# Recent Progress in Approximation Algorithms for the Traveling Salesman Problem

Lecture 4: *s*-*t* path TSP for graph TSP

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# Improvements

No improvement on Hoogeveen's algorithm for $s$-$t$ path TSP, until just the last few years.

| | | |
|---|---|---|
| An, Kleinberg, Shmoys | 2012 | 1.618 |
| Sebő | 2013 | 1.6 |
| Vygen | 2015 | 1.599 |
| Gottschalk and Vygen | 2015 | 1.56 |
| Sebő and Van Zuylen | 2016 | 1.52 |

Today: Look at the case of graph TSP instances (e.g. input is undirected graph, cost $c(i,j)$ is number of edges in shortest $i$-$j$ path)

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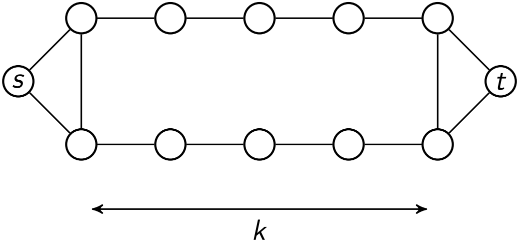
Note that we have shown $\mu \leq 1.618$, since we find a tour of cost at most $1.618 \cdot OPT_{LP}$.

# Integrality Gap

We can show a lower bound on the integrality gap using an instance of *graph TSP*: input is a graph $G = (V, E)$, cost $c_e$ for $e = (i, j)$ is number of edges in a shortest $i$-$j$ path in $G$.



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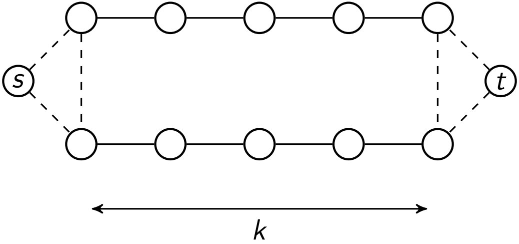


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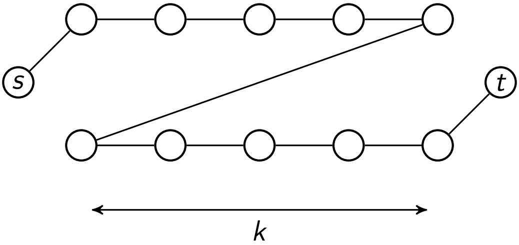


$$OPT \approx 3k$$

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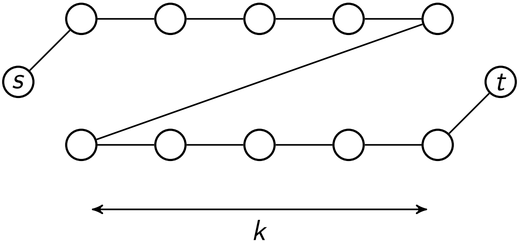



$$\frac{OPT}{OPT_{LP}} \rightarrow \frac{3}{2} \text{ as } k \rightarrow \infty$$

# Graph Instances

Sebő and Vygen (2014) show that for graph TSP instances of $s$-$t$ path TSP, can get a $\frac{3}{2}$-approximation algorithm (i.e. the algorithm produces a solution of cost at most $\frac{3}{2}OPT_{LP}$), so the integrality gap is tight for these instances.

We'll present a simplified version of this result due to Gao (2013).

# Graph Instances

Given the input graph $G = (V, E)$ and an optimal solution, can replace any edge $(i, j)$ in the optimal solution with the $i$-$j$ path in $G$ since these have the same cost.

So finding an optimal solution is equivalent to finding a multiset $F$ of edges such that $(V, F)$ is connected, $deg_F(s)$ and $deg_F(t)$ are odd, $deg_F(v)$ is even for all $v \in V - \{s, t\}$, and $|F|$ is minimum.

# LP Relaxation

$$\text{Min} \quad \sum_{e \in E} x(e)$$

subject to:

$$x(\delta(S)) \geq \begin{cases} 1, & |S \cap \{s,t\}| = 1, \\ 2, & |S \cap \{s,t\}| \neq 1, \end{cases}$$

$$x(e) \geq 0, \qquad \forall e \in E.$$

Let $x^*$ be an optimal LP solution.

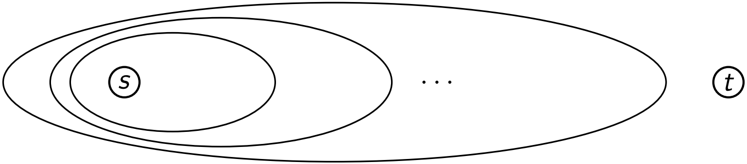
# Narrow Cuts

As before, focus on *narrow cuts* $S$ such that $x^*(\delta(S)) < 2$ (i.e. a $\tau$-narrow cut for $\tau = 1$). Recall:

**Theorem (An, Kleinberg, Shmoys (2012))**

*If $S_1, S_2$ are narrow cuts, $S_1 \neq S_2$, then either $S_1 \subset S_2$ or $S_2 \subset S_1$.*

So the narrow cuts look like $s \in S_1 \subset S_2 \subset \cdots \subset S_k \subset V$.

