Graph limits and their applications in extremal combinatorics (Part 5)

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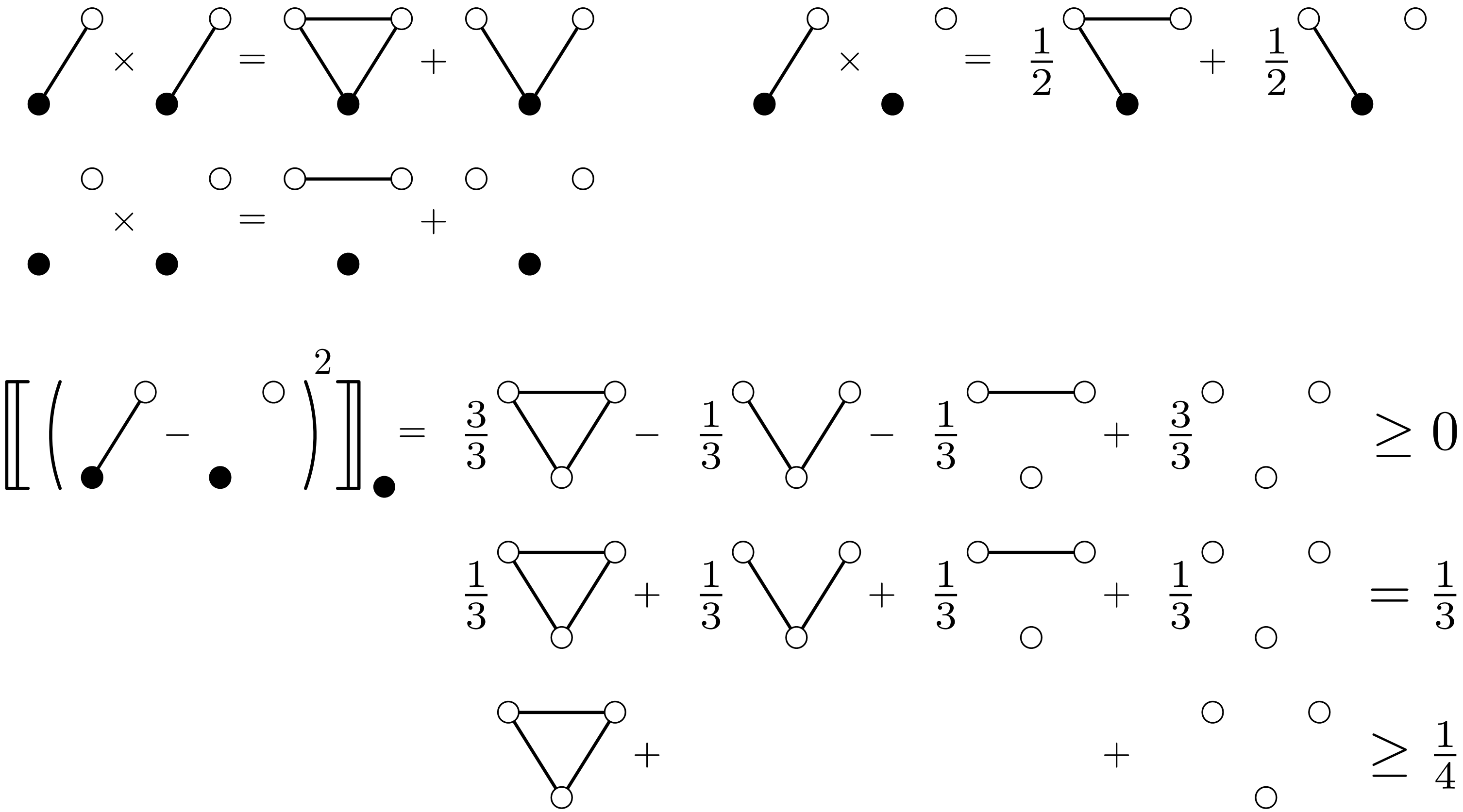
Plan for today

- Flag algebra method
  quick recap
  SDP formulation
- Limits of hypergraphs
  limit representation
- Limits of permutations
  convergence, limit representation

Flag algebras

- algebra $\mathcal{A}$ of formal linear combinations of graphs
  addition and multiplication by a scalar
- homomorphism $f_W : \mathcal{A} \to \mathbb{R}$ for a graphon W
  $f_W(\sum \alpha_i H_i) := \sum \alpha_i d(H_i, W)$
  multiplication, elements always in $\mathrm{Ker}(f_W)$
- algebra $\mathcal{A}^R$ of R-rooted graphs
  random homomorphism $f_W^R : \mathcal{A}^R \to \mathbb{R}$
  multiplication, average operator $[\![\cdot]\!]_R : \mathcal{A}^R \to \mathcal{A}$
  $\mathbb{E}_R f_W^R(x) = f_W([\![x]\!]_R)$ for every $x \in \mathcal{A}^R$

