Recent Progress in Approximation Algorithms for the Traveling Salesman Problem

Lecture 2: Graph TSP

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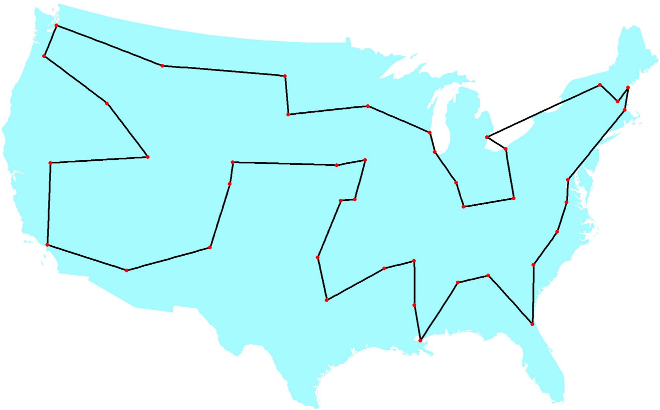
US 42 cities

From Bill Cook's site at the University of Waterloo (www.math.uwaterloo.ca/tsp)

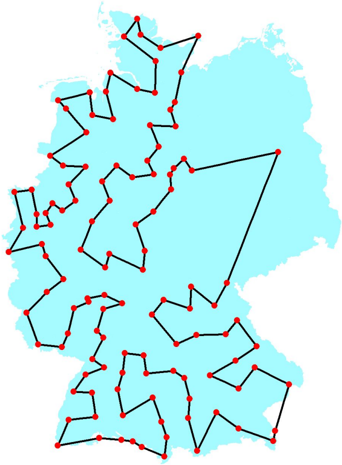


Solved by Dantzig, Fulkerson, and Johnson (1954) using linear programming.

US 33 cities

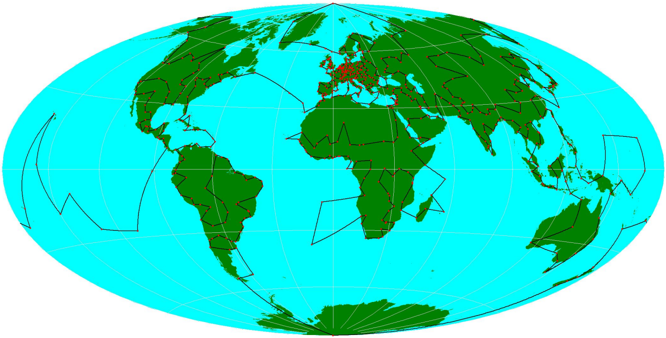
Proctor and Gamble contest in 1962.

Germany 120 cities



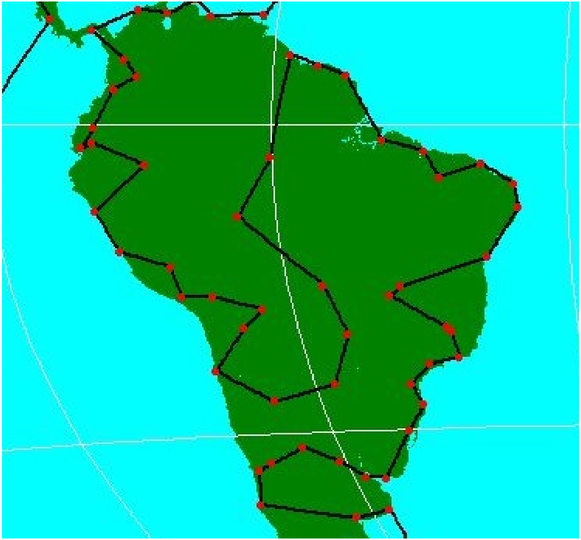
Solved by Grötschel (1977).

World tour 666 cities

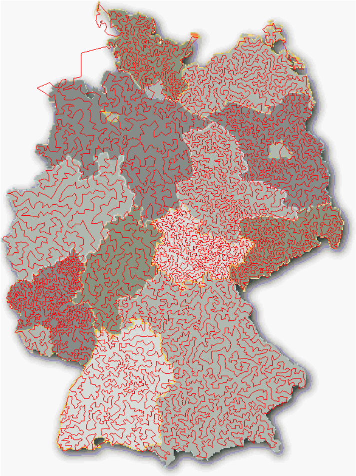


Solved by Grötschel and Holland (1987).

