Recent Progress in Approximation Algorithms for the Traveling Salesman Problem

Lecture 1: Some basic algorithms

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São Paulo School of Advanced Science on Algorithms, Combinatorics, and Optimization

Winter in Ithaca

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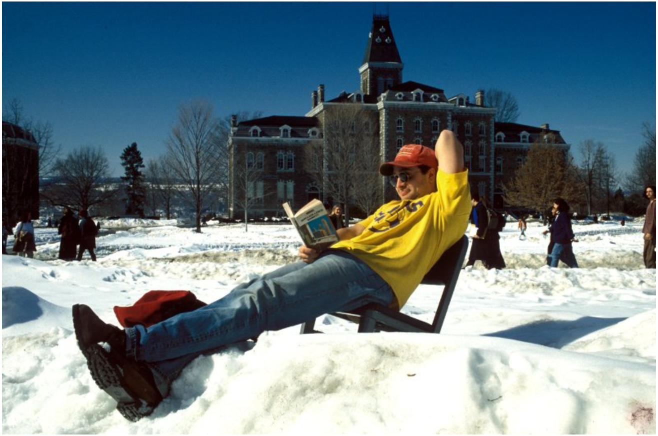


The traveling salesman problem

Traveling Salesman Problem (TSP)

Input:

- A complete, undirected graph $G = (V, E)$;
- Edge costs $c(e) \equiv c(i,j) \geq 0$ for all $e = (i,j) \in E$.

Goal: Find the min-cost tour that visits each city exactly once.

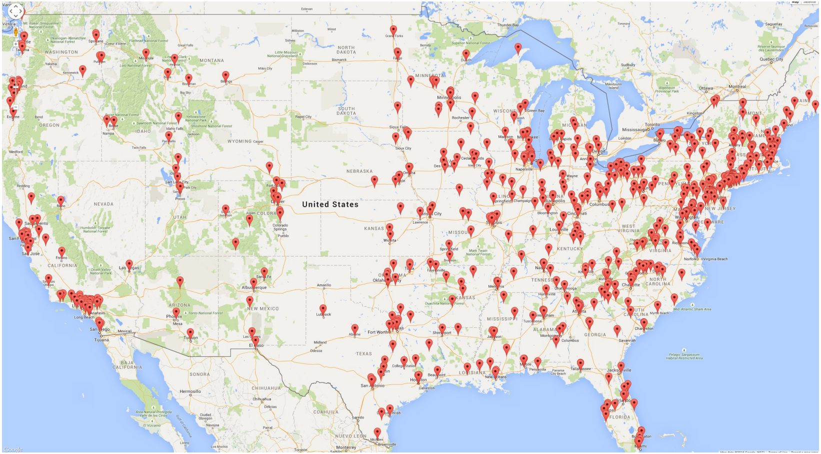
Costs are *symmetric* ($c(i,j) = c(j,i)$) and obey the *triangle inequality* ($c(i,k) \leq c(i,j) + c(j,k)$).

Asymmetric TSP (ATSP) input has complete directed graph, and $c(i,j)$ may not equal $c(j,i)$.

ATSP example

Example due to Ola Svensson (EPFL)

The traveling salesman problem



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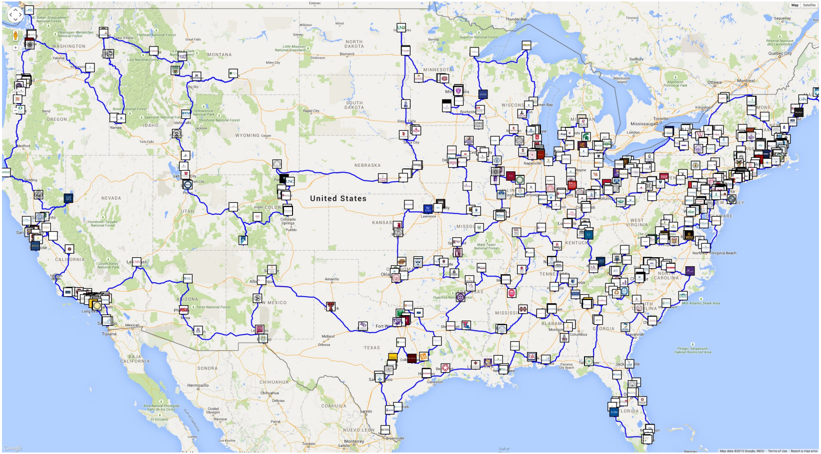

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Approximation Algorithms

Definition

An α-approximation algorithm is a polynomial-time algorithm that returns a solution of cost at most α times the cost of an optimal solution.

