Graph limits and their applications in extremal combinatorics (Part 4)

Dan Kráľ
University of Warwick

São Paulo

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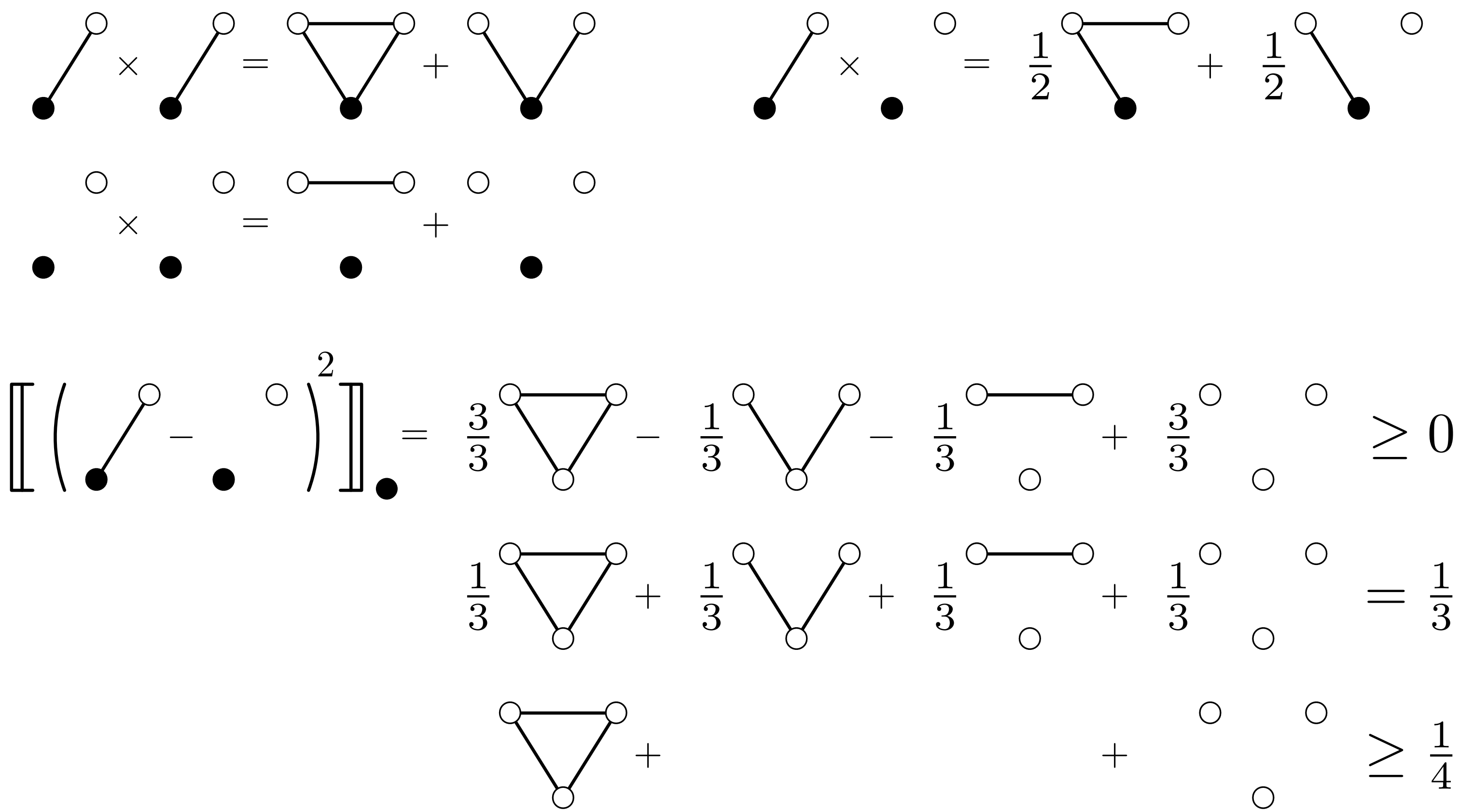
Flag algebras

- algebra $\mathcal{A}$ of formal linear combinations of graphs
  addition and multiplication by a scalar
- homomorphism $f_W : \mathcal{A} \to \mathbb{R}$ for a graphon W
  $f_W(\sum \alpha_i H_i) := \sum \alpha_i d(H_i, W)$
  multiplication, elements always in $\mathrm{Ker}(f_W)$
- algebra $\mathcal{A}^R$ of R-rooted graphs
  random homomorphism $f_W^R : \mathcal{A}^R \to \mathbb{R}$
  multiplication, average operator $[\![\cdot]\!]_R : \mathcal{A}^R \to \mathcal{A}$
  $\mathbb{E}_R f_W^R(x) = f_W([\![x]\!]_R)$ for every $x \in \mathcal{A}^R$

Computing with flags

- simple applications yields results such as
  $f_W(K_2) > 1/2 \Rightarrow f_W(K_3) > 0$ for every W
  $f_W(K_3 + \overline{K_3}) \geq 1/4$ for every W
- shorthand notation for $x, y \in \mathcal{A}$
  $x = y \Leftrightarrow \forall W \;\; f_W(x) = f_W(y)$
  $x \geq 0 \Leftrightarrow \forall W \;\; f_W(x) \geq 0$
- What can we use in computations?
  $x^2 \geq 0$ for every $x \in \mathcal{A}$
  $[\![x^2]\!]_R \geq 0$ for every $x \in \mathcal{A}^R$

Goodman's theorem



Questions?

