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Best choice problem for powers of paths

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Basic model

- The vertices of a kth power of directed path on n vertices P_n^k appear one by one in some random order ($1 \le k \le n-1$).
- At time t the selector can see the graph induced by the vertices that have already appeared.
- The selector can accept only one vertex.
- The selector can accept only the vertex that has just appeared.

Aim: maximizing the probability of accepting the vertex with no outgoing edges (denoted by $\mathbb{1}$, $v_1 = \mathbb{1}$).

Example for P_9^2

Random permutation of vertices here is $\pi = (v_2, v_9, v_4, v_7, v_3, v_1, v_5, v_8, v_6)$. Orange vertex is the one that the selector can accept at time t.

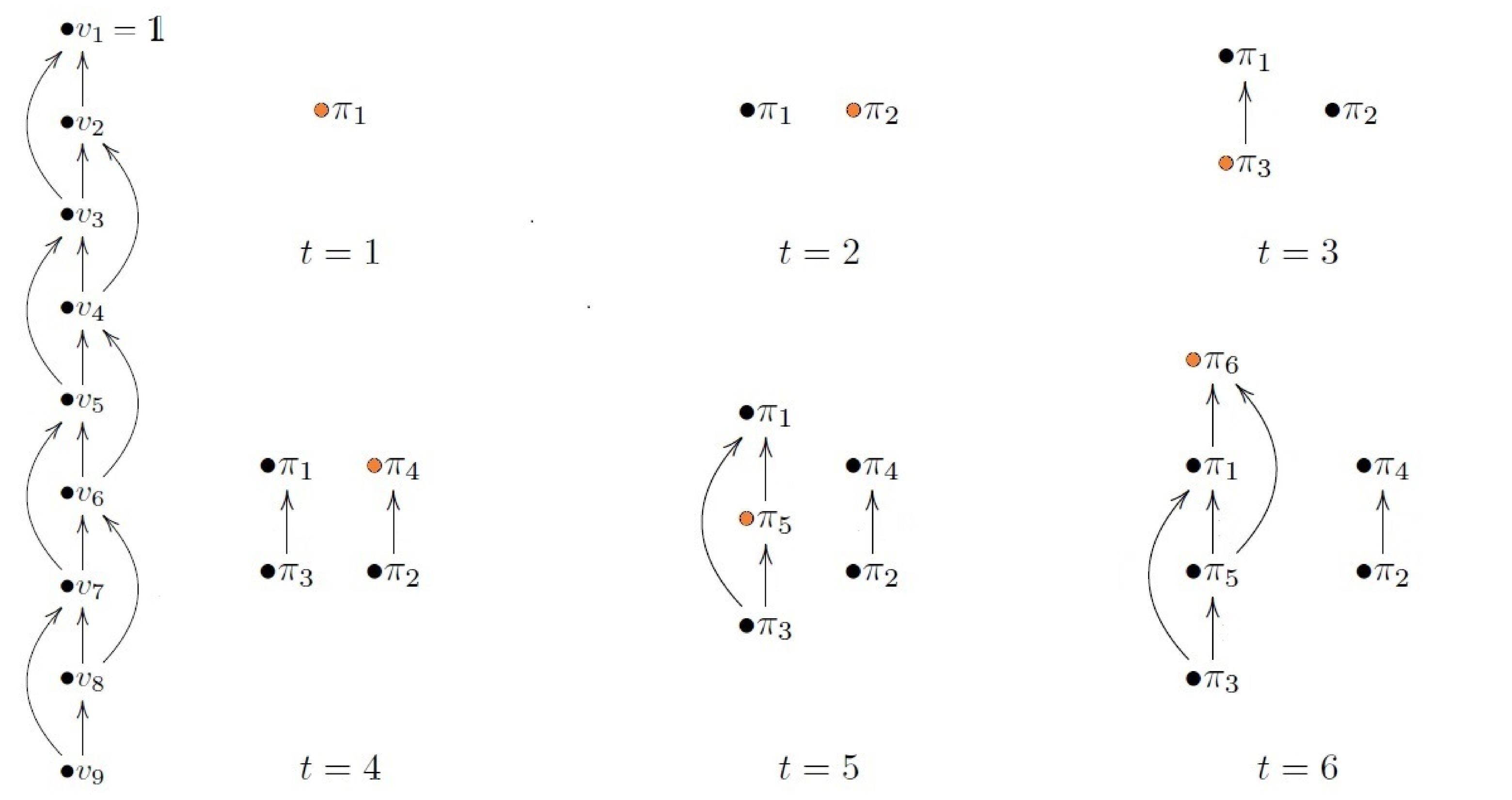

Figure : The sequence of induced graphs that the selector sees at time $t = 1, 2, \dots, 6$.

