Containers and entropy of graph limits

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Main results

- We obtain multicolour containers for general dense, hereditary properties, generalising the results of Saxton and Thomason. We further give extensions of our results to cover directed graphs, oriented graphs, tournaments, multipartite graphs, multi-graphs, hypergraphs, and hypercubes.
- We generalise the graph limit entropy results by Hatami-Janson-Szegedy to decorated graph limits. In particular we define a cut norm for decorated graph limits, and prove compactness of the space of decorated graph limits under that norm.
- We explore a weak equivalence between the container and entropy of graph limit approach to counting and characterising graphs in hereditary properties. In one direction, we show how our multicolour containers may be used to fully recover decorated versions of the graph limit results. In the other direction, we show that our decoratedextensions of the graph limits results imply finitary container-type statements and counting, characterization and transference applications.

What are graph limits?

Consider a sequence of graphs G_n, $n \to \infty$:

The sequence converges if the probability that a uniform random mapping of any graph H preserves edge adjacencies converges.

Graph limits can be represented as symmetric measurable functions

$$W : [0,1]^2 \to [0,1]$$

Decorated graph limits are functions

$$W : [0,1]^2 \to (s_1, \dots s_k)$$

where

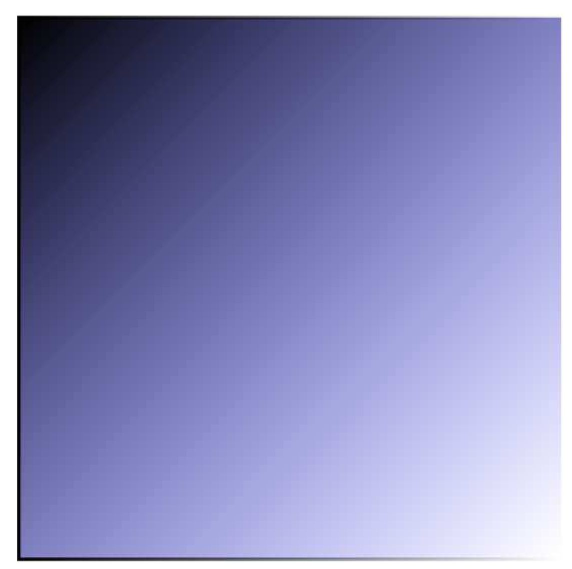
$$0 \le s_i \le 1, \quad \sum_i s_i = 1$$

and can be thought of as limits of k-edge-coloured graphs.

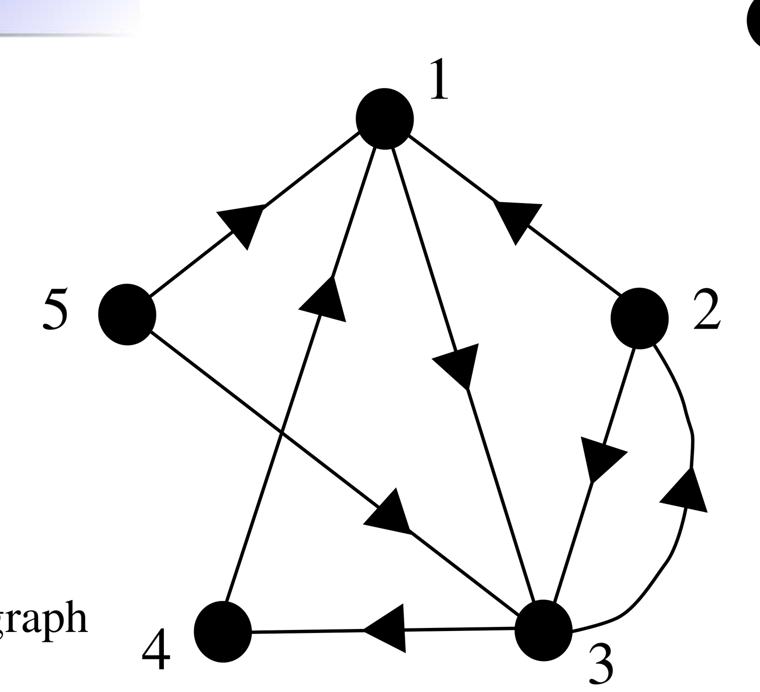
- Graph limits give us some idea of the structure of the graphs in a sequence.
- Using Hatami-Jansen-Szegedy, we can count the number of graphs in a graph property using the entropy of a graphon under certain circumstances.

What are containers?

There has been great interest in transference results, where central results in extremal combinatorics have been shown to hold in a sparse random setting. Containers are 'almost' independent sets of vertices that 'almost' cover a multigraph. In a breakthrough two years ago, Balogh-Morris-Samotji and Saxton-Thomasson developed a theory of containers that produced a spate of new and old counting and characterisation results, as well as transference results.



