Semidefinite programming techniques in combinatorial optimization (L. Tunçel)

Exercises for July 25, 2016

1. Given C, $A_i \in \mathbb{S}^n$ for $i \in \{1, 2, \ldots, m\}$ and $b \in \mathbb{R}^m$, let

$$(P) \quad \inf \langle C, X \rangle, \ \text{ s.t. } \ \langle A_i, X \rangle = b_i \quad i \in \{1, 2, \ldots, m\}, \ X \in \mathbb{S}^n_+$$

and let (D) denote its dual. Assume that both (P) and (D) have Slater points. Prove that for every pair of optimal solutions X and S for (P) and (D) respectively, we have

$$\text{rank}(X) + \text{rank}(S) \leq n.$$

2. Let $X \in \mathbb{S}^n$.

(a) Prove that $X \succ 0$ iff for every $k \in \{1, 2, \ldots, n\}$, $\det(X_{J_k}) > 0$, where $J_k := \{1, 2, \ldots, k\}$.

(b) Prove that $X \succeq 0$ iff for every nonempty $J \subseteq \{1, 2, \ldots, n\}$, $\det(X_J) \geq 0$.

3. Let $X, S \in \mathbb{R}^{n \times n}$. The *Hadamard product of* X and S is defined as

$$(X \odot S)_{ij} := X_{ij} S_{ij}, \forall i, j.$$

So, $(X \odot S)$ is an n-by-n matrix which is the pointwise product of X and S. Prove that

$$X, S \succeq 0 \Rightarrow (X \odot S) \succeq 0.$$

4. Prove that the set of extreme rays of the cone of n-by-n symmetric positive semidefinite matrices are as follows:

$$\text{ext}(\mathbb{S}^n_+) = \left\{ hh^\top : h \in \mathbb{R}^n, \|h\|_2 = 1 \right\}.$$

5. Prove that

$$\mathbb{S}^n_+ = \left\{ S \in \mathbb{S}^n : \ \text{Tr}(XS) \geq 0 \ \forall X \in \mathbb{S}^n_+ \right\}.$$

I.e., the cone of symmetric positive semidefinite matrices is self-dual.

6. Prove that for every $\bar{S} \in \mathbb{S}^n_{++}$ and for every $\alpha \in \mathbb{R}_{++}$ the sets

$$\{X \in \mathbb{S}^n_+ : \langle \bar{S}, X \rangle \leq \alpha\}$$

and

$$\{X \in \mathbb{S}^n_+ : \langle \bar{S}, X \rangle = \alpha\}$$

are nonempty, convex and compact. Also, prove that the interior of the first set is nonempty.

Exercises for July 26, 2016

7. Let $X, S \in \mathbb{S}^n$ such that $S \succeq X \succeq 0$. Prove that $S^{1/2} \succeq X^{1/2}$.
Then, prove or disprove:

$$S^2 \succeq X^2.$$

8. Let $X, S \in \mathbb{S}^n_+$ be nonsingular matrices. Prove that

$$X \succeq S \iff S^{-1} \succeq X^{-1}.$$

That is, $(\cdot)^{-1}$ is *order reversing*.

9. Consider the maximum cardinality version of the Max Cut problem (when all the weights on all the edges of the given graph G are one). Find an optimal solution for the SDP relaxation when G is the 5-cycle and when G is the 7-cycle. What are the ratios of the optimal MAX CUT objective values to the optimal values of the SDP relaxations? Prove your claims without relying on any software.

10. Prove that the quadratic optimization problem

$$\max \left\{ x^\top W x : x \in \{-1, 1\}^n \right\},$$

where $W \in \mathbb{S}^n_+$ includes the Max Cut problem on the graphs with n nodes (where every edge in G has a nonnegative weight) as a special case.

Exercises for July 27, 2016

11. Given a simple graph $G = (V, E)$, define $n := |V|$. Let $A \in \mathbb{S}^n_+$, and $p \in [1, +\infty]$ are also given. Recall the notion of smallest radius ellipsoid (with respect to A and the given p-norm) which contains a unit distance representation of G:

$$\mathcal{E}_p(G; A) := \inf \left\{ \left\| \left\langle u^{(i)}, Au^{(i)} \right\rangle_{i \in V(G)} \right\|_p : \quad u^{(1)}, u^{(2)}, \ldots, u^{(n)} \text{ form a unit distance repr. of } G \right\}.$$

In this context, the dimension of G, denoted $\dim(G)$, is the smallest integer k for which G has a unit distance representation in $\mathbb{R}^k$.

In this generality, prove that

$$\mathcal{E}_p(G; A) = 0 \text{ iff } \dim(G) \leq \dim(\mathrm{Null}(A)).$$

12. Recall the smallest radius Euclidean ball and smallest radius hypersphere unit distance representations of graphs (SDP formulations respectively given below).

$$\begin{array}{rlrcll} t_{\text{ball}}(G) := & \min & & t & & \\ & \text{subject to:} & \mathrm{diag}(X) - t\bar{e} & \leq & 0, & \\ & & X_{ii} - 2X_{ij} + X_{jj} & = & 1, & \forall \{i,j\} \in E, \\ & & X & \in & \mathbb{S}^V_+; & \end{array}$$

$$\begin{array}{rlrcll} t_{\text{sphere}}(G) := & \min & & t & & \\ & \text{subject to:} & \mathrm{diag}(X) - t\bar{e} & = & 0, & \\ & & X_{ii} - 2X_{ij} + X_{jj} & = & 1, & \forall \{i,j\} \in E, \\ & & X & \in & \mathbb{S}^V_+. & \end{array}$$

Prove that

$$t_{\text{ball}}(G) = t_{\text{sphere}}(G) \quad \text{for every graph } G.$$

Hint: If you get stuck, check out Theorem 3.5 in the paper: M. K. de Carli Silva and L. T., Optimization problems over unit distance representations of graphs, *Electronic Journal of Combinatorics* 20 (1) 2013, #P43.

Exercises for July 28, 2016

13. Prove that for every graph G,

$$\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{CLQ}(G) \subseteq \mathrm{FRAC}(G).$$

14. A convex set $S \subset \mathbb{R}^d$ is called a *convex corner* if

- $S \subseteq \mathbb{R}^d_+$ and
- for every $x \in S$, every y satisfying $0 \leq y \leq x$ is also in S.

Prove that for every graph G, $\mathrm{STAB}(G)$, $\mathrm{LS}_+(G)$, $\mathrm{TH}(G)$, $\mathrm{CLQ}(G)$ and $\mathrm{FRAC}(G)$ are all convex corners.