Semidefinite and linear programming integrality gaps for scheduling identical machines

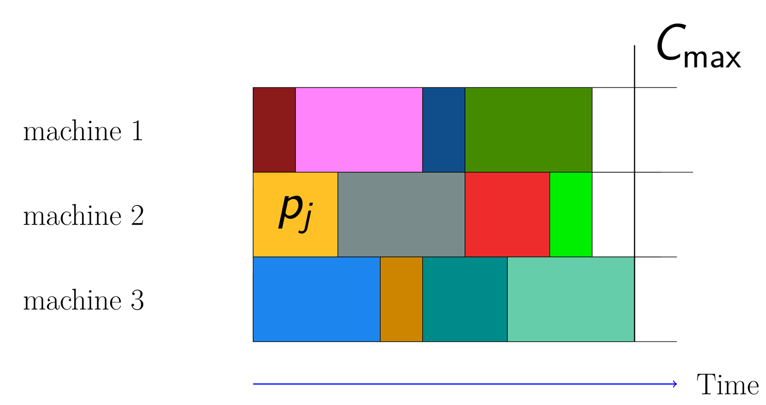
Adam Kurpisz[1] Monaldo Mastrolilli[1] Claire Mathieu[2]
Tobias Mömke[3] Victor Verdugo[2,4] Andreas Wiese[5]

IDSIA ENS U-Saarland U-Chile MPII.

Minimum makespan scheduling: $P||C_{\max}$

- A set M of identical machines,
- a set J of jobs,
- a processing time p_j for each $j \in J$.

Goal: to find an assignment from jobs to machines in order to minimize the maximum load $C_{\max}$.



- The problem is strongly NP-hard, and
- there exists a purely combinatorial PTAS, that is, for each $\varepsilon > 0$ there exists an algorithm returning a schedule with maximum load at most

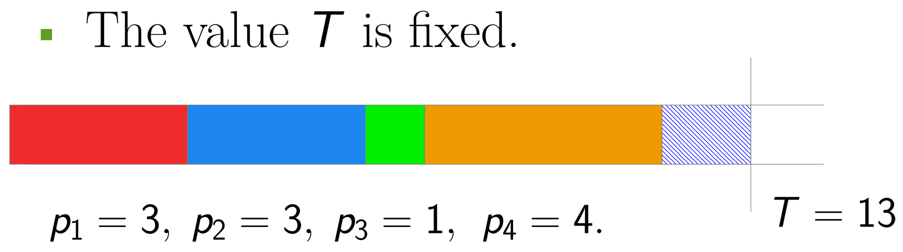
$$(1+\varepsilon)\cdot C_{\max}.$$

Question

Is it possible to obtain a polytime $(1+\epsilon)$-approximation algorithm based on known LP/SDP relaxations?

Integer program based on machine configurations

- The value T is fixed.



- Configuration: multiset $\{3, 3, 1, 4\}$.

$$\mathcal{C} = \left\{ C : \sum_{p\in\{p_j : j\in J\}} p \cdot m(p, C) \le T \right\}$$

- Every machine is in some configuration: for each $i \in M$,

$$\sum_{C\in\mathcal{C}} y_{iC} = 1.$$

- Every processing time appears enough times: for every $p \in \{p_j : j \in J\}$,

$$\sum_{i\in M}\sum_{C\in\mathcal{C}} m(p, C) y_{iC} = n_p.$$

Main Theorem

For each $n \in \mathbb{N}$, there exists an instance with n jobs and $\Theta(n)$ machines such that, even after $\Omega(n)$ rounds of Sherali & Adams LP or Lovász & Schrijver SDP hierarchies over the configuration LP, the integrality gap of the resulting relaxation is at least

$$1 + \frac{1}{1023} > 1.009\ldots$$

Convex hierarchies

- Determines a sequence of relaxations satisfying $P \supseteq P_1 \supseteq P_2 \supseteq \cdots \supseteq P_n = \mathrm{conv}(P \cap \{0,1\}^n)$.
- It is possible to optimize over P_t in time (# variables)$^{O(t)}$. Examples: LP Sherali & Adams '90 (SA), Lovász & Schrijver '91 (LS/LS$_+$) and Sum-of-Squares '00 (Parrillo, Lasserre).
- Positive results: Max-Cut, Sparsest-Cut, Knapsack and Set-Cover.
- Negative results: Independent Set, Vertex Cover, Max-3-Sat and Hypergraph Vertex Cover.