A graphon



A directed graph

Counting with graph limits

Theorem (Hatami-Jansen-Szegedy)**.** *Let $\mathcal{Q}$ be a hereditary graph property. Then*

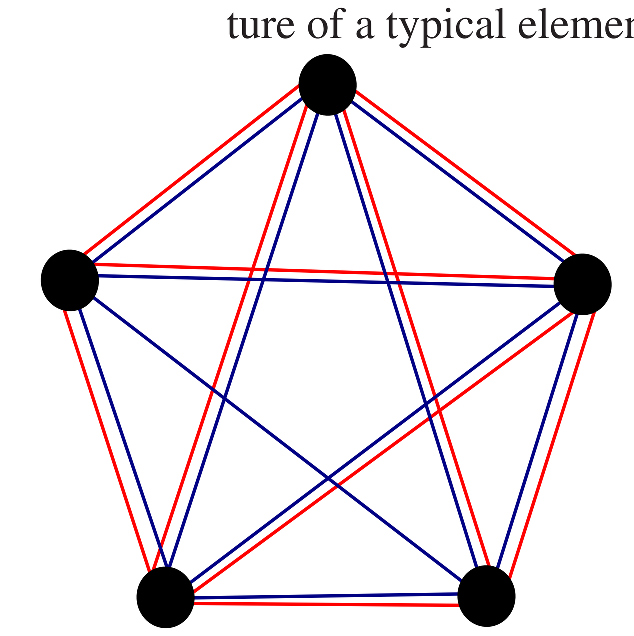
$$\lim_{n\to\infty} \frac{\log_2 |\mathcal{Q}_n|}{\binom{n}{2}} = \max_{\Gamma \in \hat{\mathcal{Q}}} Ent(\Gamma)$$

Theorem (Falgas-Ravry-O'Connell-Strömberg-Uzzell)**.** *Let $\mathcal{Q}$ be a hereditary property of k-coloured complete graphs. Then*

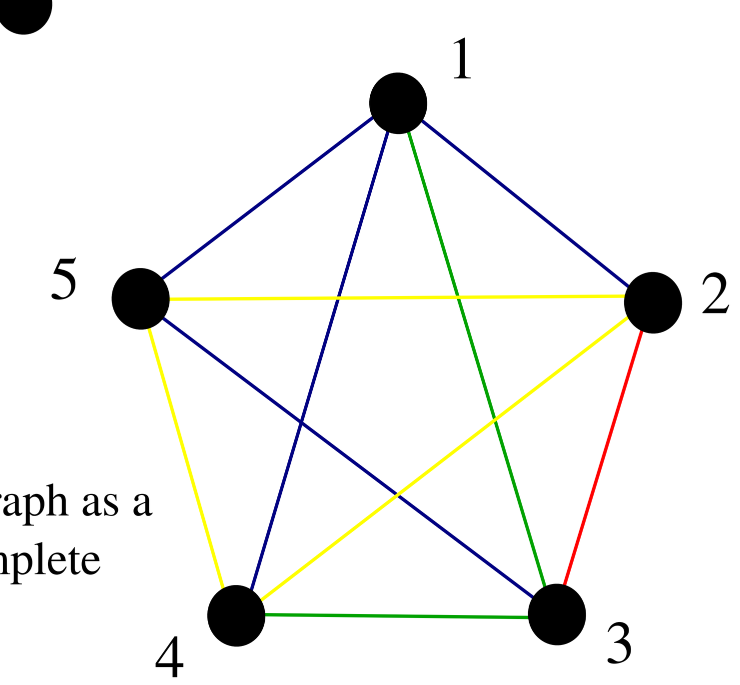
$$\lim_{n\to\infty} \frac{\log_k |\mathcal{Q}_n|}{\binom{n}{2}} = \max_{\Gamma \in \hat{\mathcal{Q}}} Ent(\Gamma)$$

Motivation

The problem of counting and characterising graphs in a given symmetric property has a long and distinguished history. The *speed* of a property was introduced by Erdõs, Kleitman and Rothschild in 1976. Together with the structural properties of a 'typical' element of the property, it has recieved considerable interest from the research community. In this paper, the we explore the relation between the speed of a property and the entropy of an entropy maximising graph limit, as well as what the structure of the entropy maximising graph limit can tell us about the structure of a typical element in the property.



A template of maximal entropy for a rainbow triangle-free graph on 5 vertices



The directed graph as a 4-coloured complete graph

Which graphs can we count?

Many natural graphs can be encoded as properties of multicoloured labelled complete graphs. To name two:

- Digraphs can be encoded by considering 4-coloured K_n, where colour 1 indicates no edge is present, colour 2 indicates that the edge ij, $i < j$ is present, colour 3 that the edge ji, $j > i$ is present, and colour 4 that edges in both directions are present.
- Touranments can be encoded similarly.

Counting graphs with no rainbow triangle

Let $k = 3$ and $\mathcal{P}$ be the property of not having a rainbow triangle, where a rainbow triangle is one with each edge a different colour. This property is clearly hereditary, so we can apply our counting result to it, if we can figure out the maximal entropy of a graph limit in the closure of the property.

Definition. *A* template *for a property is an assigment to each edge of K_n of a non-empty list of colours.*

Theorem. *The maximal entropy is $\log_3(2)$, obtained from the template with each edge list containing just the same two colours.*

Proof. (Idea) Suppose one edge of our template of K_n is coloured (s_1, s_2). Then rainbow K_3-freeness implies three possibilites for the other edges, and maximal entropy when each other edge is coloured (s_1, s_2). □

Therefore, by our counting theorem,

$$|\mathcal{P}_n| = 2^{\binom{n}{2} + o(1)}$$

That is, the 3-coloured rainbow-K_3-free complete are asymtotically the two-coloured complete graphs.

Further reading

Simple containers for simple hypergraphs

Graph properties, graph limits and entropy

Independent sets in hypergraphs

Do you want to know when we post the paper on ArXiV? Email me at johanna.stromberg@math.uu.se

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