Finitely forcible graphons

- a graphon W is finitely forcible if there exist $H_1, \ldots, H_k$ and $d_1, \ldots, d_k$ such that W is the only graphon with the expected density of H_i equal to d_i
- $W \equiv p$ is forced by densities of 4-vertex subgraphs
- Lovász, Sós (2008): Step graphons are finitely forcible.

Link to extremal combinatorics

- suppose W is forced by $d(H_i, W) = d_i$
  density calculation: $\sum(H_i - d_i)^2 = \sum \alpha_j H'_j$
  W is the unique minimizer of $\sum \alpha_j d(H'_j, W)$
- Conjecture (Lovász and Szegedy, 2011):
  Every extremal problem $\min \sum \alpha_j d(H_j, W)$
  has a finitely forcible optimal solution.
- Conjecture (Lovász and Szegedy, 2011):
  Every finitely forcible graphons has polynomial-size
  weak ε-regular partitions.

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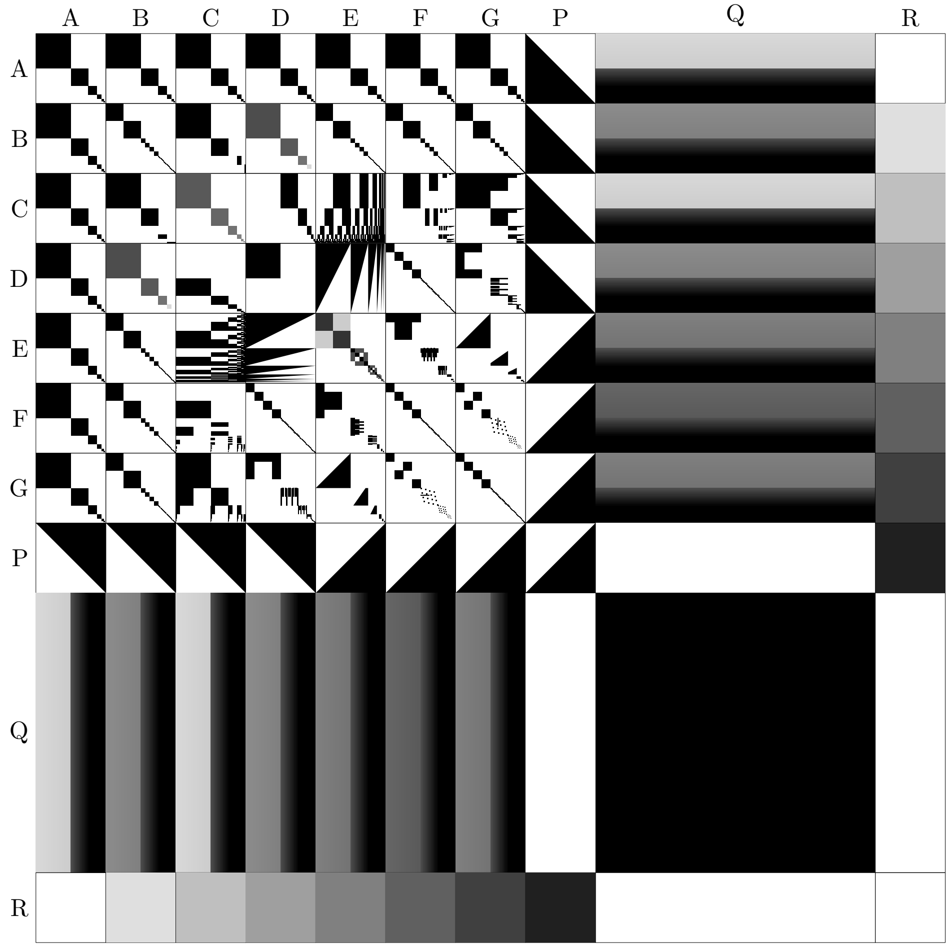
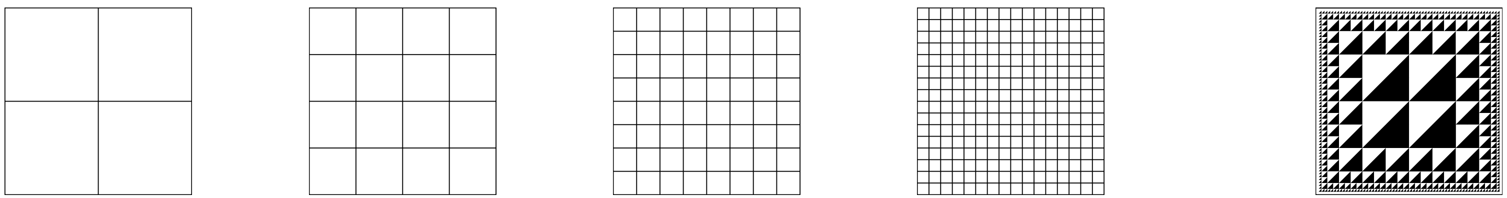
Non-regular graphon



