The regularity method and blow-up lemmas for sparse graphs

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Course outline

1. **Lecture I.** Introduction to the regularity method
2. **Lecture II.** The blow-up lemma
3. **Lecture III.** The sparse case: small subgraphs
4. **Lecture IV.** The sparse case: large subgraphs

Lecture I. Introduction to the regularity method

Introduction

Regularity method:

- ▷ A powerful method in graph theory and combinatorics and beyond.
- ▷ Shall focus on graphs, including some applications in the theory of random and pseudorandom graphs (Lecture III).
- ▷ Shall consider blow-up lemmas in the dense and sparse settings (Lectures II & IV).
- ▷ Lecture I: introduction to the regularity method

Outline (Lecture I)

1. Basic definitions
2. The regularity lemma
3. Embedding/counting lemmas
4. Example applications
5. Some words on proofs

Basic definitions

Definition 1 (Basic definition)**.** $G = (V, E)$ *a graph;* U, $W \subset V$ *non-empty and disjoint. Say* (U, W) *is* ε*-regular* (*in* G) *if*

▷ *for all* $U' \subset U$, $W' \subset W$ *with* $|U'| \geq \varepsilon|U|$ *and* $|W'| \geq \varepsilon|W|$, *we have*

$$\left| \frac{|E(U', W')|}{|U'||W'|} - \frac{|E(U, W)|}{|U||W|} \right| \leq \varepsilon.$$

Density:

$$d(U, W) = \frac{e(U, W)}{|U||W|} = \frac{|E(U, W)|}{|U||W|}$$

Basic definitions

Definition 2. *A partition* $V = V_1 \cup \cdots \cup V_t$ *is an equipartition if*

$$|V_1| \leq \cdots \leq |V_t| \leq |V_1| + 1.$$

Definition 3. *A partition* $V = V_1 \cup \cdots \cup V_t$ *is* ε*-regular if at least* $(1-\varepsilon)\binom{t}{2}$ *pairs* (V_i, V_j) $(i < j)$ *are* ε*-regular.*

Szemerédi's regularity lemma

Theorem 4 (The regularity lemma)**.** *For any* $\varepsilon > 0$ *and* $t_0 \geq 1$*, there exist* T_0 *such that any graph* G *admits an* ε*-regular equipartition* $V = V_1 \cup \cdots \cup V_t$ *with* $t_0 \leq t \leq T_0$*.*

Crucial: T_0 is independent of $|V(G)|$.

Earlier version: important lemma in Szemerédi's proof of the Erdős–Turán conjecture on arithmetic progressions in sets of integers of positive density.

Komlós:

> 'This is not a very transparent theorem, but it grows on you with time.'

Exceptional class

Sometimes consider partitions as follows:

(*i*) $V = V_0 \cup V_1 \cup \cdots \cup V_t$,

(*ii*) $|V_0| \leq \varepsilon |V|$,

(*iii*) $|V_1| = \cdots = |V_t|$.

Can even demand $|V_0| < t$. Regularity lemma delivers such partitions, as long as $|V| \geq n_0(t_0, \varepsilon)$.

Exceptional class: V_0

The embedding lemma/counting lemma (simplest version)

Set-up 5. *Let* $G = (V, E)$ *be a graph with*

(i) $V = V_1 \cup \cdots \cup V_r$ *and* $|V_i| = m$ *for all* i*,*

(ii) (V_i, V_j) ε*-regular for all* $i < j$*,*

(iii) $|E(V_i, V_j)| = \rho m^2$ *for all* $i < j$*.*

Write $G_r^{(\varepsilon)}(m, \rho)$ for a graph as above.

The embedding lemma/counting lemma (simplest version)

Let K^r have vertices $x_1, \ldots, x_r$.

Lemma 6 (Embedding lemma/Counting lemma)**.** *$\forall r \geq 2$, $\rho > 0$ and $\delta > 0$ $\exists \varepsilon > 0$, m_0: if $m \geq m_0$, then the number of homomorphisms $K^r \hookrightarrow G = G_r^{(\varepsilon)}(m, \rho)$ with $x_i \mapsto$ some $v_i \in V_i$ for all i is*

$$m^r \rho^{\binom{r}{2}} \pm \delta m^r.$$

In particular, $K^r \subset G$.

The reduced graph R

After regularization, can consider the so-called *reduced graph*.

- Vertex i for each V_i ($1 \le i \le t$),
- Edge-weight $d(V_i, V_j)$ for each $1 \le i < j \le t$ that is ε-regular and *dense* (usually, use some cut-off density as threshold).

We can use R to define a 'generalized random graph' G^{rand} on $V(G)$: replace each edge ij of R by a random bipartite graph on $V_i \times V_j$ with edge probability $d(V_i, V_j)$.

