# Tutorial on the Container Method

*A simple guide to solve (not so simple) problems*

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**Abstract**

Many important theorems and conjectures in combinatorics can be rephrased as problems about counting independent sets in some specific graphs and hypergraphs. The Container Method, whose basic idea can be traced back to Kleitman-Winston, and has recently been further developed by Balogh-Morris-Samotij and Saxton-Thomason, essentially states that hypergraphs satisfying some natural conditions have very few independent sets. Here I show some recent applications of the method, and try to give an easy recipe containing all the key ideas one needs to know to prove similar results.

## The basic problem type

Suppose we have a property $P$ that can be easily characterized by describing a list of forbidden structures, and we are interested in the number of families $\mathcal{A} \subset \mathcal{P}(n)$ satisfying this property. If this is your set-up, the Container Method was made for you!

## Examples

- Dedekinds problem: How many antichains are there in $\mathcal{P}(n)$? The property of being an antichain is monotone, and can be easily described by a list of forbidden structures (pairs $A, B$ with $A \subset B$). Hence it is a good idea to try to apply the Container Method to this problem.
- How many $t$-intersecting families are there in $\mathcal{P}(n)$? The forbidden structures are pairs of sets with intersection size less than $t$, hence it is likely that the Container Method can be successfully applied to this problem.
- How many $(p, q)$-tilted Sperner families (forbid $A, B \in \mathcal{F}$ with $p|A \backslash B| = q|B \backslash A|$)? How many $t$-error correcting codes (Sapozhenko's question)? How many cancellative codes (forbid $A, B, C \in \mathcal{F}$ with $A \cup B = A \cup C$)? How many union-free families (forbid $A_1 \cup A_2 = B$)? How many families without containing a set with two of its supersets? How many families with $|A \Delta B| \leq t$? How many families without containing a set with $17$ of its supersets AND a set with $6 \log n$ of its subsets?
- Sparse random analogues of all the above, thresholds etc. Random analogues of extremal results often reduce to counting questions, which can be attacked by containers.

## Statement of the Container Theorem

IF a (hyper-)graph **supersaturates** and **has balanced codegrees** THEN it contains very few independent sets.

*Explanation:*

A hypergraph supersaturates if every subset of its vertex set, that is just slightly larger than the maximal independent set, contains *many* hyperedges.

A hypergraph has balanced codegrees if every vertex is contained in rougly the same number of edges, every pair of vertices is contained in roughly the same number of edges, every triple ....

See more examples below.

## A step-by-step guide to solve such basic problems

**Step one:** Prove an upper bound on the size of your families.

**Step two:** Prove a supersaturation result. That is, prove that if the size of a family is slightly bigger than the upper bound given in step one, then it contains many forbidden structures.

**Step three:** Apply the Container Method to get a counting theorem.

## How to think of the Container Method

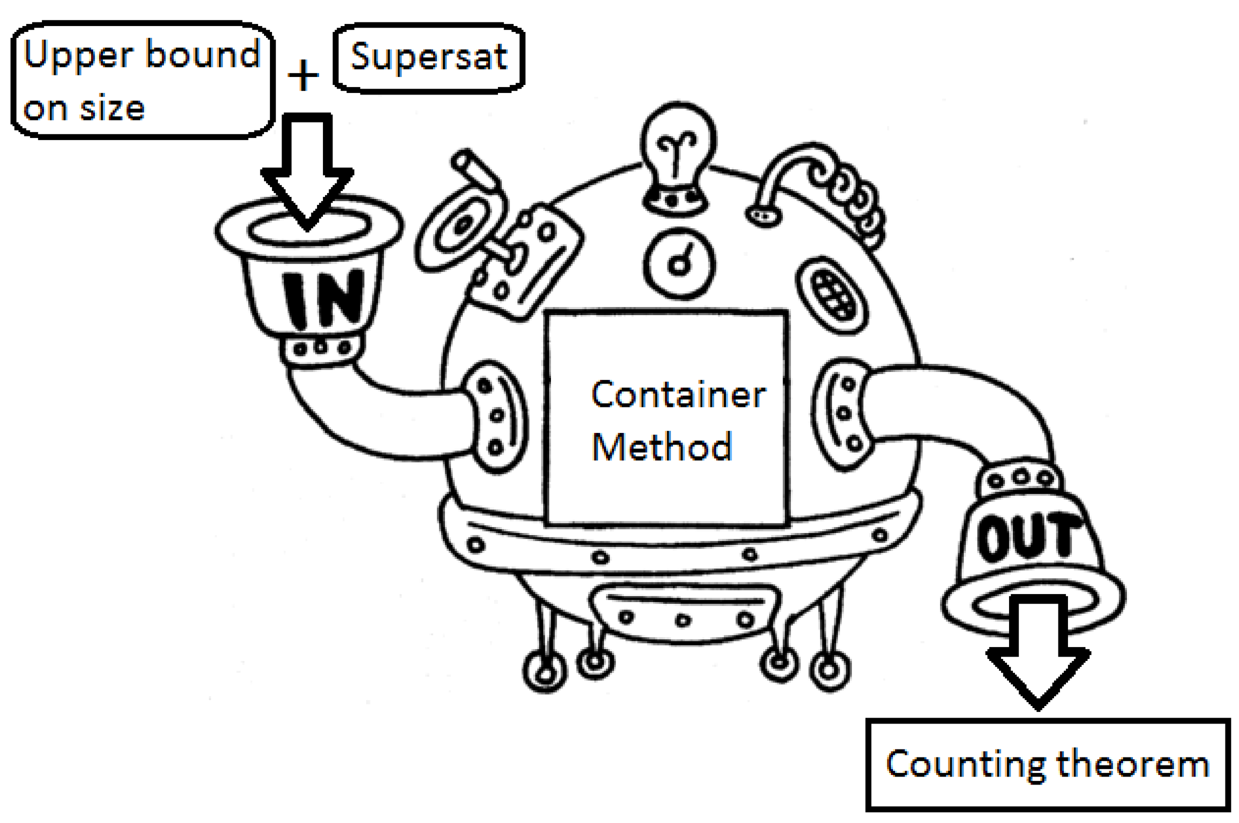


**Figure 1:** The Container Machine

In cooking, if one uses **better ingredients** then one obtains **better food**.

