The regularity method and blow-up lemmas for sparse graphs

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Course outline

1. **Lecture I.** Introduction to the regularity method
2. **Lecture II.** The blow-up lemma
3. **Lecture III.** The sparse case: small subgraphs
4. **Lecture IV.** The sparse case: large subgraphs

Lecture II. The blow-up lemma

From Lecture I

- ▷ Elements of the regularity method
- ▷ The regularity lemma

$$\forall G : G \approx G^{\mathrm{rand}} \text{ (w.r.t. small subgraph statistics)}$$

- ▷ Embedding lemmas

$$\forall \text{ bdd deg \& sufficiently small } H \;\&\; G_r^{(\varepsilon)}(m, \rho) : H \hookrightarrow G_r^{(\varepsilon)}(m, \rho)$$

Applications of the the blow-up lemma

So far: we saw the regularity method applied to problems involving subgraphs $H = H^\ell$ of some $G = G^n$ with $\ell = o(n)$.

We are now concerned with the case in which $\ell \approx n$ or even $\ell = n$ (almost spanning or spanning subgraph problems).

(*i*) The Alon–Yuster conjecture

(*ii*) The Pósa–Seymour conjecture

The Alon–Yuster theorem & conjecture

Theorem 1 (Alon and Yuster (1997))**.** $\forall \varepsilon > 0,\ h \geq 1\ \exists n_0 = n_0(\varepsilon, h)$ *such that, for every* $H = H^h$ *and* $n \geq n_0$*, any* $G = G^{hn}$ *with minimum degree* $\delta(G) \geq (1 - 1/\chi(H) + \varepsilon)hn$ *has an* H*-factor.*

Conjecture 2. $\delta(G) \geq (1 - 1/\chi(H))nh + c(h)$ *suffices.*

The Pósa–Seymour conjecture

Conjecture 3. *Suppose* $G = G^n$. *If* $\delta(G) \geq (1-1/(k+1))n$, *then* G *contains the* k*th power of a Hamilton cycle.*

Theorem 4 (Hajnal–Szemerédi (1970)). *Suppose* $G = G^n = G^{s(k+1)}$. *If* $\delta(G) \geq (1-1/(k+1))n$, *then* $G \supset sK^{k+1}$.

Theorem 5 (Corrádi–Hajnal (1963)). *Suppose* $G = G^{3s}$. *If* $\delta(G) \geq 2s$, *then* $G \supset sK^3$.

Komlós–Sárközy–Szemerédi theorems

Theorem 6. *The Alon–Yuster conjecture holds for all* $n \geq n_0(H)$*.*

Theorem 7. *The Pósa–Seymour conjecture holds for all* $n \geq n_0(k)$*.*

A key tool: the blow-up lemma

Before we proceed:

Exercise 8. *Prove that the minimum degree conditions are best possible.*

Basic definition

Definition 9 ((ε, δ)-super-regular). *Fix* ε *and* $\delta > 0$. *The bipartite graph* $G = (V_1, V_2; E)$ *with* $|V_1| = |V_2| = n$ *is an* (ε, δ)*-super-regular pair if*

▷ *for all* $W_1 \subset V_1$, $W_2 \subset V_2$ *with* $|W_1|, |W_2| \geq \varepsilon n$, *we have*

$$|d(W_1, W_2) - d(V_1, V_2)| \leq \varepsilon,$$

▷ $\deg(v) \geq \delta n$ *for all* $v \in V_1 \cup V_2$.