G and G^{rand} are 'similar'

- For any *fixed* graph H, the number of copies of H in G and in G^{rand} are approximately equal, as long as $n = |V(G)|$ is large.
- For any $\delta > 0$ and h, there are ε and ε' such that if R is obtained from SzRL with parameter ε and cut-off ε', then, for any $H = H^h$,

$$\#\{H \subset G\} = \mathbb{E}(\#\{H \subset G^{\text{rand}}\}) \pm \delta n^h.$$

- Follows from 'counting lemmas' **[exercise!]**.
- How about *large* H? E.g., $h = n$? Lectures II & IV will focus on 'embedding lemmas' for such H ('blow-up lemmas').

The embedding lemma/counting lemma (exercises)

Exercise 7 (Counting lemma)**.** *What is the CL for a general graph* $H = H^r$*? Suppose* H *has vertices* $x_1, \ldots, x_r$*. What is the number of homomorphisms* $H \hookrightarrow G_r^{(\varepsilon)}(m, \rho)$ *with* $x_i \mapsto$ *some* $v_i \in V_i$ *for all* i*?*

Exercise 8 (Counting lemma)**.** *What is the CL for a general* r*-partite graph* $H = H^\ell$*? Suppose* H *has vertices* $x_1, \ldots, x_\ell$*. Suppose* $\chi : V(H) \to [r] = \{1, \ldots, r\}$ *is an* r*-partition of* H*. What is the number of homomorphisms* $H \hookrightarrow G_r^{(\varepsilon)}(m, \rho)$ *with* $x_i \mapsto$ *some* $v_i \in V_{\chi(i)}$ *for all* i*?*

Could also consider $G_r^{(\varepsilon)}(m, (\rho_{ij})_{1 \le i < j \le r})$.

The embedding lemma/counting lemma (exercises)

Exercise 9 (Counting lemma (induced version)). *What is the induced CL for a general graph $H = H^r$? Suppose H has vertices $x_1, \ldots, x_r$. What is the number of homomorphisms $H \hookrightarrow G_r^{(\varepsilon)}(m, \rho)$ with $x_i \mapsto$ some $v_i \in V_i$ for all i such that H is isomorphic to $G[v_1, \ldots, v_r]$?*

Could also consider r-partite graphs $H = H^\ell$.

A few applications of the regularity method

(*i*) The removal lemma

(*ii*) The Erdős–Stone–Simonovits theorem

Triangle removal lemma

Theorem 10 (Triangle removal lemma)**.** $\forall \varepsilon > 0 \ \exists \delta > 0$ *for which the following holds. Suppose* $G = G^n = (V, E)$ *is such that* $G - F = (V, E \setminus F)$ *contains a triangle* $\forall F \subset \binom{V}{2}$ *with* $|F| \leq \varepsilon\binom{n}{2}$*. Then* G *contains* $\geq \delta n^3$ *triangles.*

That is: $G = G^n$ ε-far from K^3-free, then $\#\{K^3 \hookrightarrow G\} \geq \delta n^3$.

Exercise 11. *Deduce the triangle removal lemma from the regularity lemma.*

Hint 12. *Analyse the 'cleaned-up graph'* G^* (*Definition 19*)*.*

Exercise 13. *State and prove the removal lemma for general graphs* H*.*

Triangle removal lemma

Exercise 14 (Roth's theorem). *$\forall\delta > 0\ \exists n_0$ such that, for all $n \geq n_0$, if $A \subset [n]$ and $|A| \geq \delta n$, then A contains a 3-term arithmetic progression $\{a, a + d, a + 2d\}$ $(d > 0)$.*

Hint 15. *Consider a 3-partite graph $G = (X, Y, Z; E)$ defined as follows. Let $X = [n]$, $Y = [2n]$ and $Z = [3n]$ (disjoint). For each $a \in A$, put in G the edges $(x, x + a) \in X \times Y$, $(x, x + 2a) \in X \times Z$ and $(y, y + a) \in Y \times Z$. What happens if G contains a triangle **not** of the form $\{x, x + a, x + 2a\}$?*

The Erdős–Stone–Simonovits theorem

- ▷ Let $\mathrm{ex}(n, F) = \max\{e(G) : F \not\subset G \subset K^n\}$. Mantel: $\mathrm{ex}(n, K^3) = \lfloor n/2 \rfloor \lceil n/2 \rceil$.
- ▷ Mantel implies that $1/2$ is the *density threshold* for K^3: if G has "density" $1/2 + \varepsilon$, then $G \supset K^3$ (if n large). Indeed,

$$\mathrm{ex}(n, K^3) = \left(\frac{1}{2} + o(1)\right)\binom{n}{2}.$$

- ▷ Turán: what is $\mathrm{ex}(n, K^{q+1})$? Turán graph T^n_q: split n vertices into q classes as equally as possible; add all edges connecting vertices in different classes. Lower bound:

$$\mathrm{ex}(n, K^{q+1}) \geq e(T^n_q).$$

The Erdős–Stone–Simonovits theorem

Theorem 16 (Turán (1941)). *For all n and q,*

$$\mathrm{ex}(n, K^{q+1}) = e(T_q^n).$$

Density threshold for K^{q+1} is $1 - 1/q$.