A simple approximation algorithm for TSP

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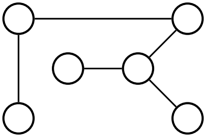

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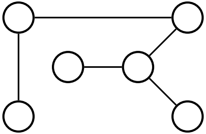
- Compute a minimum spanning tree (MST) F on G.
- Double every edge in F. The result is an *Eulerian graph*: it is connected, and every vertex has even degree.



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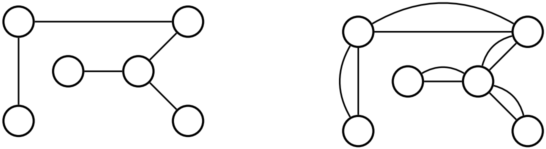

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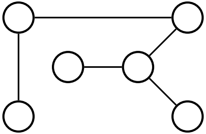

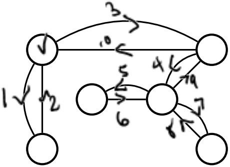

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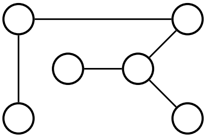
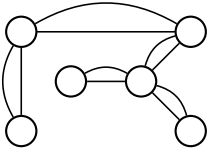
- Compute the traversal, and follow it; if the next edge goes back to a previously visited vertex, *shortcut it*, and go on to the next vertex in the traversal.





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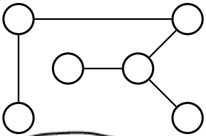

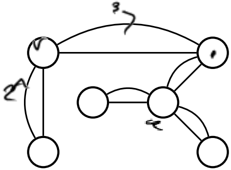

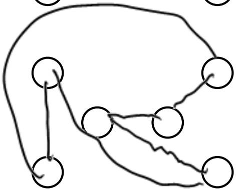

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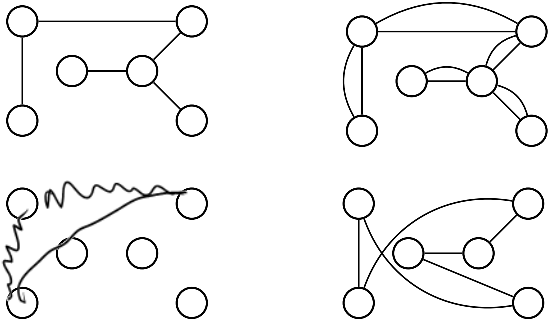

Theorem

Theorem

This algorithm is a 2-approximation algorithm for the TSP.

Let $c(F)$ be the cost of the edges in the MST. Let OPT be the cost of the optimal tour.

Lemma

$$c(F) \leq OPT.$$

Proof.

Take out one edge from optimal tour.
Result is a tree.
Min-cost spanning tree costs at most this much.

□

Proof of Theorem

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The cost of the tour given by the algorithm is at most $2c(F) \leq 2OPT$. □

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Next: a better approximation algorithm due to Christofides (1976).

Christofides' algorithm

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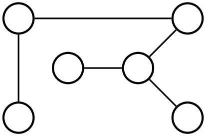
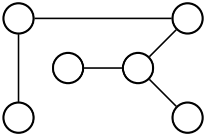


Christofides' algorithm

- Compute minimum spanning tree (MST) F on G
- Compute a *minimum-cost perfect matching* M on odd-degree vertices of F



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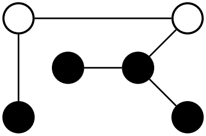

