The Local Stability Method and Its Applications: New Hypergraph Turán Numbers

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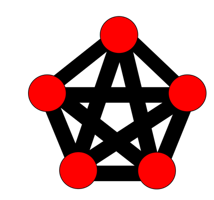
Background

For any r-uniform graph (shortly, r-graph) G, the **Turán number** is

$$\mathrm{ex}(n, G) = \max\{\mathrm{e}(F) : \mathrm{v}(F) = n \text{ and } G \not\subseteq F\}$$

and the **Turán density** is $\pi(G) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,G)}{\binom{n}{r}}$.

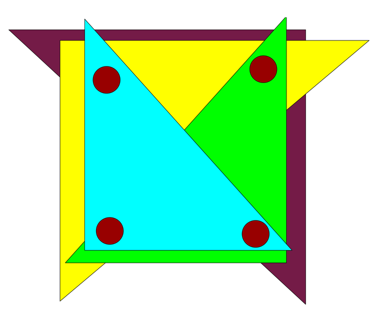
Theorem [Turán, 1941] : For every $t \geq 3$, $\mathrm{ex}(n, K_{t+1}) = \left(1 - \frac{1}{t}\right)\frac{n^2}{2}$ and the unique extremal graph is the balanced blowup of K_t (this graph is known as Turán graph and denoted by $T_t(n)$).



$T_5(n)$ is the largest K_6-free graph

Theorem [Erdős-Stone, Simonovits, 1946]: If $\mathcal{F}$ is a finite family of graphs with $\min_{F\in\mathcal{F}} \chi(F) = t$, then $\mathrm{ex}(n, \mathcal{F}) = \left(1 - \frac{1}{t-1}\right)\binom{n}{2} + o(n^2)$.

Note that this theorem determines the asymptotic behaviour of Turán numbers for all families of graphs except the ones containing some bipartite graph, but for hypergraphs similar problems become much harder!

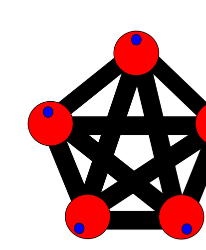


$K_4^{(3)}$

Question: [Turán, 1961] What is $\mathrm{ex}(n, K_t^{(r)})$, where $K_t^{(r)}$ is the complete r-graph on t vertices? (*None of these numbers are known even assymptotically for any* $t > r \geq 3$).

The (Classical) Stability Method

Stability Theorem for Cliques: [Erdős, Simonovits, 1966] For every $\varepsilon > 0$ there exists $\delta > 0$ such that if n is sufficiently large and G is a K_{t+1}-free graph on n vertices with at least $(1-\delta)\,\mathrm{ex}(n, K_{t+1})$ edges then $|G \triangle T_t(n)| \leq \varepsilon n^2$.



"Almost" largest K_6-free graph

For any r-graph G, typically we want to determine $\pi(G)$ (**density result**) and $\mathrm{ex}(n,G)$ (**exact result**). The stability method is a tool to derive the second from the first. The classical scenario is to prove an approximate structure theorem for graphs with density close to the maximum possible and then to find the exact structure that has maximum size among the approximate structures. One of the first applications of this method for hypergraphs was determining the Turán number of the the Fano plane (independently by Keevash, Sudakov [2005] and Füredi, Simonovits [2004]).

Our Stability

The Setting: Given two families of r-graphs - $\mathcal{F}$ and $\mathcal{H}$ (one should think of $\mathcal{F}$ being the family of all G-free r-graphs for some r-graph G and $\mathcal{H}$ being the family of (conjectured) extremal examples), we define

- the **edit distance** between a graph F and the family $\mathcal{H}$ as $d_{\mathcal{H}}(F) := \min_{\substack{H\in\mathcal{H} \\ \mathrm{v}(F)=\mathrm{v}(H)}} |F \triangle H|$
- $m(\mathcal{H}, n) := \max_{\substack{H\in\mathcal{H}, \\ \mathrm{v}(H)=n}} |H|$

Our goal: If $F \in \mathcal{F}$ and has maximum number of edges $\Rightarrow F \in \mathcal{H}$.

Classical Stability: $\mathcal{F}$ is $\mathcal{H}$-**stable** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all sufficiently large n if $F \in \mathcal{F}$ with $\mathrm{v}(F) = n$ then

$$|F| \geq m(\mathcal{H}, n) - \delta n^r \Rightarrow d_{\mathcal{H}}(F) \leq \varepsilon n^r.$$

Example: For Erdős-Simonovits stability theorem, let $\mathcal{F}$ be the family of all K_{t+1}-free graphs and $\mathcal{H}$ be the family of all blowups of K_t.

Our Stability: $\mathcal{F}$ is $\mathcal{H}$-**stable** if there exists some $\alpha > 0$ such that for all sufficiently large n if $F \in \mathcal{F}$ with $\mathrm{v}(F) = n$ then

$$d_{\mathcal{H}}(F) \leq \frac{m(\mathcal{H}, n) - |F|}{\alpha}.$$

Remark: Note that if $\mathcal{F}$ is $\mathcal{H}$-stable using our notion then $m(\mathcal{H}, n) \geq m(\mathcal{F}, n)$ for sufficiently large n.

Local Stability: $\mathcal{F}$ is $\mathcal{H}$-**locally stable** if there exists some $\varepsilon > 0$ such that the subfamily $\mathcal{F}' = \{F \in \mathcal{F} : d_{\mathcal{H}}(F) \leq \varepsilon n^r\}$ is $\mathcal{H}$-stable.

