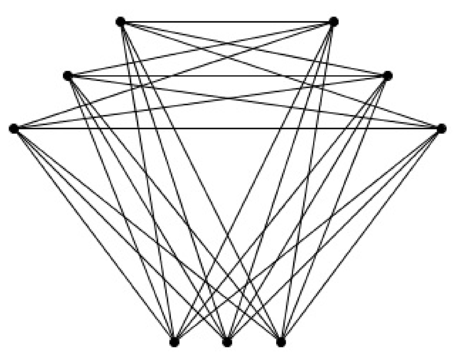
# Advances in a Hypergraph Coloring Conjecture


**Universidade Federal do Rio Grande do Sul**

Author: Lucas de Oliveira Contiero (Advisor: Carlos Hoppen)


## 1. General description and main goals

This research project deals with extremal problems for discrete structures, especially graphs and hypergraphs. Most extremal problems in graph theory consist of determining graphs that are maximal or minimal with a prescribed structure among all graphs with a given number of vertices. For instance, a simple problem in this direction is to decide, among all triangle-free $n$-vertex graphs, which is the graph with the maximum number of edges. This question was answered by Mantel in 1907 and was addressed in a general framework by Turán, who is widely considered as the father of Extremal Combinatorics. For any fixed graph $F$, we say that a graph $G$ is $F$-free if it does not contain $F$ as a subgraph. Finding the maximum number of edges $\mathrm{ex}(n, F)$ among all $F$-free $n$-vertex graphs, and determining the class of $n$-vertex graphs that achieve this number is known as the *Turán problem* associated with $F$. The figure below depicts the Turán $K_4$-free graph with $9$ vertices.



A hypergraph $H = (V, E)$ consists of an arbitrary finite set of vertices $V$, and of a set of hyperedges $E \subseteq 2^V$. A hypergraph is called $k$-uniform if $E \subseteq \binom{V}{k}$. An extremal result in hypergraphs has been proved in 1961 by Erdős, Ko and Rado. Their theorem says that, if $n \geq 2k$, the $n$-vertex $k$-uniform hypergraph $H = (V, E)$ with maximum number of edges satisfying $|f \cap g| \geq 1$, for every $f, g \in E$ is isomorphic to the hypergraph $S(1) = (V, E)$, where $E = \{F \in \binom{V}{k} : 1 \in F\}$. More generally, if $n$ is large enough in terms of $k$ and $t$, the $n$-vertex $k$-uniform hypergraph $H = (V, E)$ with maximum number of edges satisfying $|f \cap g| \geq t$, for every $f, g \in E$ is isomorphic to the hypergraph $S(t) = (V, E)$, where $E = \{F \in \binom{V}{k} : [t] \subseteq F\}$.

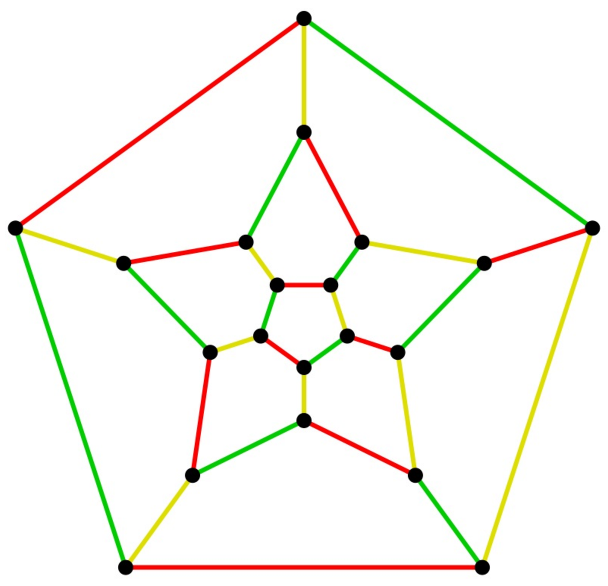
## 2. Coloring problems

In connection with a classical question of Erdős and Rothschild, we investigate the following variant of the Turán problem. Instead of looking for $F$-free $n$-vertex graphs, they were interested in *edge colorings* of graphs on $n$ vertices such that *every color class is $F$-free*. More precisely, given an integer $q \geq 1$ and a graph $F$, one considers the function that associates, with a graph $G$, the number $c_{q,F}(G)$ of $q$-colorings of the edge set of $G$ for which there is no monochromatic copy of $F$. The problem consists of finding $c_{q,F}(n)$, the maximum of $c_{q,F}(G)$ over all $n$-vertex graphs $G$. The main motivation for considering this function is its connection with the Turán number:

