Graph limits and their applications in extremal combinatorics (Part 2)

Dan Král'
University of Warwick

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Plan for today

- Limits of dense graphs

  recap of yesterday's lecture

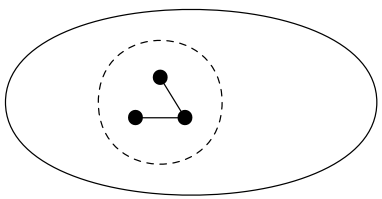
  existence of a limit graphon

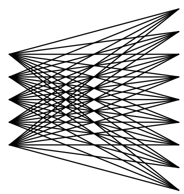
  graph quasirandomness

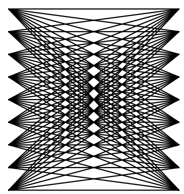
  some other applications

Dense graph convergence

- $d(H,G)$ = probability $|H|$-vertex subgraph of G is H
- a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is convergent
  if $d(H,G_n)$ converges for every H
- examples of convergent sequences:
  complete and complete bipartite graphs K_n and $K_{\alpha n,n}$
  Erdős-Rényi random graphs $G_{n,p}$
  sequences of sparse graphs (planar graphs)

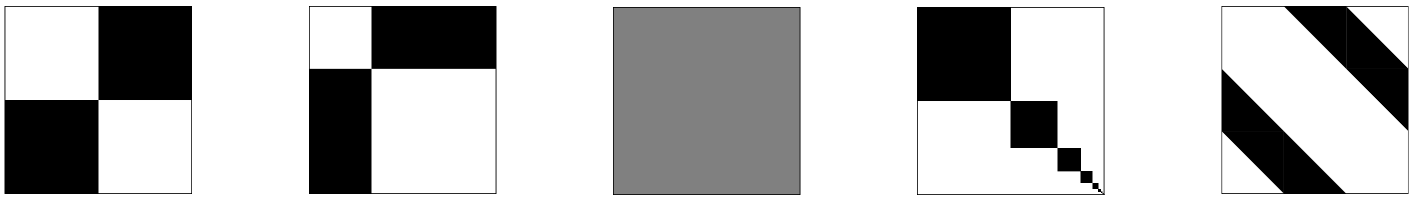






LIMIT OBJECT: GRAPHON

- graphon $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$
- W-random graph of order n
  random points $x_i \in [0,1]$, edge probability $W(x_i, x_j)$
- $d(H,W)$ = prob. $|H|$-vertex W-random graph is H
- W is a limit of $(G_n)_{n\in\mathbb{N}}$ if $d(H,W) = \lim_{n\to\infty} d(H,G_n)$



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Graph regularity

- Frieze-Kannan regularity, Szemerédi regularity
- $\forall \epsilon > 0\ \exists K_\varepsilon$ such that every graph G has an ε-regular equipartition $V_1, \ldots, V_k$ with $k \leq K_\varepsilon$
  $||V_i| - |V_j|| \leq 1$ for all i and j
- equipartition $V_1, \ldots, V_k \to$ density matrix $A_{ij} = \frac{e(V_i, V_j)}{|V_i||V_j|}$
- $\forall \delta > 0, H\ \exists \varepsilon > 0$ such that the density matrix of an ε-regular partition determines $d(H, G)$ upto an δ-error
- the lemma holds with prepartitions

Existence of limit graphon

- fix a convergent sequence G_i, $i \in \mathbb{N}$, of graphs
- set $\varepsilon_j = 2^{-j}$ and fix ε_1-regular partition of G_i
  fix ε_{j+1}-regular partition refining the ε_j-regular one
- take a subsequence G'_i of G_i such that all but finitely many ε_j-regular partitions have the same num. parts
- let A^{ij} be the density matrix for G_i and ε_j
- take a subsequence G''_i of G'_i such that
  A^{ij} coordinate-wise converge for every j

Existence of limit graphon

- a convergent sequence G_i, density matrices A^{ij}
  let A^j be the coordinate-wise limit of A^{ij}
- interpret A^j as a random variable on $[0,1]^2$ and
  apply Doob's Martingale Convergence Theorem
  in this way, we obtain a graphon W
- relate $d(H,W)$ to the density of H based on A^j

Questions?

