Minimal Unit Circular-Arc Models are Integer

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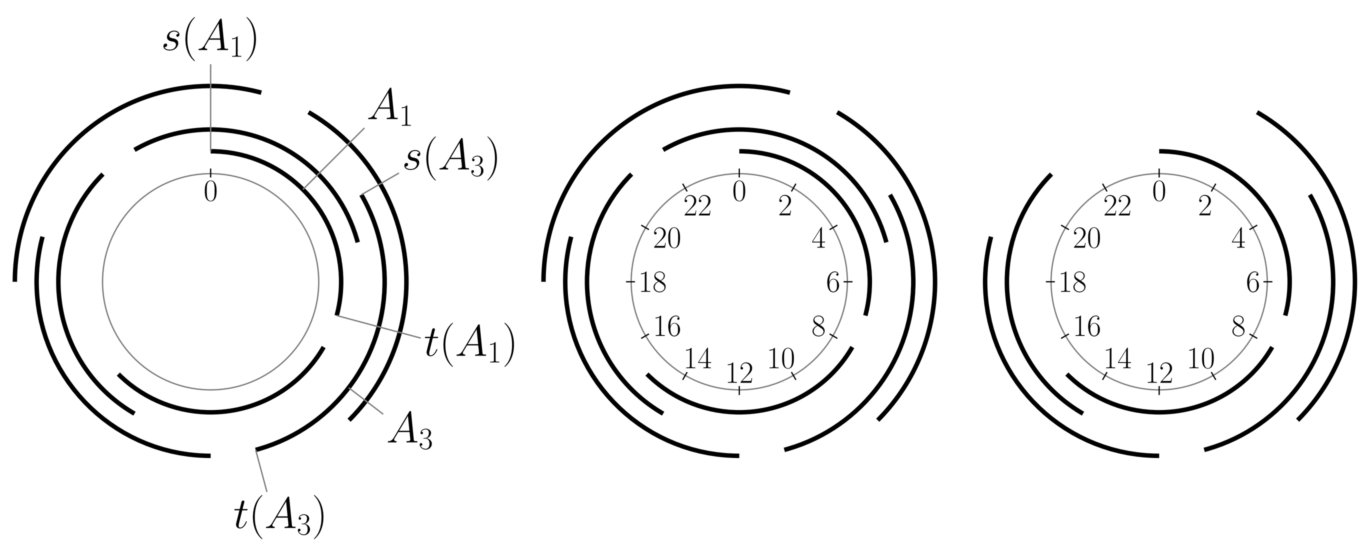
Introduction

- A proper circular-arc (PCA) model $\mathcal{M}$ is defined based on the endpoints $\mathbf{s(A)}$ and $\mathbf{t(A)}$ of its arcs on a circle $\mathbf{C}$.



PCA model　　$\mathbf{(24, 7)}$-CA model　　$\mathbf{7}$-IG model

- $\mathbf{(c, \ell)}$-CA model: $\mathbf{C}$ has circumference $\mathbf{c}$, arcs have length ℓ.
- ℓ-IG model: arcs have length ℓ, no arc crosses $\mathbf{0}$.

Problem: Find the Minimal Model

- Given a PCA model, find an equivalent "minimal" model.
- A $\mathbf{(c, \ell)}$-CA model $\mathcal{M}$ is weakly minimal when
  1. $\ell_{\mathcal{M}} \leq \ell_{\mathcal{N}}$ and
  2. $\mathbf{c}_{\mathcal{M}} \leq \mathbf{c}_{\mathcal{N}}$.

  For any equivalent model $\mathcal{N}$. If also

  3. $\mathbf{s}_{\mathcal{M}}(\mathbf{A_i}) \leq \mathbf{s}_{\mathcal{N}}(\mathbf{A_i})$ for $\mathbf{1} \leq \mathbf{i} \leq \mathbf{n}$,

  then $\mathcal{M}$ is *strongly minimal*.
- **Remark:** the existence of minimal models is not obvious because 1–3 must hold **simultaneously**.
- **Pirlot**: every PIG model is equivalent to an integer strongly minimal model.
- **Soulignac**: every UCA model is equivalent to a weakly minimal model. Is it integer?
- **Conjecture**: $\mathbf{c}, \ell \in \mathbb{N}$ for every minimal $\mathbf{(c, \ell)}$-CA model.

