that the following holds. If  $\chi(G - F) \ge k$  for any  $F \subset E(G)$  with  $|F| \le \delta n^2$ , then there is  $H \subset G$  with  $\chi(H) \ge k$  and  $|V(H)| \le C$ . Can you guarantee many such 'witnesses' H?

# Local characterization for regularity

**Setup.** B = (U, W; E), a bipartite graph with |U| = |W| = m. Consider the properties

(PC) for some constant p, have  $\mathfrak{m}^{-1}\sum_{\mathfrak{u}\in U}|\operatorname{deg}(\mathfrak{u})-\mathfrak{pm}|=o(\mathfrak{m})$  and

$$\frac{1}{m^2} \sum_{u,u' \in U} |\deg(u,u') - p^2 m| = o(m).$$
 (28)

(R) (U, W) is o(1)-regular (classical sense).

**Theorem 35.** (PC) and (R) are equivalent.

#### A proof of Theorem 33

Let a graph F=(U,E) with |U|=m and  $|E|\leq \eta\binom{m}{2}$  be given. Suppose we select a d-set N uniformly at random from U. We are then interested in giving an upper bound for e(F[N]), the number of edges that the set N will induce in F.

**Lemma 36.** For every  $\alpha$  and  $\beta > 0$ , there exist  $\eta_0 = \eta_0(\alpha, \beta) > 0$  such that, whenever  $0 < \eta \le \eta_0$ , we have

$$\mathbb{P}\left(e(\mathsf{F}[\mathsf{N}]) \ge \alpha \binom{d}{2}\right) \le \beta^d. \tag{29}$$

Proof. Exercise!

Exercise 37. Use Lemma 36 and Theorem 35 to prove Theorem 33.

# The hereditary nature of sparse regularity

**Definition 38**  $((\varepsilon, \mathfrak{p})\text{-lower-regularity})$ . Suppose  $0 < \varepsilon < 1$  and  $0 < \mathfrak{p} \le 1$ . A bipartite graph B = (U, W; E) is  $(\varepsilon, \mathfrak{p})\text{-lower-regular}$  if for all  $U' \subset U$  and  $W \subset W$  with  $|U'| > \varepsilon |U|$  and  $|W'| > \varepsilon |W|$ , we have

$$\frac{e(\mathbf{U}', \mathbf{W}')}{|\mathbf{U}'||\mathbf{W}|} \ge (1 - \varepsilon)\mathbf{p}. \tag{30}$$

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**Setup.** B = (U, W; E) an  $(\varepsilon, p)$ -lower-regular bipartite graph with |U| = |W| = m and  $|E| = pm^2$  and integer d. Sample  $N \subset U$  and  $N' \subset W$  with |N| = |N'| = d uniformly at random.

**Theorem 39** (Gerke, K., Rödl & Steger 2007). For all  $0 < \beta, \epsilon' < 1$ , there exist  $\epsilon_0 = \epsilon_0(\beta, \epsilon') > 0$  and  $C = C(\epsilon')$  such that, for any  $0 < \epsilon \le \epsilon_0$  and 0 , the following holds. Let <math>G = (U, W; E) be an  $(\epsilon, p)$ -lower-regular bipartite graph and suppose  $d \ge Cp^{-1}$ . Then

$$\mathbb{P}\big((\mathsf{N},\mathsf{N}')\;\mathit{bad}\big) \leq \beta^{\mathrm{d}},\tag{31}$$

where (N, N') is bad if (N, N') is not  $(\varepsilon', p)$ -lower-regular.

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**Setup.** B = (U, W; E) an  $(\varepsilon, p)$ -lower-regular bipartite graph with |U| = |W| = m,  $|E| = pm^2$ , and  $p \ge \alpha q$ . Also, suppose we have two other bipartite graphs  $A = (U', U; E_A)$  and  $C = (W, W'; E_C)$ , also  $(\varepsilon, p)$ -lower-regular.

**Corollary 40** (Quite imprecise...). Suppose  $A \cup B \cup C \subset G(\mathfrak{n},\mathfrak{q})$  and  $\mathfrak{u}' \in U'$  and  $w' \in W'$  are 'typical' vertices. If  $\mathfrak{pm} \gg 1/\mathfrak{p}$  then  $B[\Gamma_A(\mathfrak{u}'),\Gamma_C(w')]$  is  $(f(\varepsilon),\mathfrak{p})$ -lower-regular  $(f(\varepsilon) \to 0 \text{ as } \varepsilon \to 0)$ .

# Local characterization for sparse regularity

**Remark:** (very imprecisely) a similar statement to Theorem 35 may be proved for subgraphs of random graphs.