How about arbitrary F?

The Erdős–Stone–Simonovits theorem

F: an arbitrary graph ($e(F) \geq 1$). Lower bound for $\mathrm{ex}(n, F)$?

$\chi(F)$: the *chromatic number* of F, that is, the smallest q such that $F \subset T_q^\infty$

Lower bound: $\mathrm{ex}(n, F) \geq e\left(T^n_{\chi(F)-1}\right) = \left(1 - \frac{1}{\chi(F)-1} + o(1)\right)\binom{n}{2}$

Theorem 17 (Erdős & Stone 1946, Erdős and Simonovits 1966)**.** *For every graph F, we have*

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right)\binom{n}{2}.$$

Density threshold for F is $1 - \frac{1}{\chi(F)-1}$.

The Erdős–Stone–Simonovits theorem

Exercise 18. *Deduce the Erdős–Stone–Simonovits theorem from Turán's theorem and the regularity lemma.*

Outline: $G = G^n$ with $e(G) \geq (1 - 1/(q-1) + \eta)\binom{n}{2}$, where $q = \chi(G)$.

(*i*) *Regularize* G: apply Szemerédi's regularity lemma to G (use $\varepsilon \ll \eta$)

(*ii*) Analyse the 'cleaned-up graph' G^* (Definition 19). Deduce from Turán's theorem that $G_q^{(\varepsilon)}(m, \rho) \subset G$.

(*iii*) Apply the counting lemma.

Terminology: *Cleaned-up graph* G^*

Definition 19. *After regularization of G, have $V = V_1 \cup \cdots \cup V_t$. Remove all edges in $G[V_i, V_j]$ for all i and j such that*

1. (V_i, V_j) *is not ε-regular,*
2. $|E(V_i, V_j)| \le f(\varepsilon)|V_i||V_j|$ *(suitable f with $f(\varepsilon) \to 0$ as $\varepsilon \to 0$).*

Resulting graph: cleaned-up graph G^*.

In G^*, every $G^*[V_i, V_j]$ is regular and 'dense'. Usually, lose very little.

A Ramsey–Turán result

Exercise 20 (Szemerédi 1972)**.** *Suppose* $G = G^n$ *has independence number* $o(n)$ *and* $e(G) \geq (1/8+\varepsilon)n^2$ *for some fixed* $\varepsilon > 0$*. Prove that* G *contains a* K^4*.*

Further exercises

Notation/definitions:

(*i*) $G \to H$: if $E(G) = B \cup R$, then $H \subset G[B]$ or $H \subset G[R]$.

(*ii*) $G \xrightarrow{\text{ind}} H$: if $E(G) = B \cup R$, then there is an induced subgraph H' of G isomorphic to H with $H' \subset G[B]$ or $H' \subset G[R]$.

(*iii*) The (2-*colour*) *Ramsey number* of H is

$$r(H) = \min\{|V(G)| \colon G \to H\} = \min\{n \colon K^n \to H\}.$$

Exercise 21. *Prove that, for every* H*, there is* G *such that* $G \xrightarrow{\text{ind}} H$*.*

Another application

Theorem 22 (Chvátal, Rödl, Szemerédi and Trotter 1983). *For every Δ, there is $c = c_\Delta$ such that, for every graph $H = H^\ell$ with $\Delta(H) \leq \Delta$, we have*

$$r(H) \leq c\ell.$$

That is, bounded degree graphs have linear Ramsey numbers.

Recall

$$2^{\ell/2} \leq r(K^\ell) \leq 4^\ell.$$

Theorem 22: typical application of the regularity method.

Positive proportion EL/CL

Important: suppose x_i to be embedded in G, and $Y \subset N_H(x_i)$ already embedded. Then have the set of *candidate vertices*

$$C(x_i) = V_{\chi(i)} \cap \bigcap_{y \in Y} N_G(y).$$

More precisely, have the set of *available vertices*

$$A(x_i) = V_{\chi(i)} \cap \bigcap_{y \in Y} N_G(y) \setminus \mathrm{Im}\,\psi,$$

where ψ is the current partial embedding. Can make the sets $C(x_i)$ shrink in a controlled way: $|C(x_i)| \approx \rho^{|Y|}|V_i|$. When $H = H^\ell$ and $\ell = O(1)$, then $A(x_i) \approx C(x_i)$. How about if $\ell \to \infty$?