For container enthusiasts this translates to: **the stronger the upper bound and supersat, the better the counting theorem** will be that the machine spits out.

## A worked example

Dedekind's problem: How many antichains are there in $\mathcal{P}(n)$?

**Step one (upper bound):** An antichain has size at most $\binom{n}{n/2}$.

**Step two (supersaturation):** If $\mathcal{F} \subset \mathcal{P}(n)$ is a family of size $\binom{n}{n/2} + x$ then it contains at least $\frac{n}{2} \cdot x$ comparable pairs. – *Proving this is highly non-trivial. Luckily for us Kleitman proved this five decades ago!*

**Step three (apply containers):** We obtain containers $C_1, C_2, \ldots \subset \mathcal{P}(\backslash)$, so that every antichain is contained in at least one of them. The number of containers if $\binom{2^n}{2^n/(n/2)} = 2^{o\left(\binom{n}{n/2}\right)}$, each container has size $(1 + o(1))\binom{n}{n/2}$. Hence the number of antichains is $2^{(1+o(1))\binom{n}{n/2}}$.

## Some other results proved using similar methods

| Original theorem | Counting analog |
|---|---|
| If $p < q$ coprime then a $(p, q)$-tilted Sperner family (forbidding solutions to $p|A \backslash B| = q|B \backslash A|$) has size at most $(q - p + o(1))\binom{n}{n/2}$. (Leader-Long, 2014) | If $p < q$ coprime then the number of $(p, q)$-tilted Sperner families is $2^{(q-p+o(1))\binom{n}{n/2}}$. (Balogh-Treglown-W, 2016) |
| Let $\mathcal{A} \subset \mathcal{P}(n)$, and let $R \cup W$ be a two-coloring of $[n]$. If $\mathcal{A}$ does not contain a pair of sets which are comparable with monochromatic difference then $|\mathcal{A}| \leq \binom{n}{n/2}$. | The number of families $\mathcal{A}$ for which there exists a colouring $R \cup W = [n]$ such that there is no comparable pair $A, B \in \mathcal{A}$ with monochromatic difference is $2^{(1+o(1))\binom{n}{n/2}}$. (Balogh-Treglown-W, 2016) |
| The maximum size of a $t$-error correcting code is $H(n, t) := 2^n / \binom{n}{\leq t}$. (Hamming bound) | The number of $t$-error correcting codes, if $t \leq n^{1/3}$, is $2^{(1+o(1))H(n,t)}$. (Balogh-Treglown-W, 2016, partial answer to Sapozhenko's question) |
| Sperner's theorem | Random analog: If $p \gg 1/n$ then whp the size of the largest antichain in $\mathcal{P}(n, p)$ (i.e. keeping elements with probability $p$) is $(1 + o(1))p\binom{n}{n/2}$ (Balogh-Mycroft-Treglown, 2014) |

## What if there is no supersaturation?

With some additional ideas the container method can sometimes be made to work! E.g. the number of $C_4$-free graphs is between $2^{0.5n^{3/2}}$ and $2^{1.09n^{3/2}}$ (Kleitman-Winston (1982), Balogh-W (2015)).

## What if the codegrees are not balanced?

One needs even more ideas.

Option 1: find a nice subhypergraph with balanced codegrees (e.g. Morris-Saxton, 2015).

Option 2: adapt the proof of the Container Theorem to your problem.

E.g. a family is *union-free* if it does not contain solution to $A \cup B = C$. Kleitman proved that the largest union-free family in $\mathcal{P}(n)$ has size $(1 + o(1))\binom{n}{n/2}$. Burosch-Demetrovics-Katona-Kleitman-Sapozhenko showed that the number of union-free families is at most $2^{2\sqrt{2}\binom{n}{n/2}(1+o(1))}$ and conjectured that one can get rid of the $2\sqrt{2}$ factor. Define the 3-uniform hypergraph $\mathcal{H}$ on vertex set $\mathcal{P}(n)$, edges corresponding to solutions of $A \cup B = C$. Given $B$ and $C$ there can be few or many solutions to $A \cup B = C$, depending on $B$ and $C$! So **codegrees are not balanced**. Bad news.

The crucial (albeit trivial) observation is that given $A$ and $B$ there is only one solution to $A \cup B = C$! Hence **in one direction the codegrees are equal**. By proving a directed version of the container theorem, and a different kind of supersaturation we managed to show that the number of union-free families is $2^{(1+o(1))\binom{n}{n/2}}$. (Balogh-W, 2016)

## Conclusions

- When dealing with nice hypergraphs the hardest part of container proofs is to obtain a strong supersaturation.
- Applying the container method to nice hypergraphs has become a standard technique in combinatorics. We are only starting to explore extensions of the method to other hypergraphs.