Brazil closeup

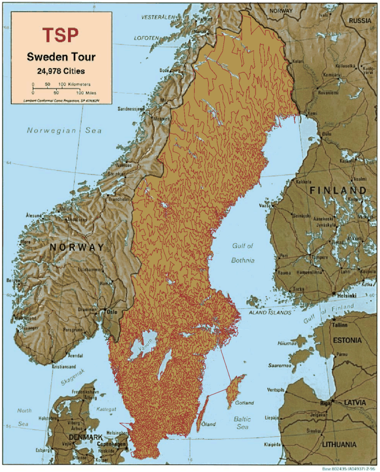


Germany 15112 cities



Solved by Applegate, Bixby, Chvátal, and Cook (2001)).

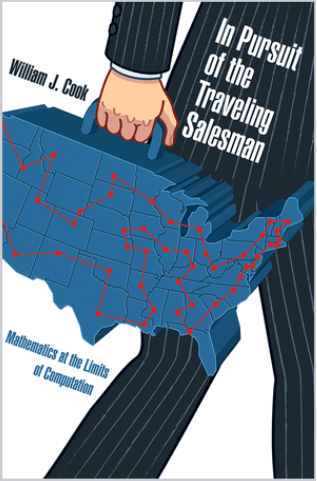
Sweden 24978 cities



Solved by Applegate, Bixby, Chvátal, Cook, and Helsgaun (2004).

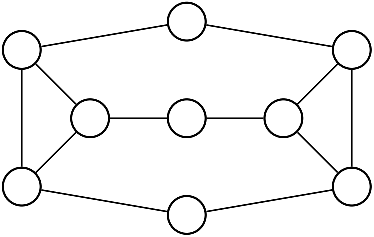
Book

To learn more about the history of the TSP, read



Graph TSP

Recall *Graph TSP*: Input is connected graph $G = (V, E)$ and cost $c(i, j)$ is number of edges in shortest path from i to j in G.



Graph TSP

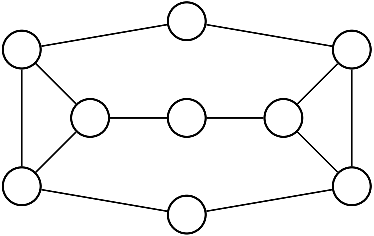
There has been recent progress on the case of graph TSP.

Oveis Gharan, Saberi, Singh	2010	$3/2 - \epsilon$	
Mömke, Svensson	2011	1.461	
Mömke, Svensson	2011	$4/3$	if graph *subcubic*
Mucha	2011	$13/9$	
Sebő and Vygen	2012	1.4	

Another perspective

Equivalent problem: Find Eulerian multigraph of (V, E) with the fewest number of edges. Recall *Eulerian* means every vertex has even degree, and the graph is connected.

Given any tour, replace any non-edge (i, j) with all edges in the shortest i-j path.



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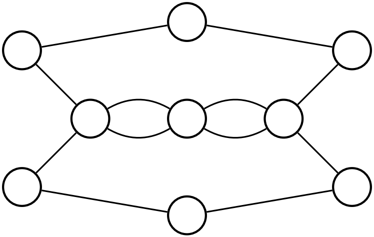

Today

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Cubic means all vertices have degree three. *2-vertex-connected* means that removing any one vertex (and its incident edges) from the graph does not disconnect the graph.

Idea

First idea: Since all vertices are odd-degree, add a matching to G. Then all vertices have even degree and G connected.

Lemma

Lemma (Naddef, Pulleyblank (1981))

Given a 2-edge-connected, cubic graph G with costs $c(e)$ on the edges, there is a perfect matching of cost at most $\frac{1}{3}\sum_{e\in E} c(e)$.

Suppose we set $c(e) = 1$ for all $e \in E$. Then cost of the graph plus matching is at most

$$|E| + \frac{1}{3}|E| \quad =$$

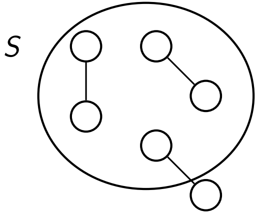
Proof of Naddef-Pulleyblank lemma

Lemma (Naddef, Pulleyblank (1981))

Given a 2-edge-connected, cubic graph G with costs $c(e)$ on the edges, there is a perfect matching of cost at most $\frac{1}{3}\sum_{e\in E} c(e)$.