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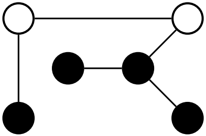
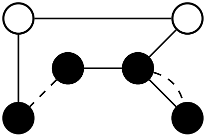
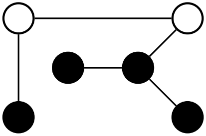
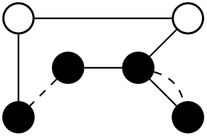




Christofides' algorithm

- Compute minimum spanning tree (MST) F on G
- Compute a *minimum-cost perfect matching* M on odd-degree vertices of F
- "Shortcut" Eulerian traversal in resulting Eulerian graph of $F \cup M$





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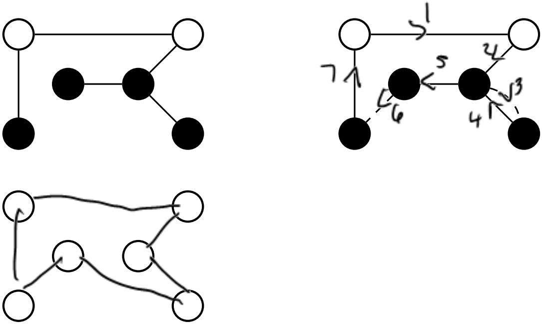

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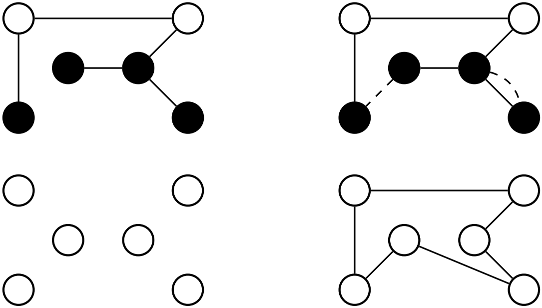

Theorem

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Christofides' algorithm is a $\frac{3}{2}$-approximation algorithm for the TSP.

Let $c(F)$ be the cost of the edges in the MST, $c(M)$ the cost of the edges in the matching Let OPT be the cost of the optimal tour.

Lemma

$$c(F) \leq OPT.$$

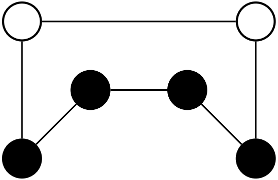
Lemma

$$c(M) \leq \frac{1}{2}OPT.$$

Then the cost of the algorithm's tour is at most

$$c(F) + c(M) \leq OPT + \frac{1}{2}OPT = \frac{3}{2}OPT.$$

Proof of Lemma



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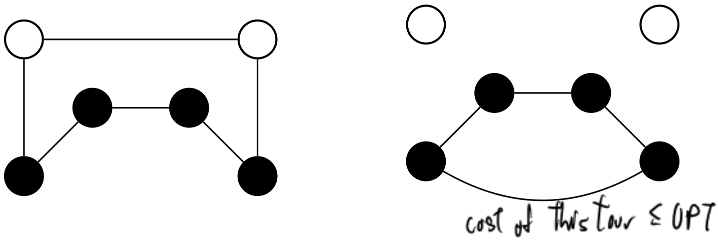

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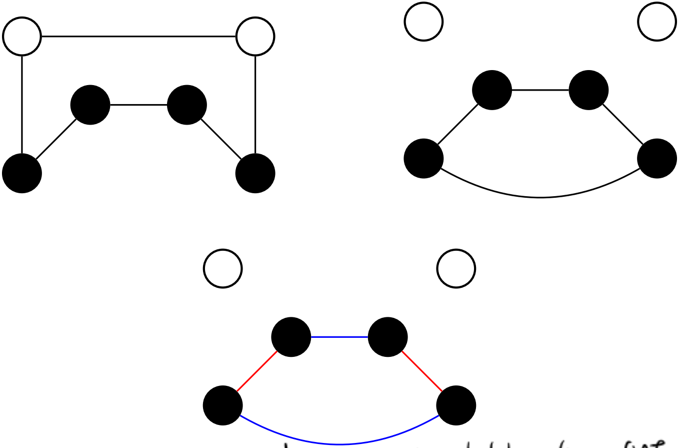

Two matchings, R and B, s.t. total cost $\leq$ OPT

$\Rightarrow$ one of them has cost $\leq \frac{1}{2}$ OPT

Linear programs

A tool we'll start using frequently: *integer* and *linear programming*. We want to devise an integer program for the traveling salesman problem.