Terminology:

- ‘regular’: edges uniformly distributed
- ‘super’: high minimum degree

Exercises

Exercise 10. *Prove that, for any positive* ρ *and* δ, *there are* $\varepsilon > 0$ *and* n_0 *for which the following holds. Suppose* $G = (U, W; E)$ *is an* (ε, δ)*-super-regular pair with* $|U| = |W| \geq n_0$ *and density* $d(U, W) = e(U, W)/|U||W| \geq \rho$. *Then* G *has a perfect matching.*

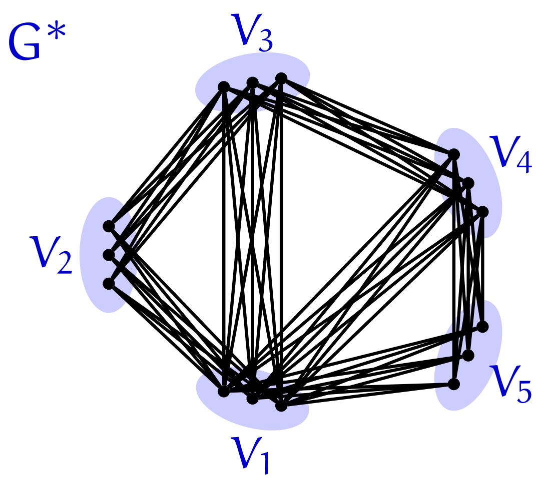
You can also try your teeth on the following:

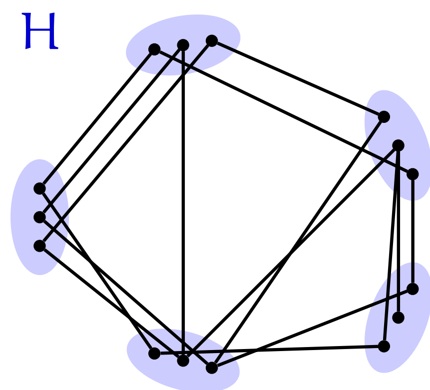
Exercise 11. *Prove that, for any positive* ρ *and* δ, *there are* $\varepsilon > 0$ *and* n_0 *for which the following holds. Suppose* $G = (U, W; E)$ *is an* (ε, δ)*-super-regular pair with* $|U| = |W| \geq n_0$ *and density* $d(U, W) = e(U, W)/|U||W| \geq \rho$. *Then* G *has a Hamilton path/cycle.*

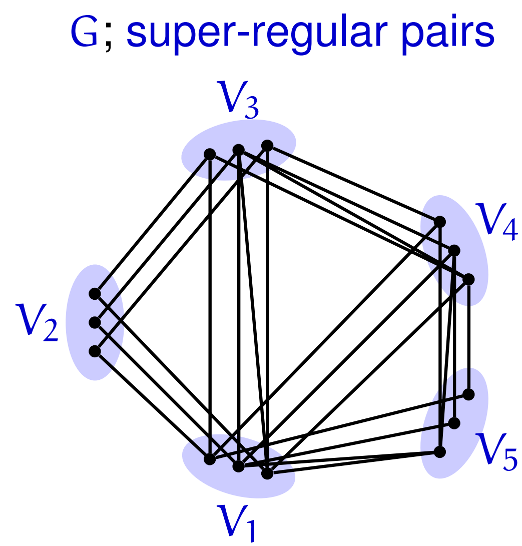
The Blow-up Lemma

Theorem 12 (The Blow-up Lemma (Komlós, Sárközy & Szemerédi 1997))**.** *$\forall \delta > 0$, Δ, $r \in \mathbb{N}$ $\exists \varepsilon > 0$ for which the following holds. Let $G^* = (V_1, \dots, V_r; E^*)$ and $G = (V_1, \dots, V_r; E)$ be two r-partite graphs. Suppose $R \subset \binom{[r]}{2}$ is such that if $ij \in R$, then (V_i, V_j) is a complete bipartite graph in G^* and (V_i, V_j) is an (ε, δ)-super-regular pair in G. If H with $\Delta(H) \leq \Delta$ can be If H with $\Delta(H) \leq \Delta$ can be* [. . .]