$$q^{\mathrm{ex}(n,F)} \leq c_{q,F}(n) \leq q^{q\,\mathrm{ex}(n,F)}.$$

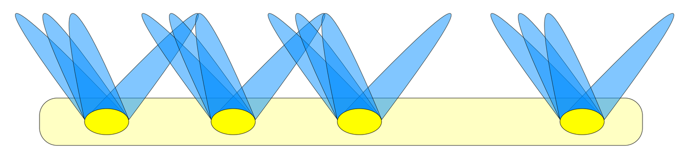
Erdős and Rothschild were interested in instances for which the first inequality is tight. The work of important combinatorialists including Alon, Balogh, Keevash, Pikhurko, Sudakov and Yuster showed that this is typically the case when $q \in \{2, 3\}$, but never the case for $q \geq 4$. However, very few extremal graphs are known in the latter case.

More recently, Balogh considered a generalization of this problem by looking at $q$-colorings of the edge set of a graph $G$ that do not contain a copy of $F$ colored *according to a fixed pattern*. He described the extremal graphs in the case of 2-colorings of complete graphs $F = K_\ell$. The figure below depicts a $K_3$-free $3$-coloring of a graph (which in this case is a $K_3$-free graph).



## 3. The coloring problem related with the Erdős-Ko-Rado Theorem

A $(q, t)$-coloring of a $k$-uniform hypergraph $H$ with $n$ vertices is a function associating every hyperedge of $H$ with a color in $[q]$ satisfying that hyperedges with the same color must have intersection of size at least $t$. We define $\kappa_{q,t}(H)$ as the number of $(q, t)$-colorings of $H$ and $\mathrm{KC}_{n,k,q,t}$ is the maximum number of $(q, t)$-colorings that some $k$-uniform hypergraph with $n$ vertices can assume. For a cover set $C = \{t_1, \ldots, t_c\} \subseteq \binom{[n]}{t}$, let $H = ([n], \mathcal{F}_C)$ be the $(C, k)$-*complete hypergraph*, where $\mathcal{F}_C(n, k) = \{e \in \binom{[n]}{k} : \exists i \in [c]$ for which $t_i \subseteq e\}$. The figure below depicts this hypergraph $H$



Hoppen, Kohayakawa and Lefmann proved the following theorem.

**Theorem 1 (Hoppen-Kohayakawa-Lefmann, 2012)** *Given positive integers $k, q$ and $t$, there is $n_0 > 0$ such that, for all $n > n_0$, the equality $\kappa_{q,t}(H) = \mathrm{KC}(n, k, q, t)$ implies that $H$ is isomorphic to $H = ([n], \mathcal{F}_C)$, with $C = \{t_1, t_2, \ldots, t_{c(q)}\}$, $|C| = c(q) = \lceil \frac{q}{3} \rceil$, and:*

*(a) If $q \in \{2, 3\}$ or if $q \geq 5$ and $k \geq 2t - 1$, then the sets from $C$ are mutually disjoint.*

*(b) If $q = 4$, then $|t_1 \cap t_2| = t - 1$.*

*(c) If $q \geq 5$ and $k < 2t - 1$, then $|t_i \cup t_j| > k$, for all $1 \leq i < j \leq c(q)$.*

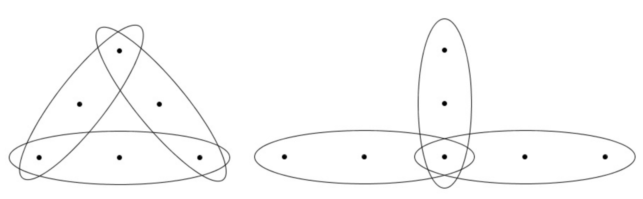
***Remark.*** *In case (a), for $q = 2$ the hypergraph in Theorem 1 is optimum, for every $n \geq k$.*

Note that, as in the graph case, for $q \in \{2, 3\}$ and a large enough $n$, the optimum hypergraph is the extremal hypergraph for the Erdős-Ko-Rado Theorem. Moreover, for $q \geq 4$, this hypergraph is far from being optimum. Also, note that case (c) in theorem does not say precisely which hypergraph is optimum, since there are many hypergraphs satisfying this property. The authors proposed a conjecture about additional properties satisfied by the optimum hypergraph in this case.