Results

Let $p_{n,k}$ denote the probability of accepting $\mathbb{1}$ under the optimal algorithm. In 2015 A.Grzesik, M.Morayne and M.Sulkowska showed that $p_{n,k}$ is of the order $n^{-1/(k+1)}$. We have found upper and lower bounds for the asymptotic behaviuor of $p_{n,k} n^{1/(k+1)}$.

	Lower bound	Upper bound
k is constant	$(k+2)^{-1/(k+1)} \frac{k+1}{k+2}$	$\Gamma\left(1+\frac{1}{k+1}\right)$
k(n) $\to \infty$ and k(n) $= o(\log n)$	1	1
k(n) $= c \log n$	$c \le (\log \frac{e}{e-1})^{-1}$ $(1-e^{1/c}) \log(1-e^{-1/c})$; $c > (\log \frac{e}{e-1})^{-1}$ $e^{1/c-1}$	$1 - 1/(2e^{1/c})$
k(n) $= \omega(\log n)$	$1/e$	$1/2$

Table : Lower bounds for $\liminf_{n\to\infty} n^{1/(k+1)}\ p_{n,k}$ and upper bounds for $\limsup_{n\to\infty} n^{1/(k+1)}\ p_{n,k}$.

The optimal algorithm is still not known!

Obtaining upper bound for $\limsup_{n\to\infty} n^{1/(k+1)}\ p_{n,k}$

We consider a *labelled model* in which the selector obtains extra information. While the vertices are being revealed, each edge of the graph induced by the observed vertices is labelled with the distance in P_n^k between its endpoints. The selector sees those labels. The optimal algorithm for accepting $\mathbb{1}$ in this case was found in 2015 by A.Grzesik, M.Morayne and M.Sulkowska.

Optimal algorithm for accepting $\mathbb{1}$ in labelled model

Accept π_m if it is maximal at time m and the probability that $\mathbb{1}$ is still to come is equal to 0. If it never happens, accept π_n.

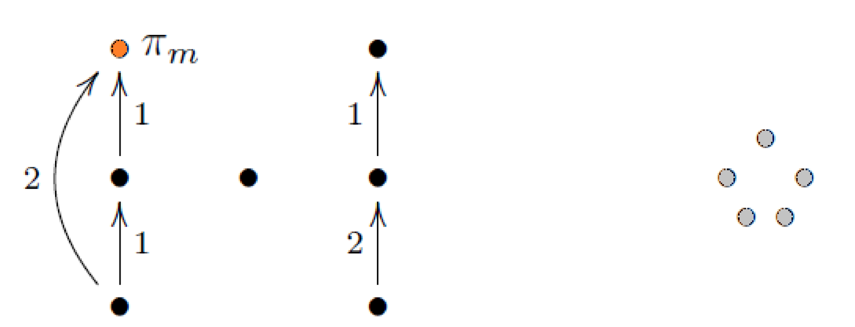

Figure : Game on P_{12}^2: example of the moment in which optimal algorithm for labelled model accepts π_m. Grey vertices haven't appeared yet.

When the length of the edges in the induced graph are known, the selector can take at least as efficient decision as when they are not known. The probability of success $\tilde{p}_{n,k}$ of the optimal algorithm in the labelled model is the upper bound for the probability of success $p_{n,k}$ of the optimal algorithm in the basic model. The order of $\tilde{p}_{n,k}$ is also $n^{-1/(k+1)}$ (A. Grzesik, M. Morayne, M. Sulkowska, 2015). We found the exact values of $\lim_{n\to\infty} \tilde{p}_{n,k} n^{1/(k+1)}$ for the whole range of k ($1 \le k \le n-1$) - see Table. To find them, we analysed, among others, the process of vertex percolation for a sequence of kth powers of a directed path.

Obtaining lower bound for $\liminf_{n\to\infty} n^{1/(k+1)}\ p_{n,k}$

Optimal algorithm performs at least as well as any other stopping time. Lower bound was obtained by analysing effectiveness of the following randomized algorithm.

Randomized algorithm for accepting $\mathbb{1}$ in basic model

Flip an asymmetric coin, having some probability p of coming down tails, n times. If it comes down tails M times, reject the first M elements. After this time accept the first element which is maximal in the induced graph. If it never happens, accept π_n.