Positive proportion EL/CL

Natural restriction. Suppose $H = H^\ell$ has bounded maximum degree: $\Delta(H) = O(1)$ (i.e., $\Delta = \Delta(H)$ independent of $n = |V(G)| \to \infty$), but allow $\ell \to \infty$.

Let $H = H^\ell$ be r-partite and suppose $V(H) = \{x_1, \ldots, x_\ell\}$. Suppose

$$\chi\colon V(H) \to [r] = \{1, \ldots, r\}$$

is an r-partition of H. We think of r as fixed and $\ell \to \infty$.

Positive proportion EL/CL

Lemma 23 (Embedding lemma/Counting lemma)**.** $\forall r \geq 2$, $\ell \geq 1$, $\rho > 0$, $\delta > 0$ *and* Δ $\exists \varepsilon > 0$, m_0: *if* $\Delta(H) \leq \Delta$ *and* $m \geq m_0$, *then the number of homomorphisms* $H \hookrightarrow G = G_k^{(\varepsilon)}(m, \rho)$ *with* $x_i \mapsto$ *some* $v_i \in V_{\chi(i)}$ *for all* i *is*

$$m^\ell \rho^{e(H)} \pm \delta m^\ell.$$

In particular, $H \subset G$*.*

Suffices: $m_0 \gg \ell \rho^{-\Delta}$

Exercise 24. *Deduce the Chvátal–Rödl–Szemerédi–Trotter theorem from Lemma 23.*

Almost spanning embedding lemma

Challenge 25. *State and prove an almost spanning embedding lemma: a version of the embedding lemma in which the vertex classes $X_1, \ldots, X_r$ of H are each of size $.99m$. Here, we suppose $r = O(1)$ and $\Delta(H) = O(1)$.*

Proof of the regularity lemma (rough sketch)

Let $G = (V, E)$ and $\varepsilon > 0$ be fixed. Let $P = (C_i)_{0 \le i \le k}$ be an equitable partition of V ($V = C_0 \cup \cdots \cup C_k$). For each ε-irregular pair (C_s, C_t) with $1 \le s < t \le k$, choose $X(s,t) \subset C_s$, $Y(s,t) \subset C_t$ witnessing this fact.

For fixed $1 \le s \le k$, the sets $X(s,t)$ in

$$\{X(s,t) \subset C_s : 1 \le t \le k \text{ and } (C_s, C_t) \text{ is not } \varepsilon\text{-regular}\}$$

define a natural partition of C_s into at most 2^{k-1} blocks, called *atoms*. Put all of these atoms together to form an equitable partition $Q = (C'_i)_{0 \le i \le k'}$, with $k' = k4^k$ and $|C'_0| \le |C_0| + n4^{-k}$.

Proof of the regularity lemma (rough sketch)

Definition 26. *The index* $\mathrm{ind}(R)$ *of an equitable partition* $R = (C_i)_0^r$ *of* V *is*

$$\mathrm{ind}(R) = \frac{2}{r^2} \sum_{1 \le i < j \le \ell} d(C_i, C_j)^2.$$

Trivially, $0 \le \mathrm{ind}(R) < 1$.

Proof of the regularity lemma (rough sketch)

Recall we constructed an equipartition $Q = Q(P)$ from P.

Lemma 27. *If P is not ε-regular, then*

$$\mathrm{ind}(Q) \geq \mathrm{ind}(P) + 10^{-2}\varepsilon^5.$$

Conclusion: we can't iterate more than $10^2/\varepsilon^5$ times!

Proof of the regularity lemma (rough sketch)

Where does the gain come from?

Lemma 28. *Let* $y_1, \ldots, y_v \geq 0$ *be given. Suppose* $0 \leq \rho = u/v < 1$, *and* $\sum_{1\leq i\leq u} y_i = \alpha\rho \sum_{1\leq i\leq v} y_i$. *Then*

$$\sum_{1\leq i\leq v} y_i^2 \geq \frac{1}{v}\left(1 + (\alpha-1)^2 \frac{\rho}{1-\rho}\right)\left\{\sum_{1\leq i\leq v} y_i\right\}^2.$$

Each irregular pair contributes $\approx \varepsilon^4$ to the index. An ε-fraction of the pairs contributes that much; total is ε^5, as claimed.

Summary

- ▷ The regularity method
- ▷ The regularity lemma + embedding lemmas
- ▷ Example applications

Lecture II:

- ○ The blow-up lemma: embedding large graphs