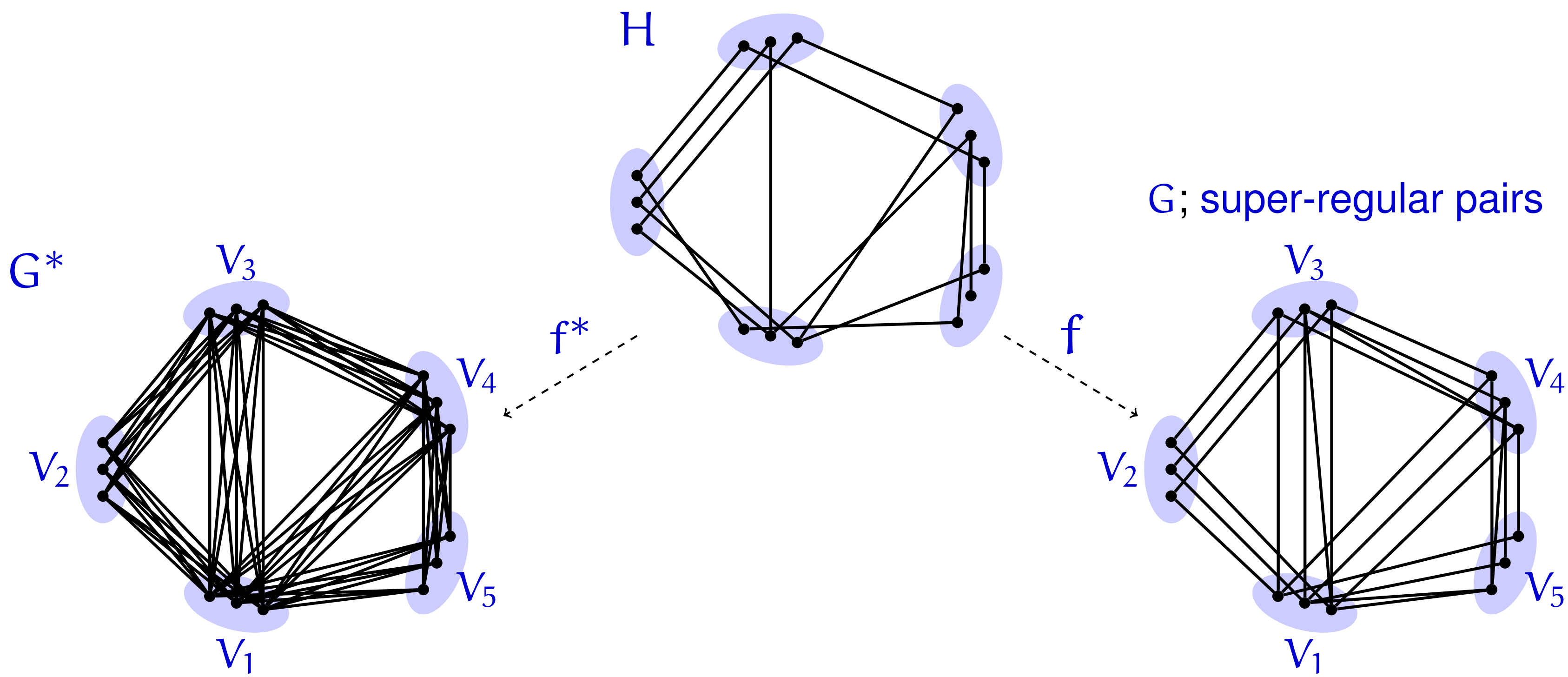
The Blow-up Lemma







The Blow-up Lemma



If f^* exists, the f exists.