LS$_+$ hierarchy

Consider

$$Q = \{(a, z) : z/a \in \mathrm{clp}\}$$

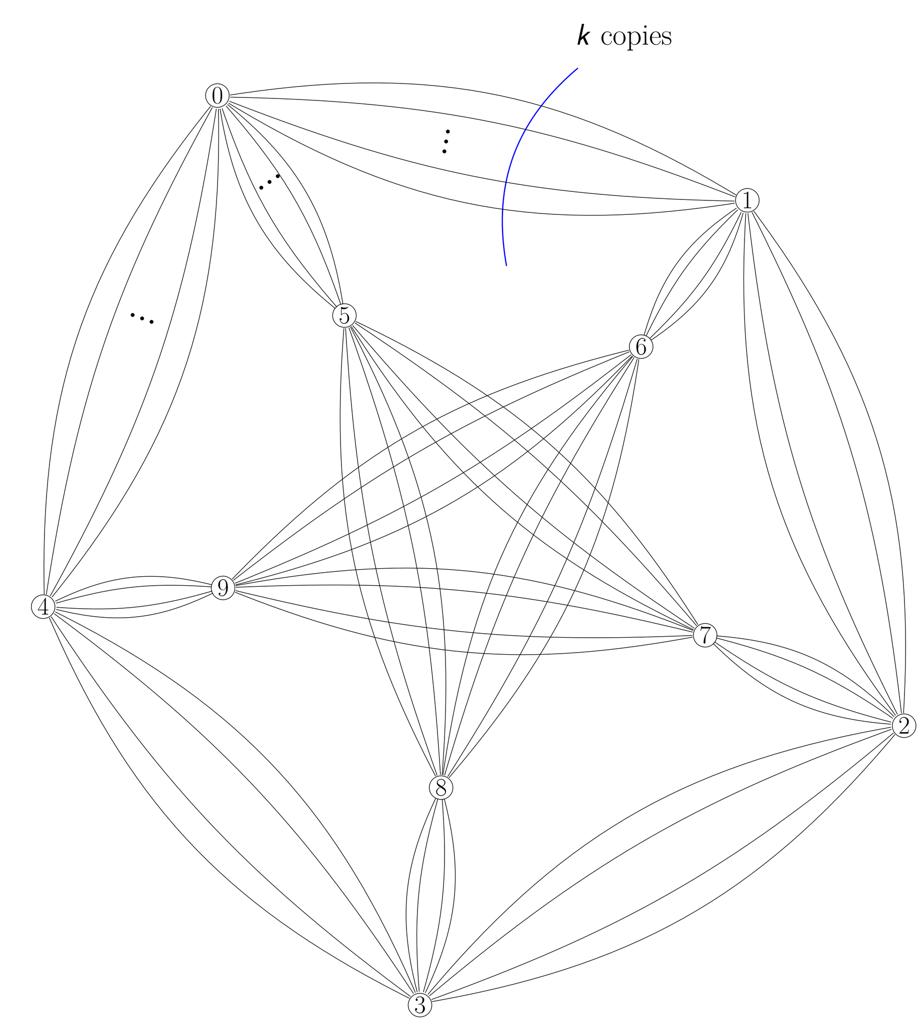
for $T = 1023$. We define an operator N_+ on convex cone Q as follows: $y \in N_+(Q)$ if and only if there exists a symmetric matrix Y such that

1. $y = Y \cdot e_0 = \mathrm{diag}(Y)$,
2. for all (i, C), $Y \cdot e_{(i,C)} \in Q$,
3. Y is positive semidefinite.

LS$_+$ hierarchy

For any $r \ge 0$, level r of the LS$_+$ hierarchy, $N_+^r(Q)$, is defined recursively by: $N_+^0(Q) = Q$ and $N_+^r(Q) = N_+(N_+^{r-1}(Q))$.

Gap instance



Proof scheme

- Let k be odd;
- $C_{\max} \ge 1023$;
- $C_{\max} = 1023 \Rightarrow$ graph is 1-factorizable; makespan is integer, then $C_{max} \ge 1024$,
- k-th level of the hierarchy is feasible for $T = 1023$.

References

Semidefinite and Linear Programming Integrality Gaps for Scheduling Identical Machines. A. Kurpisz, M. Mastrolilli, C. Mathieu, T. Mömke, V. Verdugo and A. Wiese. In proceedings of Integer Programming and Combinatorial Optimization, 152–163, 2016.

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- Email: victor.verdugo@ens.fr
- Web: https://sites.google.com/site/vicverdu/