Finitely forcible graphons

- Theorem (Cooper, K., Lopes Martins)
  If W is a computable graphon, then there exists a finitely forcibly graphon W_0 such that $W_0(x/100, y/100) = W(x, y)$ for every $(x, y) \in [0, 1]^2$.
- methods developed by Glebov, Klimošová, K., Volec
  non-regular graphon due to Cooper, Kaiser, K., Noel



Step graphons

- Lovász, Sós (2008): Step graphons are finitely forcible.
- step graphon W: k parts, i.e. $[0,1] = J_1 \dot\cup \cdots \dot\cup J_k$
  W is equal to D_{ij} on every $J_i \times J_j$, $i,j \in [k]$
- we prove the theorem under two additional assumptions
  all parts have the same size, say $J_i = [(i-1)/k, i/k)$
  $D_{i1} + \cdots + D_{ik} \neq D_{i'1} + \cdots + D_{i'k}$ for $i \neq i'$

Step graphons

- graphon W_0, k parts, equal to D_{ij} on $J_i \times J_j$
  $f_W(H) = f_{W_0}(H)$ for all $(8k+4)$-vert. $H \Rightarrow W = W_0$
- degree in the i-th part: $\delta_i = (D_{i1} + \cdots + D_{ik})/k$
  we have assumed that $\delta_i \neq \delta_j$ for $i \neq j$
- let $e \in \mathcal{A}^\bullet$ be the rooted K_2
  $\mathbb{E}_x f_W^x \left(\prod_{i=1}^{k} (e - \delta_i)^2 \right) = 0 \Rightarrow f_W^x(e) \in \{\delta_1, \ldots, \delta_k\}$ a.s.
  $\int W(x,z) \mathrm{d}z \in \{\delta_1, \ldots, \delta_k\}$ for a.e. $x \in [0,1]$

Step graphons

- graphon W_0, k parts, equal to D_{ij} on $J_i \times J_j$
  $f_W(H) = f_{W_0}(H)$ for all $(8k+4)$-vert. $H \Rightarrow W = W_0$
- $f_W^x(e) \in \{\delta_1, \ldots, \delta_k\}$ for a.e. $x \in [0,1]$
  $J_i = \{x \text{ s.t. } f_W^x(e) = \delta_i\}$
- $\mathbb{E}_x f_W^x \left(\prod_{i=1, i\neq j}^{k} (e - \delta_i)^2 \right) = \frac{1}{k} \prod_{i=1, i\neq j}^{k} (\delta_j - \delta_i)^2 \Rightarrow$
  the measure of J_i is $1/k$, w.l.o.g. $J_i = [(i-1)/k, i/k)$
- $f_W^x(e) = f_{W_0}^x(e)$ for a.e. $x \in [0,1]$

Step graphons

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  $f_W(H) = f_{W_0}(H)$ for all $(8k+4)$-vert. $H \Rightarrow W = W_0$
  $f_W^x(e) = f_{W_0}^x(e)$ for a.e. $x \in [0,1]$
- 4-vertex decorated graph H
  the i-th vertex must belong to part ℓ_i
- the density $d(H,W)$ is equal to normalization factor $\times$

$$\int\limits_{J_{\ell_1} \times \cdots \times J_{\ell_4}} \prod_{v_i v_j} W(x_i, x_j) \prod_{\overline{v_i v_j}} (1 - W(x_i, x_j)) \, \mathrm{d}x_1 \cdots x_4$$

Step graphons

- graphon W_0, k parts, equal to D_{ij} on $J_i \times J_j$
  $f_W(H) = f_{W_0}(H)$ for all $(8k+4)$-vert. $H \Rightarrow W = W_0$
  $f_W^x(e) = f_{W_0}^x(e)$ for a.e. $x \in [0,1]$
- 4-vertex decorated graph H with $\ell_1, \ldots, \ell_4$
- $H_0 =$ the graph H with all vertices roots
  $H_i =$ sum of 5-vert. H_0-rooted graphs with the i-th root adjacent to the non-root vertex
- $d(H,W) = \dfrac{\mathbb{E}_{H_0} f_W^{H_0}\left(\prod_{i=1}^{4} \prod_{j=1, j\neq\ell_i}^{k} (H_i - \delta_j)^2\right)}{\prod_{i=1}^{4} \prod_{j=1, j\neq\ell_i}^{k} (\delta_{\ell_i} - \delta_j)^2}$

Questions?