Parameter testing

- graph parameter $\mathcal{P}$: graphs $\to \mathbb{R}$
- large input data, not possible to process
  providing an estimate based on a small sample
- $\mathcal{P}$ is testable if there exists a randomized algorithm that estimates the parameter $\mathcal{P}$ within the additive error ε based on a sample of size $f(\varepsilon)$ with probability $\geq 1 - \varepsilon$
- $\mathcal{P}$ is testable $\Leftrightarrow$ $\mathcal{P}$ is continuous on the graphon space

Graphon entropy

- Hatami, Janson, Szegedy (2013)
  Falgas-Ravry, O'Connell, Strömberg, Uzzell
- How many graphs resemble a graphon W?
  the number $\approx 2^{cn^2/2+o(n^2)}$, what is c?
  $c = \lim\limits_{\varepsilon\to 0}\lim\limits_{n\to\infty} \frac{\log|n\text{-vertex graphs } \varepsilon\text{-close to } W|}{n^2/2}$
- graphon entropy $\mathrm{Ent}(W) = \int h(W(x,y))\mathrm{d}xy$
  where $h(p) = -p\log_2 p - (1-p)\log_2(1-p)$
- the constant c is $\mathrm{Ent}(W)$

Quasirandom graphs

- Thomason, and Chung, Graham and Wilson (1980's)
- a sequence G_i is quasirandom if $d(H, G_i) \approx d(H, G_{n,p})$
  G_i converges to the constant graphon W_p
- $d(H, G_i) \to d(H, W_p)$ for every H if and only if
  $h(K_2, G_i) \to p$ and $h(C_4, G_i) \to p^4$
- $h(H, G) =$ prob. inj. map $H \to G$ is a homomorphism
  $h(\cdot, G)$ and $d(\cdot, G)$ for k-vertex subgraphs
  determine each other

Quasirandom graphs

- $d(H, G_i) \to d(H, W_p)$ for every H if and only if $h(K_2, G_i) \to p$ and $h(C_4, G_i) \to p^4$
- $\Rightarrow$ easy
- $\Leftarrow$ not that easy
  let G_i be such that $h(K_2, G_i) \to p$ and $h(C_4, G_i) \to p^4$
  let G_i' be a convergent subsequence and W its limit
  we show that W is equal to p almost everywhere

Quasirandom graphs

- $d(K_2, W) = p$ and $h(C_4, W) = p^4 \implies W = W_p$
  where $h(C_4, W) = \frac{1}{3}d(C_4, W) + \frac{1}{3}d(K_4^-, W) + d(K_4, W)$

- degree of a vertex $z \in [0, 1]$: $w(z) = \int W(z, x)\mathrm{d}x$
- $\int w(z)\mathrm{d}z = d(K_2, W) = p$
- apply Cauchy-Schwarz Inequality
  $$\int w(z)^2\mathrm{d}z \cdot \int 1\mathrm{d}z \geq \left(\int_{[0,1]} w(z)\mathrm{d}z \right)^2 = p^2$$

Quasirandom graphs

- $d(K_2, W) = p$ and $h(C_4, W) = p^4 \implies W = W_p$
  $w(z) = \int W(z,x)\mathrm{d}x \qquad \int w(z)^2 \mathrm{d}z \geq p^2$

- $0 \leq \int \left(\int W(x,z)W(y,z)\mathrm{d}z - p^2\right)^2 \mathrm{d}xy =$
  $\cdots = p^4 - 2p^2 \int w(z)^2 \mathrm{d}z + p^4$
  so, we get that $\int w(z)^2 \mathrm{d}z = p^2$

- recall the Cauchy-Schwarz Inequality we used
  $w(z) = p$ for almost every $z \in [0,1]$

Quasirandom graphs

- $d(K_2, W) = p$ and $h(C_4, W) = p^4 \implies W = W_p$
  $w(z) = \int W(z,x)\mathrm{d}x \qquad w(z) = p$ a.e.
  $0 = \int \left(\int W(x,z)W(y,z)\mathrm{d}z - p^2\right)^2 \mathrm{d}xy$

- $\int W(x,z)W(y,z)\mathrm{d}z = p^2$ for a.e. $x, y \in [0,1]^2$
  $\int W(x,z)^2\mathrm{d}z = p^2$ for a.e. $x \in [0,1]$
  $\int W(z,x)^2\mathrm{d}x = p^2$ for a.e. $z \in [0,1]$

- we apply the Cauchy-Schwarz Inequality again
  $p^2 = \left(\int W(z,x)\mathrm{d}x\right)^2 \leq \int W(z,x)^2\mathrm{d}x \cdot \int 1\mathrm{d}x = p^2$
  for a.e. z, $W(z,x) = p$ for a.e. x

Questions?