The Blow-up Lemma

Theorem 13 (The Blow-up Lemma (Komlós, Sárközy & Szemerédi 1997))**.** $\forall \delta > 0$, Δ, $r \in \mathbb{N}$ $\exists \varepsilon > 0$ *for which the following holds. Let* $G^* = (V_1, \ldots, V_r; E^*)$ *and* $G = (V_1, \ldots, V_r; E)$ *be two* r*-partite graphs. Suppose* $R \subset \binom{[r]}{2}$ *is such that if* $ij \in R$*, then* (V_i, V_j) *is a complete bipartite graph in* G^* *and* (V_i, V_j) *is an* (ε, δ)*-super-regular pair in* G*. If* H *with* $\Delta(H) \leq \Delta$ *can be embedded into* G^**, then it can also be embedded into* G*.*

Remarks

- Already interesting in the case in which

$$G^* = (V_1^*, \ldots, V_r^*; E^*) \quad \text{and} \quad |V_i^*| = .99|V_i| \text{ for all } i.$$

- Super-regularity not required in this case.
- Recall the challenge from Lecture I (almost spanning embedding lemma).

Proof of the blow-up lemma (rough sketch)

(*i*) Almost spanning embedding lemma

(*ii*) A König–Hall argument

Shall prove (*i*). Shall only mumble a few words about (*ii*).

Almost spanning embedding lemma

Blow-up lemma with $G^* = (V_1^*, \dots, V_r^*; E^*)$ and $|V_i^*| = .99|V_i|$ for every i. Suppose H has vertex classes $X_1, \dots, X_r$.

A trick (Alon–Füredi trick):

- ▷ Make the X_i 2-independent (or even more).
- ▷ Achieve this by refining the partition (X_i): use a proper colouring of the square of H.
- ▷ Can use Hajnal–Szemerédi (complementary form) to make the new vertex classes of H of the same size (up to 1).
- ▷ Refine the partition (V_i) randomly, watching out for the sizes (want the vertex partitions of H and G to be *compatible*).

Almost spanning embedding lemma

Suppose:

(*i*) X_i 2-independent in H and $|X_i| = (1-\mu)m$ for all $1 \leq i \leq r$.

(*ii*) $|V_i| = m$ for all i.

(*iii*) Set aside $V_i^q \subset V_i$ with $|V^q| = (\mu/2)m$ for each i.

(*iv*) $x_1, \ldots, x_\ell$ is some ordering, say τ, of $X = V(H)$.

(*v*) Shall embed the x_i basically following τ.

(*vi*) Partial embeddings will be $\psi_0 = \emptyset, \psi_1, \psi_2, \ldots$.

Almost spanning embedding lemma

▷ We'll map X_i into V_i. If $x \in X_i$, then let $V(x) = V_i$.

Further details: for $t = 0, 1, \dots,$ we let

(*i*) $C_t(x) = V(x) \cap \bigcap \{N_G(\psi_t(y))\colon y \in N_H(x) \text{ and } y \in \mathrm{Dom}(\psi_t)\}$,

(*ii*) $A_t(x) = C_t(x) \setminus \mathrm{Im}(\psi_t)$,

(*iii*)

$$\begin{aligned} B_t(x) = \{v \in A_t(x) : \exists y \in N_H(x),\ & y \notin \mathrm{Dom}(\psi_t), \\ & \text{and } |N_G(v) \cap A_t(y) \setminus V^q| < (\rho - \varepsilon)|A_t(y) \setminus V^q| \\ & \text{or } |N_G(v) \cap C_t(y) \cap V^q| < (\rho - \varepsilon)|C_t(y) \cap V^q|\}, \end{aligned}$$

(*iv*) $\pi_t(x) = |\mathrm{Dom}(\psi_t) \cap N_H(x)|$.

Algorithm 1: Random greedy algorithm

Input : G and H with compatible partitions; an ordering τ on $X = V(H)$

$t := 0$; $\psi_0 := \emptyset$; $Q_0 := \emptyset$;
repeat
    let $x \in X \setminus (\mathrm{Dom}(\psi_t) \cup Q_t)$ be the next vertex in the order τ;
    choose $v \in A_t(x) \setminus B_t(x) \setminus V^q$ uniformly at random;
    $\psi_{t+1} := \psi_t \cup \{x \mapsto v\}$; $Q_{t+1} := Q_t$;
    foreach $y \in X \setminus \mathrm{Dom}(\psi_{t+1})$ **do**
        if $|A_{t+1}(y) \setminus V^q| < \frac{1}{2}\mu(\rho - \varepsilon)^{\pi_{t+1}(y)}|V(y)|$ **then**
            $Q_{t+1} := Q_{t+1} \cup \{y\}$;
    $t := t + 1$;
until $\mathrm{Dom}(\psi_t) \cup Q_t = X$;
embed $H[Q_t]$ into $G[V^q]$, respecting ψ_t built above;

