Study of k-biclique edge-choosability in some graph classes

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1. Introduction

When we talk about *graph coloring* we usually think about for which value of $k \in \mathbb{N}$ a graph has a proper vertex or edge coloring with k colors. However it's possible to add two concepts to change this problem by creating another version of graph coloring.

non-monochromatic structure color the edges of a graph such that there is no monochromatic biclique or star, for example.

k**-choosability** check if the graph is k-colorable for any list assignment of size k.

Putting these two concepts together we have the problem k-biclique (star) edge-choosability as to color the edges of graph such that there is no monochromatic biclique (star) for any list assignment of size k. We can present them as :

Problem: k**-biclique edge-choosability**
Instance: Given a connected graph G and $k \in \mathbb{N}$
Question: Is G k-biclique edge-choosable ?

Problem: k**-star edge-choosability**
Instance: Given a connected graph G and $k \in \mathbb{N}$
Question: Is G k-star edge-choosable ?

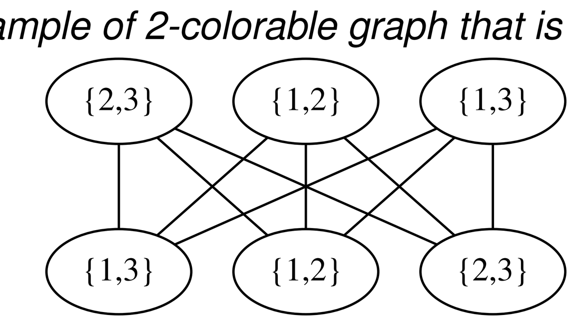
We'll present the motivation and the objective of this work in the following sections. Please note that this research is still in progress so we'll show the results we got so far.

1.1 Motivation

Basically the motivation of this work comes from two facts :

1. k-colorable does not necessarily implies em k-choosable [3, 1].

Figure 1: *Example of 2-colorable graph that is not 2-choosable*



2. Extend the results [2] of biclique edge-coloring to biclique edge-choosability.

The paper [2] shows that K_3-free is 2-star edge-colorable, chordal bipartite is 2-biclique edge-colorable, power of cycles and paths are biclique edge-colorable using at most 4 colors. We shall investigate how the results of [2] behaves when the choosability concept is used. This work is also based on the results of [7, 4, 6, 5, 8].

1.2 Objectives

- Find the complexity of k-biclique edge-choosability in general case;
- Prove K_3-free is 2-star edge-choosable and 2-biclique edge-choosable, if possible;
- Show that chordal bipartite is 2-biclique edge-choosable;
- And prove that power of cycles and paths is biclique edge-choosable for at most 4 colors.

2. Preliminary Results

As this research is still going on then we'll show two results we got. They were obtained by adapting the proofs of biclique edge-coloring from [2] to biclique edge-choosability. Basically we had to show it's possible to color the edges of a graph for any list assignment of size 2 such that there is no monochromatic star for K_3-free and biclique for chordal bipartite. We show this in mode details in the following subsections.

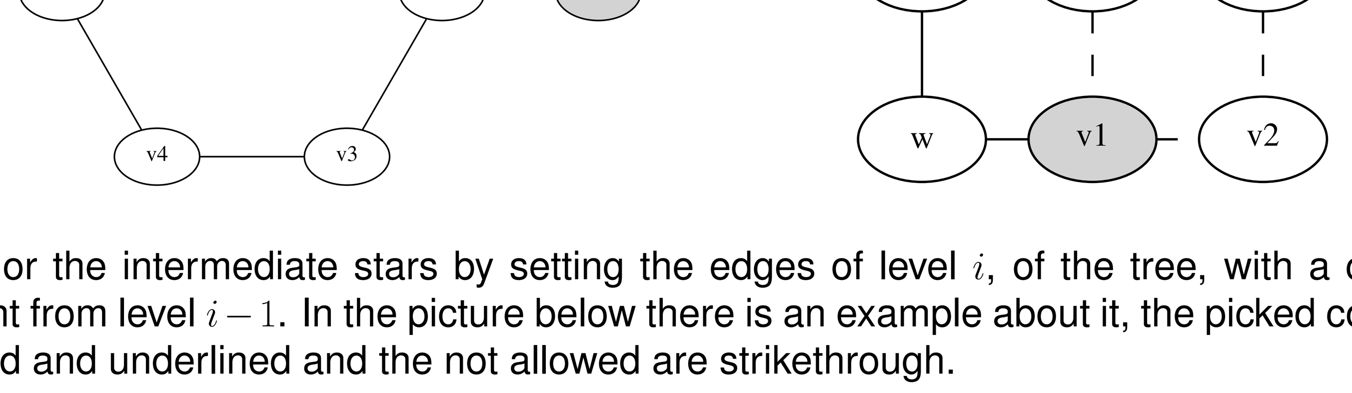
2.1 K_3-free 2-star edge-choosable

We'll show a short version of the proof showing that K_3-free is 2-star edge-choosable.

