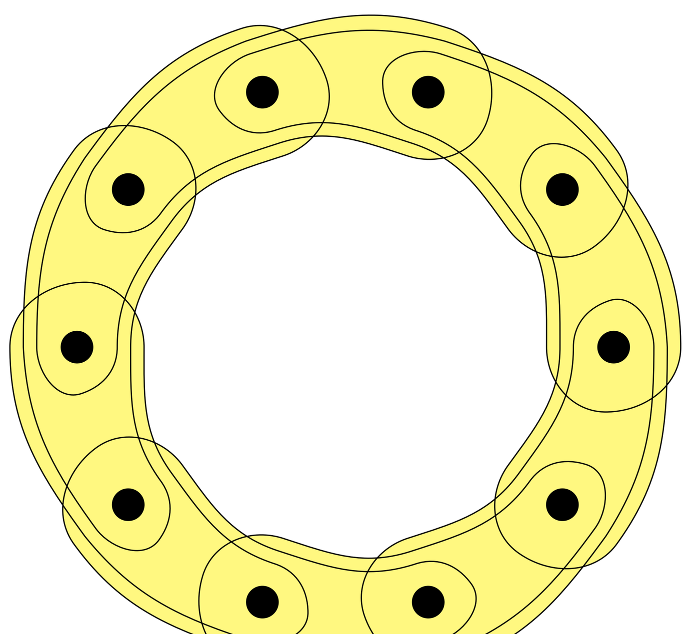
Minimum codegree threshold for covering k-uniform hypergraphs with tight cycles

Nicolás Sanhueza-Matamala (nicolas@sanhueza.net)
University of Birmingham, United Kingdom

k-graphs, tight cycles and codegree

A k-**graph** $\mathcal{H} = (V, E)$ consists of a vertex set V and an edge set E, where each edge $e \in E$ is a subset of V of size exactly k. Given a k-graph $\mathcal{H} = (V, E)$ and $S \subseteq V$, let $\deg_{\mathcal{H}}(S)$ denote the number of edges of $\mathcal{H}$ containing the set S. We define the **minimum codegree** $\delta_{k-1}(\mathcal{H})$ **of** $\mathcal{H}$ to be the minimum of $\deg_{\mathcal{H}}(S)$ over all $(k-1)$-element sets S.



C^3_{10}, a tight cycle on 10 vertices.

Given $k \le s$, we say that a k-graph C^k_s is a **tight cycle** on s vertices if there is a cyclic ordering of its vertices such that every k consecutive vertices under this ordering form an edge, and no other edges are present.

Covering problem

Given hypergraphs $\mathcal{H}$ and $\mathcal{F}$, we say that $\mathcal{H}$ has an $\mathcal{F}$**-covering** if every vertex in $\mathcal{H}$ is contained in a copy of $\mathcal{F}$. Define

$$c(n, \mathcal{F}) = \max\{\delta_{k-1}(\mathcal{H}) : |V(\mathcal{H})| = n \text{ and } \mathcal{H} \text{ is a } k\text{-graph without an } \mathcal{F}\text{-covering}\}.$$

This value is called **minimum codegree covering threshold for** $\mathcal{F}$. It was introduced by Falgas-Ravry and Zhao [2], who studied $c(n, C^3_4)$ and $c(n, C^3_5)$.

Question: Can we determine $c(n, C^k_s)$?

Theorem (Han–Lo–S., 2016⁺)

For every $s \ge 2k^2$ and $k \ge 2$,

$$c(n, C^k_s) \le \begin{cases} o(n) & \text{if } s \equiv 0 \bmod k, \\ \left(\frac{1}{2} + o(1)\right) n & \text{if } s \not\equiv 0 \bmod k. \end{cases}$$