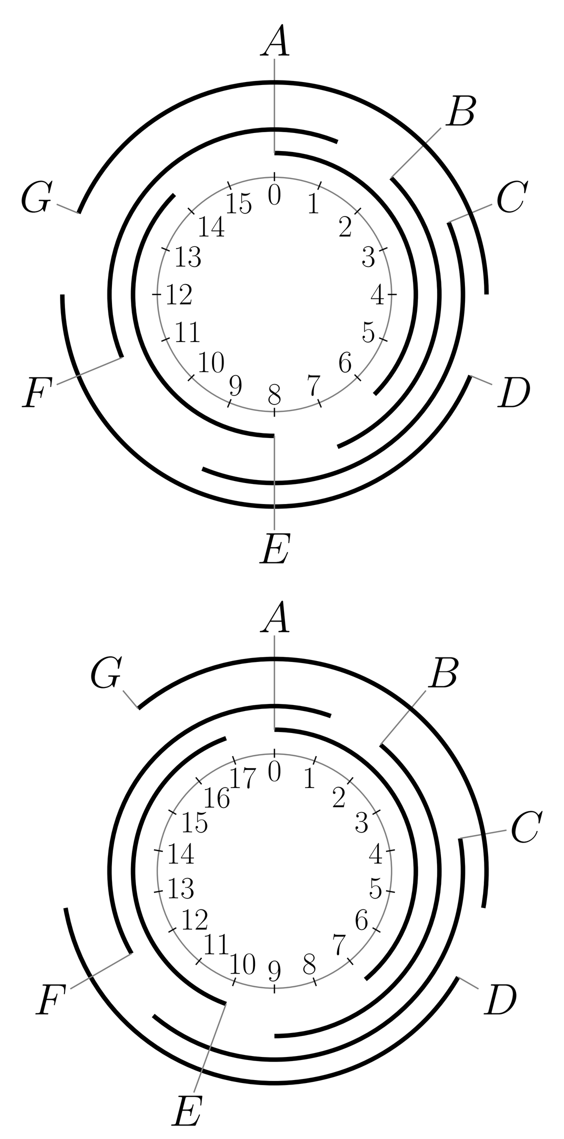


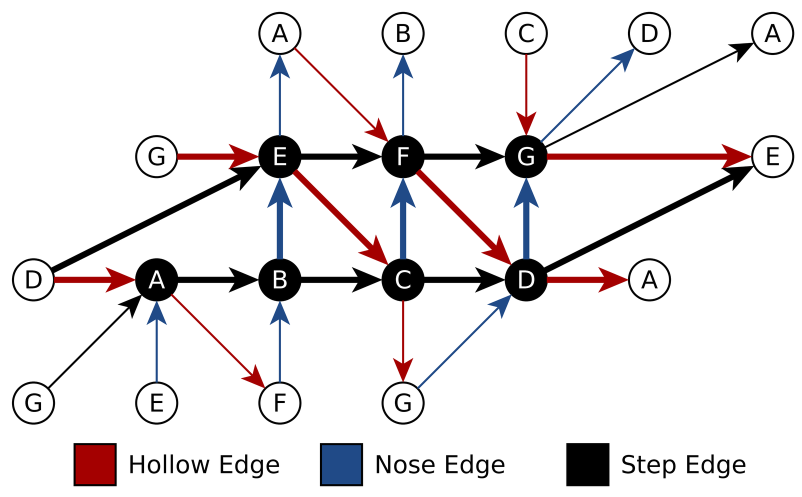
Figure 1: a PCA model and its equivalent minimal model.

Synthetic Graphs (Pirlot, Soulignac)

- For a PCA model $\mathcal{M}$ with arcs $\mathbf{A_1} < \ldots < \mathbf{A_n}$:
  - One vertex for each arc $\mathbf{A_1}, \ldots, \mathbf{A_n}$.
  - *steps*: $\mathbf{A_i} \to \mathbf{A_{i+1}}$ for $\mathbf{1} \leq \mathbf{i} \leq \mathbf{n}$.
  - *noses*: $\mathbf{v(A_i)} \to \mathbf{v(A_j)}$ when $\mathbf{t(A_i)s(A_j)}$ are consecutive in $\mathcal{M}$.
  - *hollows*: $\mathbf{v(A_i)} \to \mathbf{v(A_j)}$ when $\mathbf{s(A_i)t(A_j)}$ are consecutive in $\mathcal{M}$.
  - *weights*: separation of the endpoints in any equivalent $\mathbf{(c, \ell)}$-CA model.
- **Theorem:** equivalent to $\mathbf{(c, \ell)}$-CA iff no cycle has a positive weight.

Mitas' plane drawing of the synthetic graph

- Vertices: positions in a circular matrix (first column follows last column).
- Internal edges (thick): advance to the right and induce a plane drawing.
- External edges (thin): escape through the top and last rows.



- Consequently, every cycle has exactly one more hollow than nose.
- Hence, every cycle $\mathcal{W}$ has weight $(\ell + \mathbf{1}) + |\mathcal{W}|$
  - implying that the minimum feasible ℓ is integer.
- We exploit this drawing to settle Soulignac's conjecture.

A new characterization of UCA models

- Intervalization $\mathcal{I}(\mathcal{M}, \mathbf{k})$: copy $\mathbf{k}$ times the arcs of $\mathcal{M}$ in a PIG model.
  - **Nose (hollow) path**: contain no hollows (no noses).
  - **Nose-(hollow-)like**: more ext. noses (hollows) than ext. hollows (noses).
  - **Greedy nose (hollow):** prefer noses (hollows) to steps.
- Characterization of UCA models.
  - $\mathcal{M}$ is equivalent to a UCA model.
  - every nose-like circuit intersects every hollow-like circuit.
  - any greedy nose cycle intersects any greedy hollow cycle.
- Proof. Take advantage of the plane drawing of $\mathcal{S}(\mathcal{I}(\mathcal{M}, \mathbf{k}))$.