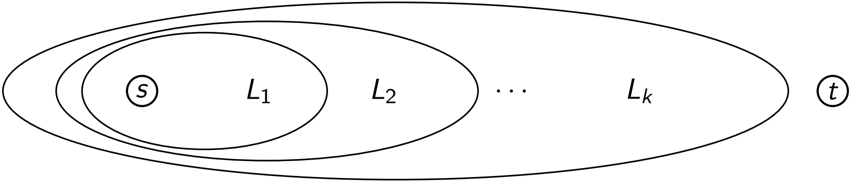



Let $S_0 \equiv \emptyset$, $S_{k+1} \equiv V$, $L_i \equiv S_i - S_{i-1}$.

# Key Idea

Find a tree spanning $L_i$ in the support of $x^*$ for each $i$. Connect each of these via a single edge from $L_i$ to $L_{i+1}$. Let $F$ be the resulting tree, $T$ the vertices in $F$ whose parity needs changing.

Then $|F| = n - 1$ and $|\delta(S_i) \cap F| = 1$ for each narrow cut $S_i$.



# Key Lemma

Recall:

**Lemma**

*Let $S$ be an odd set. If $|S \cap \{s,t\}| = 1$, then $|F \cap \delta(S)|$ is even.*

$$
\begin{array}{lll}
\text{Min} & \displaystyle\sum_{e \in E} c(e)x(e) & \\
\text{subject to:} & x(\delta(S)) \geq 1, & \forall S \subseteq V, |S \cap T| \text{ odd} \\
 & x(e) \geq 0, & \forall e \in E.
\end{array}
$$

**Lemma**

*$y = \frac{1}{2}x^*$ is feasible for the the $T$-join LP.*

# Gao (2013)

**Theorem (Gao (2013))**

*For spanning tree $F$ constructed by the algorithm, let $J$ be a minimum-cost $T$-join. Then $c(F \cup J) \le \frac{3}{2} OPT_{LP}$.*

$$\text{Min} \quad \sum_{e \in E} x(e)$$

subject to:

$$x(\delta(S)) \ge \begin{cases} 1, & |S \cap \{s,t\}| = 1, \\ 2, & |S \cap \{s,t\}| \neq 1, \end{cases}$$

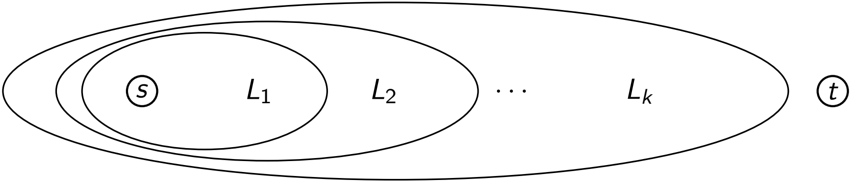
$$x(e) \ge 0, \qquad \forall e \in E.$$

# Last Lemma

Let $E(x^*) = \{e \in E : x^*(e) > 0\}$ be the *support* of LP solution $x^*$, $H = (V, E(x^*))$ the support graph of $x^*$, $H(S)$ the graph induced by a set $S$ of vertices.

**Lemma (Gao (2013))**

*For $1 \leq p \leq q \leq k+1$, $H\left(\bigcup_{p \leq i \leq q} L_i\right)$ is connected.*



# The Big Question

Is there a $\frac{3}{2}$-approx. alg. for $s$-$t$ path TSP for general costs?

# One Idea

Idea: Construct a spanning tree $F$ just as in Gao's algorithm for the graph case. Then again $y = \frac{1}{2}x^*$ is feasible for the $T$-join LP, and the overall cost of $F$ plus the $T$-join is at most $c(F) + \frac{1}{2}\sum_{e \in E} c(e)x^*(e)$.

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Idea: Construct a spanning tree $F$ just as in Gao's algorithm for the graph case. Then again $y = \frac{1}{2}x^*$ is feasible for the $T$-join LP, and the overall cost of $F$ plus the $T$-join is at most $c(F) + \frac{1}{2}\sum_{e\in E} c(e)x^*(e)$.

Problem: Not clear how to bound the cost of $F$. Gao (2014) has an example showing that $F$ can have cost greater than $OPT_{LP}$.

# Further directions

Best-of-Many Christofides from An et al. works with any possible decomposition of the LP solution into spanning trees. Recent improvements of Vygen (2015), Gottschalk and Vygen (2015), and Sebő and Van Zuylen (2016) all use decompositions that have particular properties.

In particular, the last two use a decomposition that gives an ordering on trees such that at every narrow cut $Q$, the first $2 - x^*(Q)$ fraction of trees in the ordering have exactly one edge in $\delta(Q)$.

# More ideas

Sebő and Van Zuylen (2016) also (like Mömke and Svensson) use an idea in which edges are sometimes removed from the tree in hopes that the $T$-join will connect the two parts together again.

# The Big Open Questions

- Beat $\frac{3}{2}$ for TSP
- Achieve $\frac{3}{2}$ for $s$-$t$ path TSP
- Achieve $\frac{4}{3}$ for graph TSP