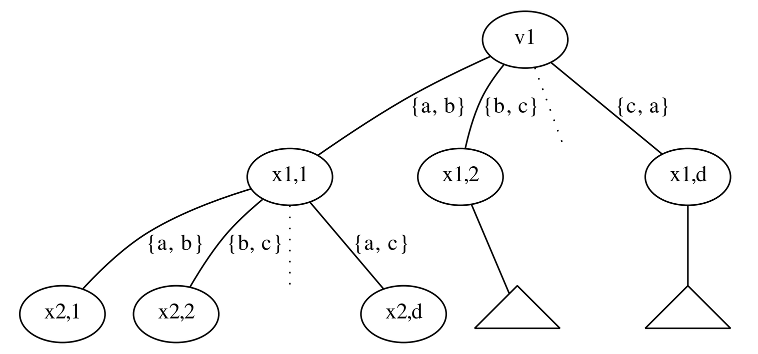
Theorem 1. *If G is a connect graph, K_3-free, not isomorphic to C_n with n odd such that $n \geq 5$ then G is 2-star edge-choosable.*

Proof. To proof the theorem 1 we used the tree produced in a *depth first search* to color the edges such that there is no monochromatic star.
First of all, if the graph G is an even cycle then it's 2-star edge-choosable since G is 2-edge choosable, according to [3]. As G can't be an odd cycle then it has an induced cycles that can colored using the following strategy :

1. First let's pick the starting vertex to build the tree. If the graph G has a leaf then set v_1. Otherwise there is a cycle $C = \{v_1, \dots, v_k, v_1\}$ such that $\{u, v_k\} \in E(G)$ and $u \notin C$ then we choose $v_1 \in C$. These two scenarios are shown in the picture below and v_1 is marked.



2. We color the intermediate stars by setting the edges of level i, of the tree, with a color different from level $i-1$. In the picture below there is an example about it, the picked colors are bold and underlined and the not allowed are strikethrough.



3. The return edges are colored by splitting them into cases.
   (a) If they come from intermediate stars then we pick a color different from the edges of the child of the root. If two are already in use then we already have a non-monochromatic star.
   (b) If the they are from the leaves to the root then we pick a color different from the edge of the tree.
   (c) After resolving these cases above we can set any color to a return edge, if any was left.

□

2.2 Chordal Bipartite 2-biclique edge-choosable

In this subsection we present a simplified version of the proof that every chordal bipartite is 2-biclique edge-choosable.

Theorem 2. *Every chordal bipartite graph is 2-biclique edge-choosable.*

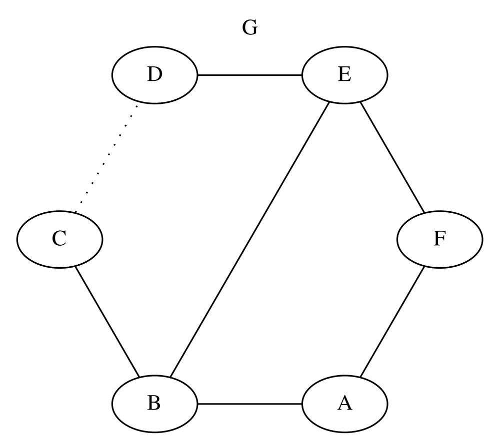
Proof. To that any chordal bipartite graph G is 2-biclique edge-choosable we'll do this by induction on its *perfect edge without vertex elimination ordering*. This order is important because it allow us to traverse all maximal bicliques of G.

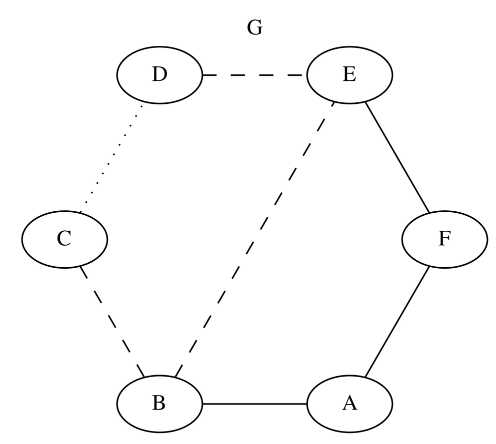
base if G has two edges then it's always possible to pick up a color for each one.

hypotheses if $uv \in E(G)$ and uv is bisimplicial then it belongs to only one biclique, therefore $G \setminus uv$ is 2-biclique edge-choosable.

step color $e \in E(G) \setminus uv$ with a color and another for uv, as list size is 2. If the color for uv is already chosen then the biclique is not monochromatic.

In the example below the first picture the bisimplicial edge is dotted. And in the second the biclique edges are dashed.





□

3. What's next

As this work is still in progress the following task is going to executed :

- Adapt the results of [2] from biclique edge-coloring to choosability, which is :
  - Find the complexity of 2-biclique edge-choosability in general case.
  - prove k-biclique edge-choosability for power of cycles and paths, with $k \leq 4$.
- Find for which k K_3-free is k-biclique edge-choosable.
- Elaborate the algorithms to compute the biclique edge-choosability for a graph and a list assignment for the classes K_3-free, chordal bipartite, power of cycles and power of paths. And also find its complexity.

References

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