Embedding queue vertices

Let t_{RGAend} be the last value of t, so that we have the partial embedding $\psi_{t_{\text{RGAend}}}$ before we try to embed $H[Q_t]$ into $G[V^q]$. A crucial lemma is as follows:

Lemma 14. *With high probability, $Q_t \cap X_i$ never gets too large.*

If $|Q_t \cap X_i| \ll \rho^{\Delta}|V^q|$, then $H[Q_t] \subset G[V^q]$ (positive proportion embedding lemma).

Corollary 15. *The embedding of $H[Q_t]$ into $G[V^q]$ succeeds with high probability.*

The main lemma

Lemma 16 (Uniform distribution of candidate sets). *Suppose* $\varepsilon \ll \mu, \rho, \kappa$ *and* $1/\Delta$ *and let* $\emptyset = \psi_0, \psi_1, \dots, \psi_T$ *be partial embeddings with each* ψ_t *obtained from* ψ_{t-1} *by embedding some* $x \in V(H) \setminus \mathrm{Dom}(\psi_{t-1})$ *to a uniform random vertex from a subset of* $C_{t-1}(x)$ *of size at least*

$$\tfrac{1}{4}\mu(\rho-\varepsilon)^{\pi_{t-1}(x)}|V(x)|.$$

The following holds with probability at least $1 - r2^{-n/r}$. *For every* $i \in [r]$ *and every set* $W \subset V_i$ *of size at least* $\kappa|V_i|$, *the number of vertices* $x \in X_i$ *such that there exists* $t = t(x)$ *when* x *is unembedded and, moreover,*

$$|C_t(x) \cap W| < (\rho-\varepsilon)^{\pi_t(x)}\,|W|$$

is at most $\kappa|X_i|$.

The queues never get too large

Proof of Lemma 14 (rough sketch). Let $W = V_i \setminus V^q \setminus \mathrm{Im}(\psi_{t_{\mathrm{RGAend}}})$. Have $|W| \geq (\mu/2)|V_i|$. Suppose $x \in X_i$ got into Q_t at time $t = t(x)$. We have

$$C_t(x) \cap W \subset A_t(x) \setminus V^q.$$

Since $x \in Q_t$, we have

$$|C_t(x) \cap W| \leq |A_t(x) \setminus V^q| \leq \frac{1}{2}\mu(\rho - \varepsilon)^{\pi_t(x)}|V(x)| \leq (\rho - \varepsilon)^{\pi_t(x)}|W|.$$

By Lemma 16, have at most $\kappa|X_i|$ such x, with high probability. $\square$

The main lemma

Proof of Lemma 16 (rough sketch). Fix i and $W \subset V_i$ with $|W| \geq \kappa|V_i|$. Suppose $x \in X_i$ is unembedded and, moreover,

$$|C_t(x) \cap W| < (\rho - \varepsilon)^{\pi_t(x)} |W|.$$

Pick the first time $t = t(x)$ such that the inequality above holds. Some $y \in N_H(x)$ was embedded to form ψ_t. This y was mapped onto a vertex v that is bad with respect to W (at most $\varepsilon|V(y)|$ choices), and v was selected from a set of size at least $\frac{1}{4}\mu(\rho - \varepsilon)^{\pi_{t-1}(y)}|V(y)|$. Probability of this event is $\lesssim \varepsilon/\mu\rho^{\Delta}$. Because of the 2-independence of X_i, the map $x \mapsto y(x)$ is injective. The probabilities multiply, and the probability that there are $\kappa|X_i|$ such x is

$$\lesssim \left(\frac{\varepsilon\Delta}{\mu\rho^{\Delta}}\right)^{\kappa|X_i|} \leq 2^{-4|X_i|}.$$

Now use union bound over all i and W. $\square$

The König–Hall argument

(*i*) Now $G^* = (V_1^*, \ldots, V_r^*; E^*)$ and $|V_i^*| = |V_i|$ for all i.