## 4. The HKL-Conjecture and advances on it

**Conjecture 2 (HKL-Conjecture)** *If $q \geq 5$, $k$ and $t$ are positive integers with $t < k < 2t - 1$, then there is $n_0 > 0$, such that, for $n > n_0$, a hypergraph $H = ([n], \mathcal{F}_C)$ which satisfies $\kappa_{q,t}(H) = \mathrm{KC}(n, k, q, t)$ also satisfies $|t_i \cup t_j| = k + 1$ for any disjoint $t_i, t_j \in C$.*

Note that, even if the HKL-Conjecture were true, we would not know precisely which hypergraphs are optimum in every case, because we still have many non-isomorphic hypergraphs satisfying the HKL-Conjecture. The figure below depicts two possible configurations satisfying the HKL-Conjecture for $k = 4$ and $t = 3$.



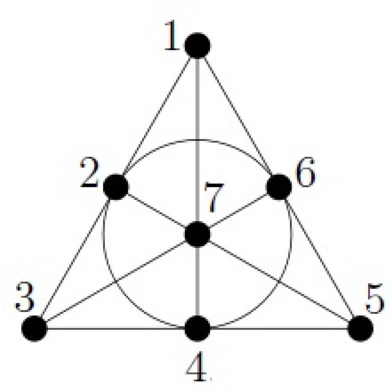
**Theorem 3** *The HKL-Conjecture is true for $q \leq 9$.*

To prove this result we start with a new hypergraph $H = ([n], \mathcal{F}_C)$, where $C$ satisfies item (c) in the theorem. We assume that $C$ does not satisfy the HKL-Conjecture. We create a shifting function $\varphi$, adapted from the proof of the Erdős-Ko-Rado Theorem, which associates $H = ([n], \mathcal{F}_C)$ with a new hypergraph $\widehat{H} = ([n], \mathcal{F}_{\widehat{C}})$. Then, we prove that $\kappa_{q,t}(H) < \kappa_{q,t}(\widehat{H})$ using another function $R$ that associates colorings of $H$ with colorings of $\widehat{H}$ in an injective, but not surjective way. Unfortunately, our approach does not prove the HKL-conjecture in general, since our shifting function cannot be applied to arbitrary hypergraphs.

Note that, since HKL-Conjecture is true for $q \leq 9$, we have found the optimal hypergraph when $q \in \{5, 6\}$, because there is only one configuration for the cover $C$ in this case.

## 5. Another problem

We consider the well known $3$-uniform hypergraph *Fano plane*, which is illustrated below.




It contains seven vertices and seven hyperedges (in the figure, six lines plus the circle) satisfying that every two hyperedges have intersection exactly one and every two vertices are contained in some hyperedge.

Given a number $q$ of colors and a hypergraph $F$, a pattern $P$ of $q$ colors of $F$ is a partition of its hyperedge set into at most $q$ classes. A hyperedge-coloring of a host hypergraph $H$ is said to be $(F, P)$-free if $H$ does not contain a copy of $F$ in which the partition of the hyperedge set induced by the coloring is isomorphic to $P$. An example of problem in this direction has been solved by Lefmann, Person, Rödl and Schacht. Their theorem says that, if $q \in \{2, 3\}$, $F$ is the Fano plane and $P$ is the monochromatic pattern (all hyperedges in the same class), then the hypergraph with the maximum number of $(F, P)$-free $q$-colorings is the balanced complete bipartite hypergraph $B_n$, which is also the hypergraph with the maximum number of hyperedges not containing a Fano plane. The problem that we are studying is similar, but considering $R$ as the rainbow pattern, where any two hyperedges are assigned different colors.

## 6. References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