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For each edge $e \in E$, we introduce a decision variable $x(e)$, in which we want

$$x(e) = \begin{cases} 1 & \text{if tour uses edge } e \\ 0 & \text{otherwise} \end{cases}$$

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$$x(e) = \begin{cases} 1 & \text{if tour uses edge } e \\ 0 & \text{otherwise} \end{cases}$$

Then our *objective function* is to

$$\text{Minimize } \sum_{e \in E} c(e)x(e).$$

Initial constraint

Let $\delta(i)$ represent the set of all edges e that have $i \in V$ as one endpoint. Then we want

$$\sum_{e \in \delta(i)} x(e) = 2$$

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An initial integer programming formulation of the problem is then:

$$\text{Minimize} \quad \sum_{e \in E} c(e)x(e)$$

subject to:

$$\begin{aligned} \sum_{e \in \delta(i)} x(e) &= 2 & \forall i \in V, \\ x(e) &\in \{0,1\} & \forall e \in E. \end{aligned}$$

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& \sum_{e \in \delta(i)} x(e) = 2 \qquad \forall i \in V, \\
& x(e) \in \{0,1\} \qquad \forall e \in E.
\end{aligned}$$

What might be a problem with this formulation?

Initial IP

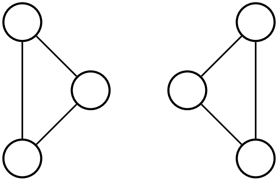
$$\text{Minimize} \quad \sum_{e \in E} c(e)x(e)$$

subject to:

$$\sum_{e \in \delta(i)} x(e) = 2 \qquad \forall i \in V$$

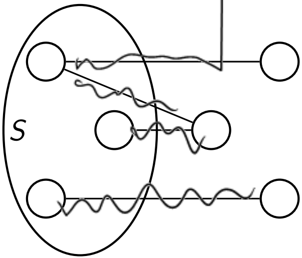
$$x(e) \in \{0,1\} \qquad \forall e \in E.$$

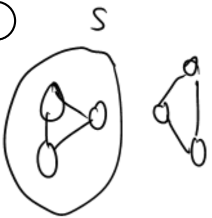
The following solution is feasible:



Subtour elimination constraints

For $S \subseteq V$, let $\delta(S)$ be the set of edges with one endpoint in S.



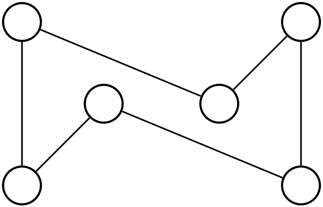


Then we want that

$$\sum_{e \in \delta(S)} x(e) \geq 2$$

for any set S.

LP relaxation



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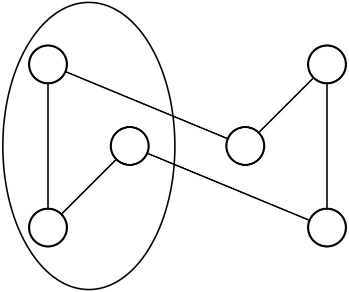

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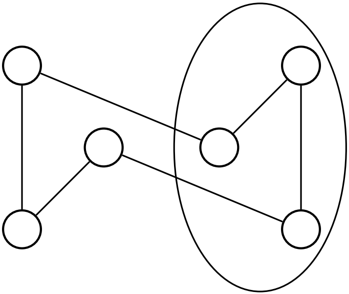

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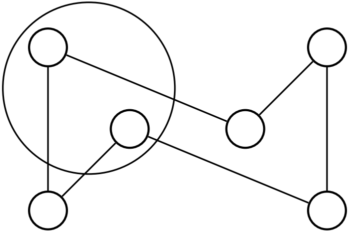

Subtour LP

Thus an initial integer programming formulation is

$$\text{Minimize} \quad \sum_{e \in E} c(e)x(e)$$

subject to:

$$\sum_{e \in \delta(i)} x(e) = 2 \qquad \forall i \in V$$

$$\sum_{e \in \delta(S)} x(e) \geq 2 \qquad \forall S \subset V, S \neq \emptyset$$

$$x(e) \in \{0, 1\} \qquad \forall e \in E.$$

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Replace $x(e) \in \{0,1\}$ by $0 \leq x(e) \leq 1$ to obtain a *linear programming relaxation* called the *Subtour LP*.