Recall that the minimum-cost perfect matching can be found as the solution to the following LP:

$$\begin{aligned} \text{Minimize} \quad & \sum_{e\in E} c(e)z(e) & \\ & z(\delta(i)) = 1 & \forall i \in V \\ & z(\delta(S)) \geq 1 & \forall S \subset V, |S| \text{ odd.} \end{aligned}$$



Proof of the Naddef-Pulleyblank lemma

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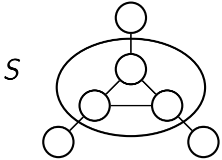
\begin{aligned}
\text{Min} \sum_{e\in E} c(e)z(e) & \\
z(\delta(i)) = 1 &\quad \forall i \in V\\
z(\delta(S)) \geq 1 &\quad \forall S \subset V, |S| \text{ odd}\\
z(e) \geq 0 &\quad \forall e \in E.
\end{aligned}
$$



New idea

Key idea: Use matchings to figure out which edges to **remove** and to add.

Removable pairing

Definition (Mömke, Svensson (2011))

Given G 2-vertex connected, $R \subseteq E$ *removable edges*, $P \subseteq R \times R$ is a *removable pairing* if:

- Any edge is in at most one pair of P;
- Edges in a pair have a common endpoint of degree at least 3;
- If we remove edges in R from G with at most one edge per pair in P removed, the resulting graph is still connected.

Theorem

Theorem (Mömke, Svensson (2011))

Given a removable pairing (R, P) and G 2-vertex-connected and cubic, there is an Eulerian multigraph with at most $\frac{4}{3}|E| - \frac{2}{3}|R|$ edges.

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Given a removable pairing (R, P) *and* G *2-vertex-connected and cubic, there is an Eulerian multigraph with at most* $\frac{4}{3}|E| - \frac{2}{3}|R|$ *edges.*

Idea: Take the graph, compute a perfect matching. If matching edge is in R, remove it, otherwise add it.

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Proof of second claim: At most $\frac{4}{3}|E| - \frac{2}{3}|R|$ edges.

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Proof of first claim: Result is connected and has even degree at all nodes.

Final Lemma

Lemma (Mömke, Svensson (2011))

In any cubic, 2-vertex-connected graph G, there is a removable pairing (R, P) with $|R| \geq |V|$.

Therefore, we can find an Eulerian graph with total number of edges at most

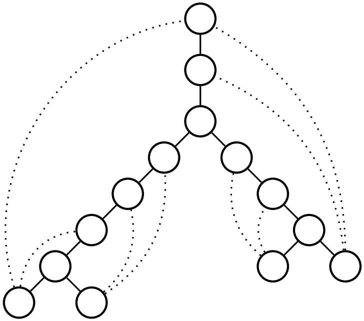
$$\frac{4}{3}|E| - \frac{2}{3}|R| \quad =$$

Proof of final lemma

Lemma (Mömke, Svensson (2011))

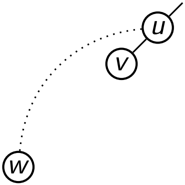
In any cubic, 2-vertex-connected graph G, there is a removable pairing (R, P) with $|R| \geq |V|$.

Start by considering a depth-first search tree T of the graph G.



Proof of final lemma

For each back edge (u, w) with u the ancestor of w, tree edge (u, v), make (u, w) and (u, v) a removable pair. For root put only one back edge in the pair.



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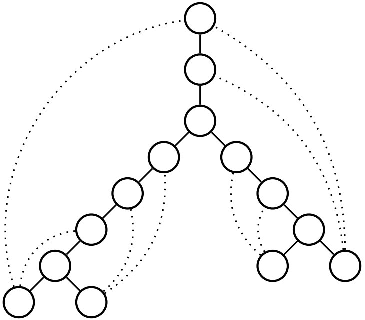

Definition again

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Proof of final lemma

Number of back edges is

$|R| =$

Proof of final lemma

Need to show that G stays connected if we remove at most one edge per pair. Prove by induction bottom up on subtrees; let T_u be subtree rooted at vertex u.

Case 1: u has two children v and w, one parent in T.

Proof of final lemma

Need to show that G stays connected if we remove at most one edge per pair. Prove by induction bottom up on subtrees; let T_u be subtree rooted at vertex u.

Case 2: u has one child v, one parent in T.

Extensions

Mömke and Svensson (2011) show that the same ideas can be extended to:

- a $\frac{4}{3}$-approximation algorithm for subcubic graphs (all degrees at most 3)
- a 1.461-approximation algorithm for all graphs (improved by Mucha (2012) to $\frac{13}{9}$)

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Sebő and Vygen (2012) add some new ideas and get a 1.4-approximation algorithm.