The Lagrangian function

Given an r-graph F, let μ be any probability distribution on $V(F)$, we call the pair (F, μ) a **weighted graph**. The **edge density** of the weighted graph (F, μ) is $\lambda(F, \mu) := \sum_{e\in F}\prod_{v\in e}\mu(v)$. The **Lagrangian** of an r-graph F is $\lambda(F) := \max_{\mu} \lambda(F, \mu)$. For a family of r-graphs, $\mathcal{F}$, $\lambda(\mathcal{F}) := \sup_{F\in\mathcal{F}} \lambda(F)$.

Example: $\lambda(K_t) = \binom{t}{2}\frac{1}{t^2} = \frac{1}{2}\left(1 - \frac{1}{t}\right)$.

Note that

1. $|F| \leq \lambda(F) n^r$,
2. If μ^* is such that $\lambda(F) = \lambda(F, \mu^*)$ and $\mathrm{supp}(\mu^*) := \{v : \mu^*(v) > 0\}$ is minimal, then $F[\mathrm{supp}(\mu^*)]$ **covers pairs**, that is, every pair of vertices is contained in an edge.
3. $\pi(F) = r!\sup_G \lambda(G)$, where supremum is taken overall F-**hom-free** G (i.e. there is no homomorphism from F to G).

Remark:

- (2) implies that for any 2-graph G, $\lambda(G) = \frac{1}{2}\left(1 - \frac{1}{t-1}\right)$, where $t := \min\{r : K_r \notin G\}$. Thus, (1) + (2) together $\Rightarrow$ Turán's theorem.
- (2)+(3) together $\Rightarrow$ Erdős-Stone, Simonovits Theorem because for any graph G, K_t is G-hom-free if and only if $\chi(G) > t$.

Stability in Weighted Setting: We say that $\mathcal{F}$ is $\mathcal{H}$-**weight-stable** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\lambda(F, \mu) \geq \lambda(\mathcal{H}) - \delta \Rightarrow d_{\mathcal{H}}(F, \mu) \leq \varepsilon.$$

Example: Let $\mathcal{F} = \{K_1, K_2, \ldots, K_t\}$ and $\mathcal{H}$ be the family of all blowups of K_t, then it is easy to see that $\mathcal{F}$ is $\mathcal{H}$-weight-stable.

Local Stability $\Rightarrow$ Global Stability

Theorem: If both $\mathcal{F}$ and $\mathcal{H}$ are clonable (i.e. closed under taking blowups) and

- $\mathcal{F}$ is $\mathcal{H}$-locally stable,
- the subfamily $\mathcal{F}^* \subseteq \mathcal{F}$ of graphs which cover pairs is $\mathcal{H}$-weight-stable ,

then $\mathcal{F}$ is $\mathcal{H}$-stable.

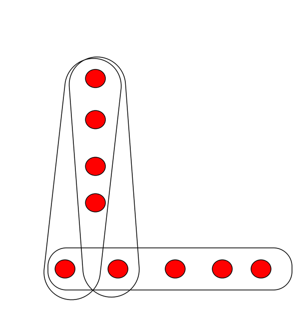
Remark:

- The idea of the above theorem comes from so-called symmetrization argument, pioneered independently by Zykov and Sidorenko.
- In fact, it is enough to prove local stability for graphs in $\mathcal{F}$ where degrees of vertices are "large enough".
- It is possible to remove the restriction $\mathcal{F}$ being clonable.

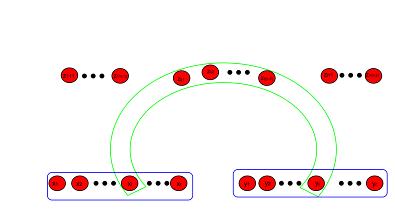
Applications

The **extension** of an r-graph F is an r-graph obtained by adding new $(r-2)$ vertices for each uncovered pair in F and putting it in an edge with this pair.

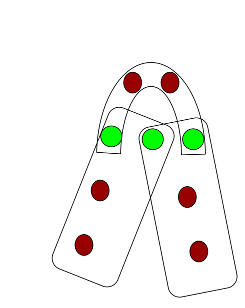
1. The extension of two r-edges sharing $(r-1)$ vertices (known as the Generalized Triangle), for $r = 5, 6$.
   - **Density result:** Frankl, Füredi [1989]
   - **Previous results:**
     $r = 2$ - Mantel [1907]
     $r = 3$ - Bollobás [1974] (density result) and Frankl, Füredi [1983] (exact result)
     $r = 4$ - Sidorenko [1987] (density result) and Pikhurko[2008] (exact result)



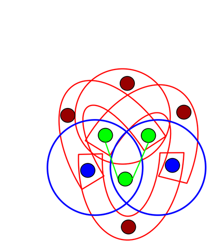
2. The extension of a matching of size two, for $r \geq 4$.
   - **Density result:** Bene Watts, Norin, Y. [2016+]
   - **Previous results:** $r = 3$ - Hefetz and Keevash [2013] (density and exact results)



3. The extension of t isolated vertices and a generalization of these graphs, for all $t > r \geq 3$.
   - **Density result:** Keevash [2011]
   - **Previous results:** Mubayi [2006] (density result) and Pikhurko [2013] (exact result) for the extension of t isolated vertices



4. The extension of r-*expansions* of sufficiently large trees that satisfy Erdős-Sós Conjecture, for all $r \geq 2$.
   - **Density result:** Sidorenko [1989]



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