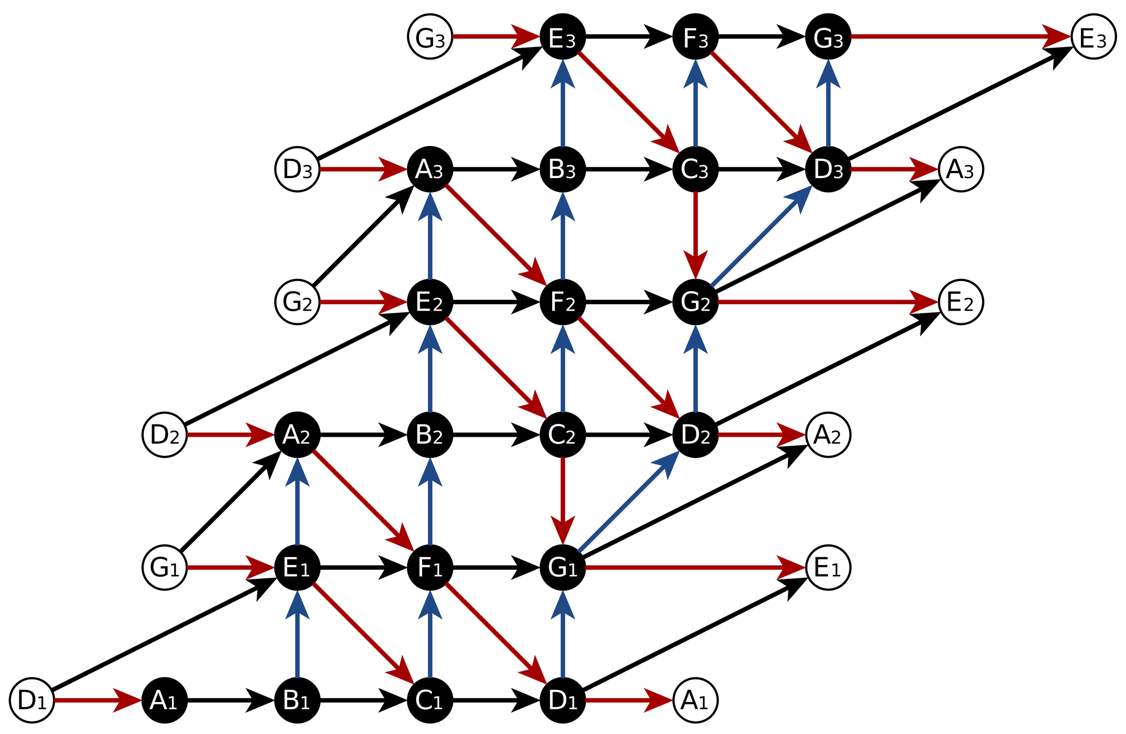


Figure 2: Plane drawing of $\mathcal{S}(\mathcal{I}(\mathcal{M}, \mathbf{3}))$ for the model $\mathcal{M}$ in Figure 1.

- Simpler $\mathbf{O(n)}$ time recognition algorithm of UCA graphs.

Minimal UCA models are integer

- Let $\mathcal{S}$ be the synthetic graph of a minimal $\mathbf{(c, \ell)}$-CA model $\mathcal{M}$.
- **Fact.** $\mathcal{S}$ has nose-like $\mathcal{W}_{\mathbf{N}}$ and hollow-like $\mathcal{W}_{\mathbf{H}}$ cycles with weight $\mathbf{0}$.
- From the common vertex of $\mathcal{W}_{\mathbf{N}}$ and $\mathcal{W}_{\mathbf{H}}$, we build a circuit of $\mathcal{S}$ with weight $\mathbf{0}$ that corresponds to a cycle of $\mathcal{S}(\mathcal{I}(\mathcal{M}, \mathbf{k}))$.
  - Hence, ℓ coincides with the minimal feasible length for $\mathcal{I}(\mathcal{M}, \mathbf{k})$.
  - Implying that ℓ is integer.
- Next, we build a nose-like circuit of $\mathcal{S}$ with one nose more than hollows.
  - Implying that $\mathbf{c}$ is integer.
- The algorithm to find $\mathcal{M}$ runs in $\mathbf{O(n^3)}$ time and $\mathbf{O(n^2)}$ space.

Conclusions

- We define the intervalization $\mathcal{I}$ of a PCA model $\mathcal{M}$.
  - Simpler characterization of UCA models.
  - Different properties about $\mathcal{S}(\mathcal{M})$ are found by studying $\mathcal{S}(\mathcal{I})$.
- We prove that ℓ and $\mathbf{c}$ are integer for minimal $\mathbf{(c, \ell)}$-CA models.
- We devise an $\mathbf{O(n^3)}$ time algorithm to compute a minimal $\mathbf{(c, \ell)}$-CA model equivalent to an input UCA model.

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