Goodman's theorem



SDP formulation

- find maximum α_0 such that $f_W(G_0) \geq \alpha_0$ if $f_W(G_i) \geq \alpha_i$ where $G_0, \ldots, G_k \in \mathcal{A}$
- What inequalities can we use?
  $f_W(G') \geq 0$ for any graph G'
  $f_W(K_1) = 1$ where K_1 expressed in n-vertex graphs
  $f_W(\llbracket x^2 \rrbracket_R) \geq 0$ for $x \in \mathcal{A}^R$
- let $H_1, \ldots, H_m$ be elements of $\mathcal{A}^R$, $h = (H_1, \ldots, H_m)$
  if $M \succeq 0$, then $f_W(\llbracket h^T M h \rrbracket_R) \geq 0$

SDP FORMULATION

- prove $f_W(G_0) \geq \alpha_0$ if $f_W(G_i) \geq \alpha_i$
- find $\gamma_i \geq 0$, $\delta_0 \in \mathbb{R}$, $\delta_i \geq 0$, $M \succeq 0$
  $G_0 = \sum_{i=1}^{k} \gamma_i G_i + \sum_{i=1}^{\ell} (\delta_0 + \delta_i) G'_i + \llbracket h^T M h \rrbracket_R$
  $\alpha_0 = \delta_0 + \sum_{i=1}^{k} \gamma_i \alpha_i$
  where $G'_1, \ldots, G'_\ell$ are all n-vert. graphs and $h \in (\mathcal{A}^R)^m$
- $\gamma_i \times f_W(G_i) \geq \gamma_i \times \alpha_i$
  $\delta_0 \times f_W(G'_1 + \cdots + G'_\ell) = \delta_0 \times 1$
  $\delta_i \times f_W(G'_i) \geq 0$
  $f_W(\llbracket h^T M h \rrbracket_R) \geq 0$

SDP EXAMPLE

- prove $f_W(\overline{K_3} + K_3) \geq \alpha_0$ for maximum α_0
- $(G'_1, \ldots, G'_4) = (\overline{K_3}, \overline{K_{1,2}}, K_{1,2}, K_3)$, $h = (\overline{K_2}^{\bullet}, K_2^{\bullet})$
- SDP: $\max \langle C, X \rangle$ s.t. $\langle A_i, X \rangle = b_i$, $X \succeq 0$, $X \in \mathbb{R}^{8\times 8}$

$$C = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad X = \begin{matrix} 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3/4 & -3/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3/4 & 3/4 \end{matrix}$$

$$A_1 = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad b_1 = 1$$

$$A_2 = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 \end{matrix} \qquad b_2 = 0$$

$$A_3 = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \end{matrix} \qquad b_3 = 0$$

$$A_4 = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \qquad b_4 = 1$$

SDP FORMULATION

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- find $\gamma_i \geq 0$, $\delta_0 \in \mathbb{R}$, $\delta_i \geq 0$, $M \succeq 0$
  $G_0 = \sum_{i=1}^{k} \gamma_i G_i + \sum_{i=1}^{\ell} (\delta_0 + \delta_i) G'_i + \llbracket h^T M h \rrbracket_R$
  $\alpha_0 = \delta_0 + \sum_{i=1}^{k} \gamma_i \alpha_i$
  where $G'_1, \ldots, G'_\ell$ are all n-vert. graphs and $h \in (\mathcal{A}^R)^m$
- SDP: $\max \langle C, X \rangle$ s.t. $\langle A_i, X \rangle = b_i$ and $X \succeq 0$
  X of size $k + 2 + \ell + m$, diagonal γ_i, $\pm\delta_0$, δ_i, M
  ℓ constraints, b_i is the coefficient of G'_i in G_0

Questions?