(*ii*) **Key:** set aside a *set of buffer vertices* in each X_i, say, $X_i^{\text{buff}} \subset X_i$. Let $|X_i^{\text{buff}}| = \mu |X_i|$.

(*iii*) Choose X_i^{buff} so that $X^{\text{buff}} = \bigcup_i X_i^{\text{buff}}$ is 4-independent.

(*iv*) Run the random greedy algorithm between $X \setminus X^{\text{buff}}$ and $V = V(G)$. This gives $\psi^- \colon H[X \setminus X^{\text{buff}}] \hookrightarrow G$.

(*v*) Embed X^{buff} into $V^{\text{buff}} = V \setminus \mathrm{Im}(\psi^-)$ by showing that $C(x) \cap V^{\text{buff}}$ ($x \in X^{\text{buff}}$) has a system of distinct representatives $(v_x)_{x \in X^{\text{buff}}}$. Extend ψ^- by mapping x onto v_x for all $x \in X^{\text{buff}}$.

(*vi*) Check König–Hall conditions: small sets (easy), medium sized sets (harder), very large sets (hard).

Example applications

(i) Very approximate Alon–Yuster

(ii) Square of almost Hamilton cycles—approximate version

Very approximate Alon–Yuster

Theorem 17. *Fix $\varepsilon > 0$ and $H = H^h$. Then any $G = G^n$ with $n \geq n_0(\varepsilon, H)$ with minimum degree $\delta(G) \geq (1 - 1/\chi(H) + \varepsilon)n$ contains $(1 - \varepsilon)n/h$ vertex disjoint copies of H.*

- Doesn't require blow-up; constant size embedding lemma suffices.

Useful to note that the clean-up graph G^* obtained from regularization may be required to satisfy a minimum degree condition.

Cleaned-up graph G^* with minimum degree condition

Definition 18. *After regularization of G, have $V = V_1 \cup \cdots \cup V_t$. Remove all edges in $G[V_i, V_j]$ for all i and j such that*

1. *(V_i, V_j) is not ε-regular,*
2. *$|E(V_i, V_j)| \le f(\varepsilon)|V_i||V_j|$ (suitable f with $f(\varepsilon) \to 0$ as $\varepsilon \to 0$).*

Resulting graph: cleaned-up graph G^*.

In G^*, every $G^*[V_i, V_j]$ is regular and 'dense'. Usually, lose very little. By allowing an exceptional class V_0 with $|V_0| \le \varepsilon|V|$, may require, for every $v \in V(G) \setminus V_0$, that

$$\deg_{G^*}(v) \ge \deg_G(v) - g(\varepsilon)n,$$

for some g with $g(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Very approximate Alon–Yuster

Exercise 19. *Prove the very approximate Alon–Yuster theorem.*

Hint 20. *Clean-up graph with mindeg condition* $+$ *Hajnal–Szemerédi.*

Squares of almost Hamilton cycles—approximate form

Theorem 21. *Suppose* $G = G^n$ *and* $n \geq n_0$. *If* $\delta(G) \geq (2/3 + \varepsilon)n$, *then* G *contains the square of a cycle with* $\geq (1 - \varepsilon)n$ *vertices.*

Exercise 22. *Prove the theorem above.*

Hint 23. *Clean-up graph with mindeg condition* $+$ *Corrádi–Hajnal* $+$ *almost spanning blow-up lemma* $+$ *image restrictions.*

Summary

- Got a glimpse into how to use the regularity method to find large subgraphs in given graphs; key tool: **blow-up lemma**.