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If OPT_{IP} is value of IP optimal, OPT_{LP} value of LP optimal, how do they compare?

$$OPT_{LP} \leq OPT_{IP}.$$

Subtour LP

We'll sometimes use the shorthand $x(F) = \sum_{e \in F} x(e)$, or $c(F) = \sum_{e \in F} c(e)$. For example, rewrite LP as

$$\text{Minimize} \quad \sum_{e \in E} c(e)x(e)$$

subject to:

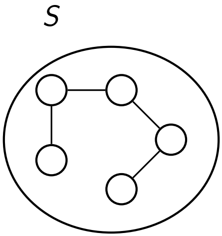
$$\begin{aligned} x(\delta(i)) &= 2 && \forall i \in V \\ x(\delta(S)) &\geq 2 && \forall S \subset V, S \neq \emptyset \\ 0 \leq x(e) &\leq 1 && \forall e \in E. \end{aligned}$$

Equivalent constraints

An equivalent way to write the subtour elimination constraints is via a constraint that says no cycles in any strict subset. Let $E(S)$ be the set of edges with both endpoints in S; then

$$x(E(S)) \leq |S| - 1$$

for all $S \subset V, |S| \geq 2$.



Equivalent LP

So an LP that's equivalent to the subtour LP is the following:

$$\text{Minimize} \quad \sum_{e \in E} c(e)x(e)$$

subject to:

$$\begin{aligned} x(\delta(i)) &= 2 && \forall i \in V \\ x(E(S)) &\leq |S| - 1 && \forall S \subset V, |S| \geq 2 \\ 0 \leq x(e) &\leq 1 && \forall e \in E. \end{aligned}$$

Christofides' again

Let OPT_{LP} be the optimal value of the linear programming. We can now (almost) prove the following.

Theorem (Wolsey (1980), Shmoys, W (1990))

Christofides' algorithm returns a tour of cost at most $\frac{3}{2}OPT_{LP}$.

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Theorem (Wolsey (1980), Shmoys, W (1990))

Christofides' algorithm returns a tour of cost at most $\frac{3}{2}OPT_{LP}$.

To prove this, we need to show that for MST F and matching M on odd-degree vertices,

Lemma

$$c(F) \leq OPT_{LP}$$

Lemma

$$c(M) \leq \frac{1}{2}OPT_{LP}$$

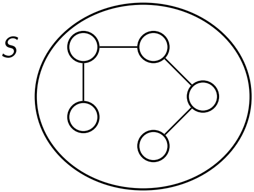
Proof of first lemma

The minimum spanning tree can be found as the solution to the following LP (Edmonds 1971):

$$\text{Minimize} \quad \sum_{e \in E} c(e) z(e)$$

subject to:

$$
\begin{aligned}
z(E) &= |V| - 1 & & \\
z(E(S)) &\leq |S| - 1 & & \forall S \subset V, |S| \geq 2 \\
z(e) &\geq 0 & & \forall e \in E.
\end{aligned}
$$



Proof of first lemma

$$c(F) \leq OPT_{LP}$$

$$\begin{aligned} \text{Min} \sum_{e \in E} c(e)x(e) & \\ x(\delta(i)) = 2 & \quad \forall i \in V \\ x(E(S)) \leq |S| - 1 & \quad \forall S \subset V, |S| \geq 2 \\ 0 \leq x(e) \leq 1 & \quad \forall e \in E. \end{aligned}$$

$$\begin{aligned} \text{Min} \sum_{e \in E} c(e)z(e) & \\ z(E) = |V| - 1 & \\ z(E(S)) \leq |S| - 1 & \quad \forall S \subset V, |S| \geq 2 \\ z(e) \geq 0 & \quad \forall e \in E. \end{aligned}$$

Let x^* be opt soln to Subtour LP. Let $z = \frac{n-1}{n} x^*$.