We set the value of p as follows:

- for k being a constant, $p = 1 - \delta_k n^{-1/(k+1)}$, where $\delta_k = (k+2)^{-1/(k+1)}$,
- for $k = k(n) \xrightarrow{n\to\infty} \infty$ such that $k(n) = o(\log n)$, $p = 1 - \delta_n n^{-1/(k+1)}$, where δ_n is a function such that $\delta_n = 1 - \frac{1}{o(k(n))}$ and $\delta_n \xrightarrow{n\to\infty} 1$,
- for $k = k(n) = c \log n$, $p = 1 - \delta/n^{1/(k+1)}$, where $\delta = e^{1/c}(1 - 1/e)$ for $c > (\log \frac{e}{e-1})^{-1}$, and δ is arbitrarily close to 1 for $c \le (\log \frac{e}{e-1})^{-1}$,
- for $k = k(n) = \omega(\log n)$, $p = 1/e$.

To estimate the probability of success of the randomized algorithm for some values of k, we analysed the process of vertex percolation for a sequence of kth powers of a directed path.

Vertex percolation on P_n^k

Declare each vertex of a graph to be *open* with some probability p and *closed* otherwise, independently of all other vertcies. The structure *percolates* if there is an open passegeway in it "from one side to the other".

- Formulas for percolation probability.
  Let $\psi_{n,k}$ denote the probability that P_n^k percolates. For $n > k+1$
  - $\psi_{n,k} = \psi_{n-1,k} - p(1-p)^k \psi_{n-k-1,k}$,
  - $p^2(1-(1-p)^k)^{n-k} \le \psi_{n,k} \le p^2(1-p(1-p)^k)^{n-k-1}$.
- Percolation versus optimal algorithm for labelled case.
$$\tilde{p}_{n,k} = \int_0^1 \frac{\psi_{n,k+1}}{p} dp$$

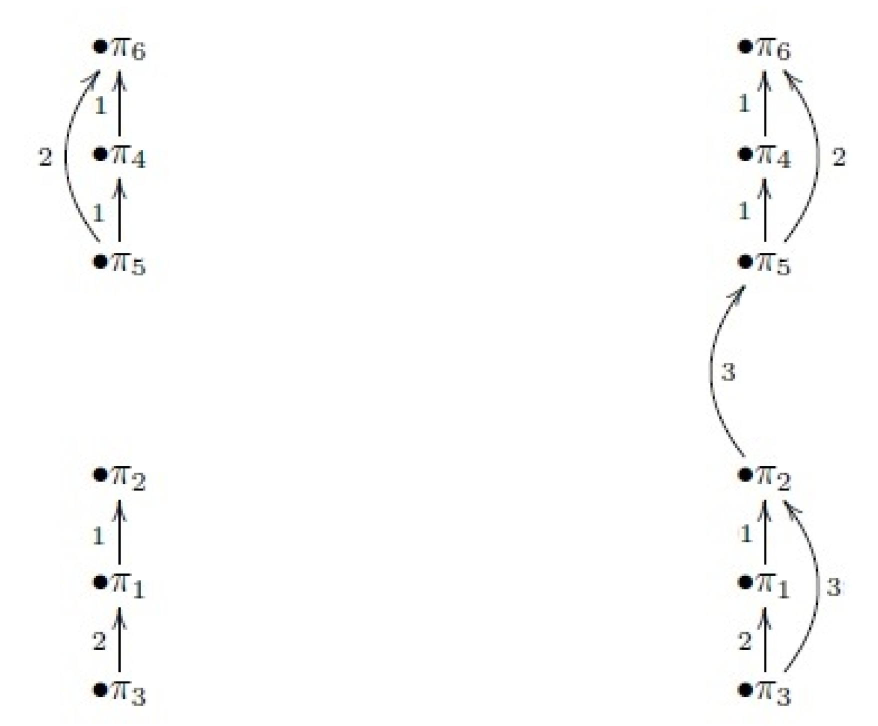

Figure : On the left, we have the induced graph at step 6 for the optimal stopping algorithm on labelled P_9^2, using the permutation $\pi = \{v_7, v_6, v_9, v_2, v_3, v_1, v_5, v_4, v_8\}$ as input; the algorithm accepts π_6. On the right, the graph induced by the same vertices in P_9^3 - P_9^3 percolates.

- Percolation versus randomized algorithm.
  Let $V_M = \{\pi_1, \pi_2, \dots \pi_M\}$, according to the definition of randomized algorithm. All vertices of P_n^k appear in V_M with probability p independently. We can associate the event of the vertex being open in the percolation model with the event of vertex being in V_M.

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