Hypergraph limits

- 3-uniform hypergraphs (for simplicity)
- convergence
  $d(H, G)$ = probab. $|H|$-vertex subhypergraph of G is H
  $(G_n)_{n\in\mathbb{N}}$ converges $\Leftrightarrow$ $d(H, G_n)$ converges for every H
- naïve analytic representation
  measurable function $W : [0,1]^3 \to [0,1]$
  symmetric: $W(x,y,z) = W(x,z,y) = W(y,x,z) = \cdots$
  W-random hypergraph

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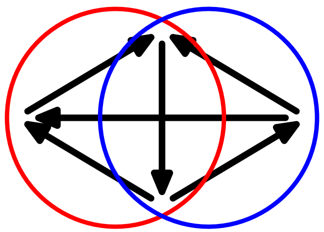
Hypergraph limits

- construct G_n as follows
  take a random directed graph on n vertices
  include an edge for each cyclically oriented triangle
- $(G_n)_{n\in\mathbb{N}}$ converges with probability one
  density uniformly 1/4 everywhere $\Rightarrow W \equiv 1/4$
- $d(K_4^{(3)}, W) = 1/256$ but every G_n is $K_4^{(3)}$-free



Hypergraph limits

- hypergraphon $W : [0,1]^6 \to [0,1]$
  $W(x,y,z,e_{xy},e_{yz},e_{xz}) = W(y,z,x,e_{yz},e_{xz},e_{xy}) = \cdots$
- sampling n-vertex W-random hypergraph
  K_n, each vertex/edge gets a random number from $[0,1]$
  ijk is an edge with prob. $W(x_i,x_j,x_k,x_{ij},x_{jk},x_{ik})$
- in the example, we define for $x < y < z$
  $W(x,y,z,\alpha,\beta,\gamma) = 1$ if $(\alpha,\beta,\gamma) \in [0,1/2)^2 \times [1/2,1]$
  $W(x,y,z,\alpha,\beta,\gamma) = 1$ if $(\alpha,\beta,\gamma) \in [1/2,1]^2 \times [0,1/2)$
  $W(x,y,z,\alpha,\beta,\gamma) = 0$ otherwise

Questions?

Permutations

- permutation of order n: order on numbers $1, \ldots, n$
  subpermutation: $4\underline{53}21\underline{6} \longrightarrow 213$
- density of a permutation π in a permutation Π:

$$d(\pi, \Pi) = \frac{\#\text{ subpermutations of } \Pi \text{ that are } \pi}{\#\text{ all subpermutations of order } \pi}$$

- $(\Pi_j)_{j \in \mathbb{N}}$ convergent if $\exists \lim\limits_{j \to \infty} d(\pi, \Pi_j)$ for every π

Representation of a limit

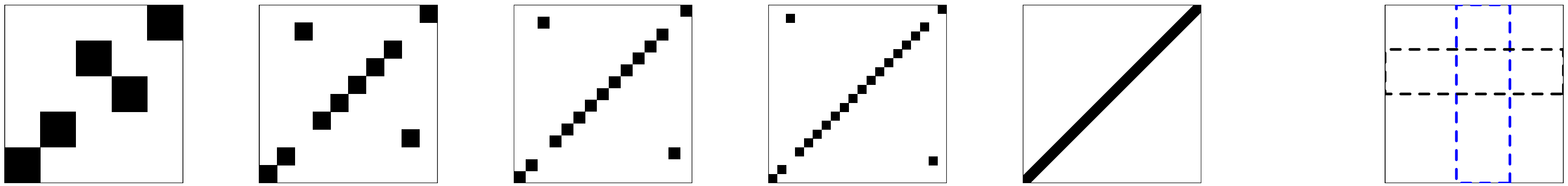
- probability measure μ on $[0,1]^2$ with unit marginals

  $\mu([a,b]\times[0,1]) = \mu([0,1]\times[a,b]) = b-a$

  Hoppen, Kohayakawa, Moreira, Ráth and Sampaio

- μ-random permutation

  choose n random points, x- and y-coordinates



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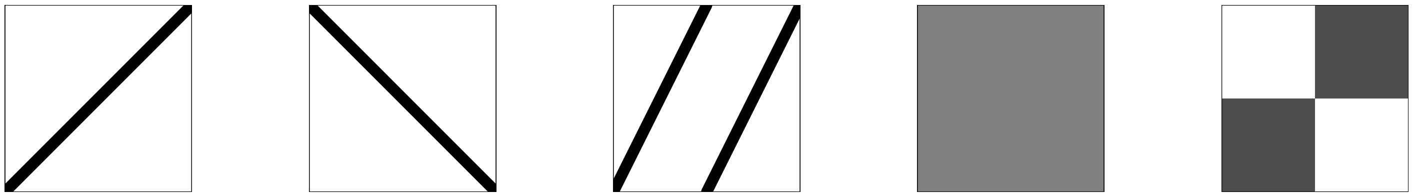

Questions?

Thank you for your attention!