Graph limits and their applications in extremal combinatorics (Part 1)

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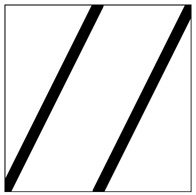
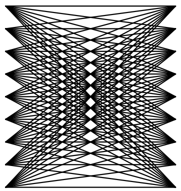
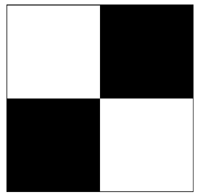
Why limits?

- asymptotic properties of large (discrete) objects
  we implicitly use limits in our considerations anyway
- How does the seq. $1, 3, \ldots, 2n-1, 2, 4, \ldots, 2n$ look like?
  How does the graph $K_{n,n}$ look like?
- Turán type questions in extremal graph theory
  How many edges can have a graph without K_3? K_5?
  What is a typical structure of an extremal graph?





$$\begin{array}{|cc|} \hline 0 & 1 \\ 1 & 0 \\ \hline \end{array}$$



Overview of the course

- Limits of dense graphs

  Survey of main concepts in the area

  Graph quasirandomness

- The flag algebra method

  Applications in extremal combinatorics

  Finitely forcible graph limits

- Limits of sparse graphs

  Various concepts, less understood

Graph limits

- concise (analytic) representation of large graphs
  avoiding calculations with smaller order terms

- convergence of a sequence of graphs

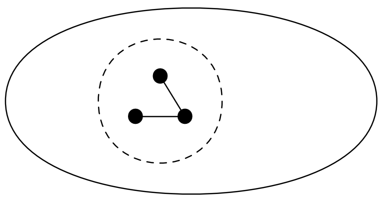
  vs.

  analytic representation of a convergent sequence

- convergence and limits of other discrete structures

Dense graph convergence

- convergence for dense graphs ($|E| = \Omega(|V|^2)$)
- $d(H, G)$ = probability $|H|$-vertex subgraph of G is H
- a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is convergent if $d(H, G_n)$ converges for every H
- extendable to other discrete structures

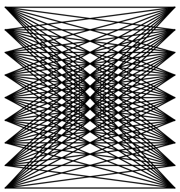


Convergent graph sequences

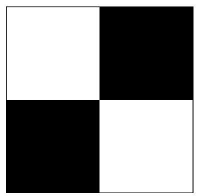
- complete graphs K_n
- complete bipartite graphs $K_{\alpha n,n}$
- Erdős-Rényi random graphs $G_{n,p}$ (exercise)
- any sequence of graphs with bounded maximum degree
- any sequence of planar graphs

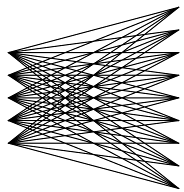
Limit object: graphon

- graphon $W : [0,1]^2 \to [0,1]$
  measurable symmetric function, i.e. $W(x,y) = W(y,x)$
- "limit of adjacency matrices" (very imprecise)
- points of $[0,1] \approx$ vertices, values of $W \approx$ edge density



$$\begin{matrix} 0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&1&1\\ 1&1&1&1&1&1&0&0&0&0&0&0\\ 1&1&1&1&1&1&0&0&0&0&0&0\\ 1&1&1&1&1&1&0&0&0&0&0&0\\ 1&1&1&1&1&1&0&0&0&0&0&0\\ 1&1&1&1&1&1&0&0&0&0&0&0\\ 1&1&1&1&1&1&0&0&0&0&0&0 \end{matrix}$$





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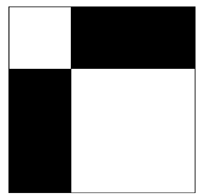
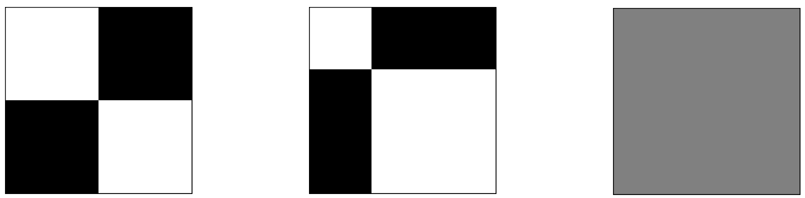


W-random graphs

- graphon $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$
- W-random graph of order n
  sample n random points $x_i \in [0,1] \approx$ vertices
  join two vertices by an edge with probability $W(x_i, x_j)$
- density of a graph H in a graphon W
  $d(H,W) =$ prob. $|H|$-vertex W-random graph is H