Claim: z is feasible for spanning tree LP. Cost of MST ≤ Cost of subtour LP.

$$z(E) = \sum_{e \in E} z_e = \frac{n-1}{n} \sum_{e \in E} x^*_e = \frac{n-1}{n} \cdot \frac{1}{2} \sum_{i \in V} x^*(\delta(i))$$

$$= \frac{n-1}{n} \cdot \frac{1}{2} \cdot (2n) = n-1 = |V|-1.$$

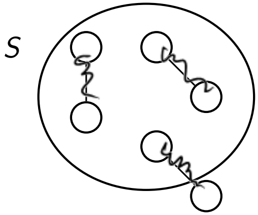
Proof of second lemma

The minimum-cost perfect matching can be found as the solution to the following LP (Edmonds 1965):

$$\text{Minimize} \quad \sum_{e \in E} c(e) z(e)$$

subject to:

$$z(\delta(i)) = 1 \qquad \forall i \in V$$
$$z(\delta(S)) \geq 1 \qquad \forall S \subset V, |S| \text{ odd.}$$



Proof of second lemma

$$c(M) \leq \frac{1}{2} OPT_{LP}$$

$$\begin{aligned} \text{Min} \sum_{e \in E} c(e)x(e) & \\ x(\delta(i)) = 2 & \quad \forall i \in V \\ x(\delta(S)) \geq 2 & \quad \forall S \subset V, S \neq \emptyset \\ 0 \leq x(e) \leq 1 & \quad \forall e \in E. \end{aligned}$$

$$\begin{aligned} \text{Min} \sum_{e \in E} c(e)z(e) & \\ z(\delta(i)) = 1 & \quad \forall i \in V \\ z(\delta(S)) \geq 1 & \quad \forall S \subset V, |S| \text{ odd} \\ z(e) \geq 0 & \quad \forall e \in E. \end{aligned}$$

Let x^* be opt soln to Subtour LP.

Claim: $z = \frac{1}{2} x^*$ is feasible for matching LP.

So cost$(M) \leq \frac{1}{2} \sum_{e \in E} c(e) z(e) = \frac{1}{2} \sum_{e \in E} c(e) x^*(e) = \frac{1}{2} OPT_{LP}$.

Missing part

We also need that the value of the subtour LP can only go down as we remove vertices from the instance (Shmoys, W 1990), so that we can consider a matching only on the odd-degree vertices.

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Theorem

$$c(F) + c(M) \leq \frac{3}{2} OPT_{LP},$$

so that Christofides' algorithm returns a solution of cost at most this much.

Update

"But I thought you were going to talk about *recent* approximation algorithms for the traveling salesman problem..."

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For TSP, no better approximation algorithm known that Christofides' algorithm.

Graph TSP

Progress made for two different special cases: *graph TSP* and the *s-t path TSP*.

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Graph TSP: Input is connected graph $G = (V, E)$ and cost $c(i, j)$ is number of edges in shortest path from i to j in G.

Oveis Gharan, Saberi, Singh	2010	$3/2 - \epsilon$	
Mömke, Svensson	2011	1.461	
Mömke, Svensson	2011	$4/3$	if graph *subcubic*
Mucha	2011	$13/9$	
Sebő and Vygen	2012	1.4	

s-t path TSP

The s-t path TSP:
Usual TSP input plus $s, t \in V$, find a min-cost path from s to t visiting all other nodes in between (an *s-t Hamiltonian path*).

Hoogeveen (1991) shows that the natural variant of Christofides' algorithm gives a $\frac{5}{3}$-approximation algorithm.

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An, Kleinberg, Shmoys	2012	1.618
Sebő	2013	1.6
Vygen	2015	1.599
Gottschalk and Vygen	2015	1.56
Sebő and Van Zuylen	2016	1.52

Agenda

For the rest of the week:

Wednesday:	Graph TSP (4/3 for cubic, 2-vertex-connected graphs)
Thursday:	s-t path TSP
Friday:	s-t path TSP for graph TSP, open questions