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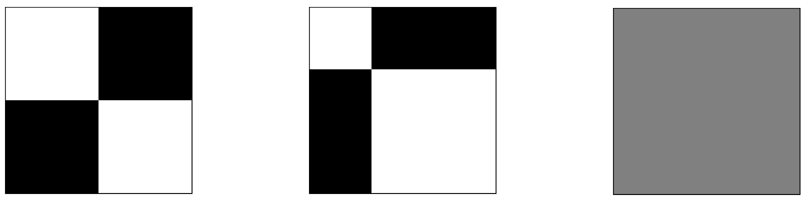
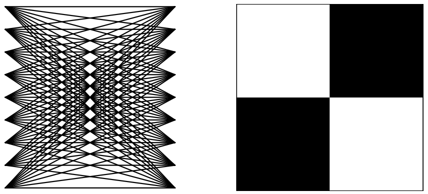
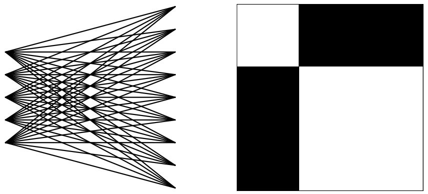
$$\frac{|H|!}{|\mathrm{Aut}(H)|}\int_{[0,1]^{|H|}} \prod_{v_iv_j} W(x_i,x_j) \prod_{\overline{v_iv_j}} (1-W(x_i,x_j))\, \mathrm{d}x_1\cdots x_n$$

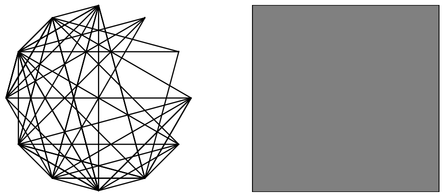


W-random graphs

- graphon $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$
- $d(H,W)$ = prob. $|H|$-vertex W-random graph is H
- W is a limit of $(G_n)_{n\in\mathbb{N}}$ if $d(H,W) = \lim_{n\to\infty} d(H,G_n)$
- examples of limits, details left as exercise







Graphons as limits

- Does every convergent sequence have a limit?
- Is every graphon a limit of convergent sequence?
- Uniqueness of a graphon representing a sequence.

Martingales

- martingale is a sequence of random variables X_n
  $\mathbb{E}(X_{n+1}|X_1, \ldots, X_n) = X_n$ for every $n \in \mathbb{N}$
- Azuma-Hoeffding inequality
  suppose that $\mathbb{E}X_n = X_0$ and $|X_n - X_{n-1}| \leq c_n$
  $\mathbb{P}\left(|X_n - X_0| \geq t\right) \leq 2e^{\frac{-t^2}{2\sum_{k=1}^{n} c_k^2}}$
- Doob's Martingale Convergence Theorem (corr.)
  if $|X_n| < K$, then $X_n \to X$ almost everywhere

W-random graphs converge

- A sequence of W-random graphs with increasing orders converges with probability one.
- fix $n \in \mathbb{N}$, a graph H and a graphon W
- $X_i =$ exp. number of H in an n-vertex W-rand. graph after fixing the first i vertices and edges between them
- apply Azuma-Hoeffding inequality with $c_i = n^{|H|-1}$
  $\mathbb{P}\left(|X_n - X_0| \geq \varepsilon n^{|H|}\right) \leq 2e^{-\varepsilon^2 n/2}$

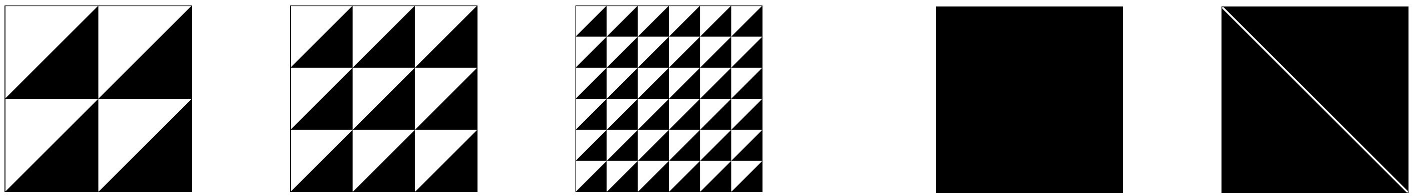
  $\mathbb{P}\left(|X_n - X_0| \geq t\right) \leq 2e^{\frac{-t^2}{2\sum_{k=1}^{n} c_k^2}}$

W-random graphs converge

- A sequence of W-random graphs with increasing orders converges with probability one.
- X_i = exp. number of H in an n-vertex W-rand. graph after fixing the first i vertices and edges between them $\mathbb{P}\left(\frac{|X_n - X_0|}{n^{|H|}} \geq \varepsilon\right) \leq 2e^{-\varepsilon^2 n/2}$
- the sum of $2e^{-\varepsilon^2 n/2}$ is finite for every $\varepsilon > 0$
- Borel-Cantelli $\Rightarrow$ the sequence converges with prob. one
- $X_0 \approx \frac{d(H,W)n^{|H|}}{|H|!} \Rightarrow$ the graphon W is its limit

Uniqueness of the limit

- $W^{\varphi}(x,y) := W(\varphi(x),\varphi(y))$ for $\varphi : [0,1] \to [0,1]$
- $d(H,W) = d(H,W^{\varphi})$ if φ is measure preserving
- Theorem (Borgs, Chayes, Lovász)
  If $d(H,W_1) = d(H,W_2)$ for all graphs H,
  then there exist measure preserving maps φ_1 and φ_2
  such that $W_1^{\varphi_1} = W_2^{\varphi_2}$ almost everywhere.



Questions?