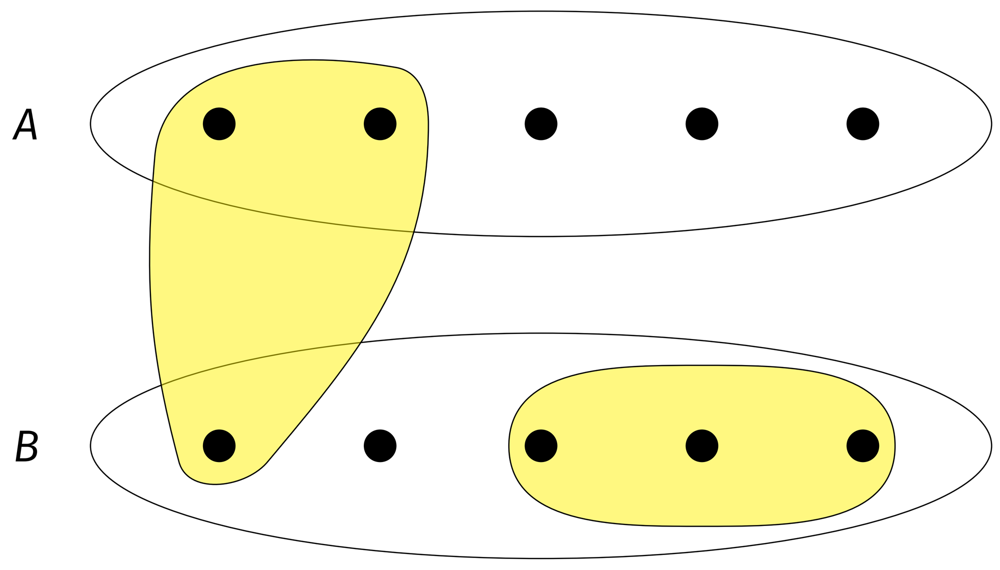
Moreover, the bound is tight if $s \equiv 0 \bmod k$ or $\gcd(s, k) = 1$.

Extremal example for $\gcd(s, k) = 1$

Let $\mathcal{H}$ be a k-graph such that

$$V(\mathcal{H}) = A \cup B, \text{ with } |A| = \lfloor n/2 \rfloor, |B| = \lceil n/2 \rceil,$$
$$E(\mathcal{H}) = \{e \subseteq V(\mathcal{H}) : |e| = k, |e \cap A| \not\equiv k \bmod 2\}.$$

Note that $\delta_{k-1}(\mathcal{H}) \ge \lfloor n/2 \rfloor - k + 1$.



Note that if $v_1 \dots, v_\ell$ is a tight cycle in $\mathcal{H}$, then $v_i \in A$ implies $v_{i+k} \in A$ for all i. If $v_1 \in A$, then $\gcd(\ell, k) > 1$. Therefore no vertex in A is contained in a C^k_s.

Proof sketch for $s \equiv 0 \bmod k$

Given $k \ge 2$ and $t \ge 1$, let $K^k(t)$ be the **complete** k**-partite** k**-graph**, that is, the k-graph with vertex set $V(K^k(t)) = V_1 \cup \dots \cup V_k$ such that $|V_i| = t$ for all $1 \le i \le k$, the sets $\{V_1, \dots, V_k\}$ are pairwise disjoint and $E(K^k(t)) = \{e \subseteq V(K^k(t)) : |e \cap V_i| = 1 \text{ for all } 1 \le i \le k\}$.

Theorem (Erdős, [1])

For every $k \ge 2$ and $t \ge 1$, any k-graph on n vertices with at least $\Omega(n^k)$ edges has a copy of $K^k(t)$.

We use this result of Erdős to prove a covering result for complete k-partite k-graphs. If k divides s, then C^k_s is a spanning subgraph of $K^k(s/k)$, so the covering threshold for C^k_s follows as a corollary.

Proposition 1

For every $k \ge 2$ and $t \ge 1$, $c(n, K^k(t)) = o(n)$. Moreover, $c(n, C^k_s) = o(n)$ if $s \equiv 0 \bmod k$.

Proof sketch for $s \not\equiv 0 \bmod k$

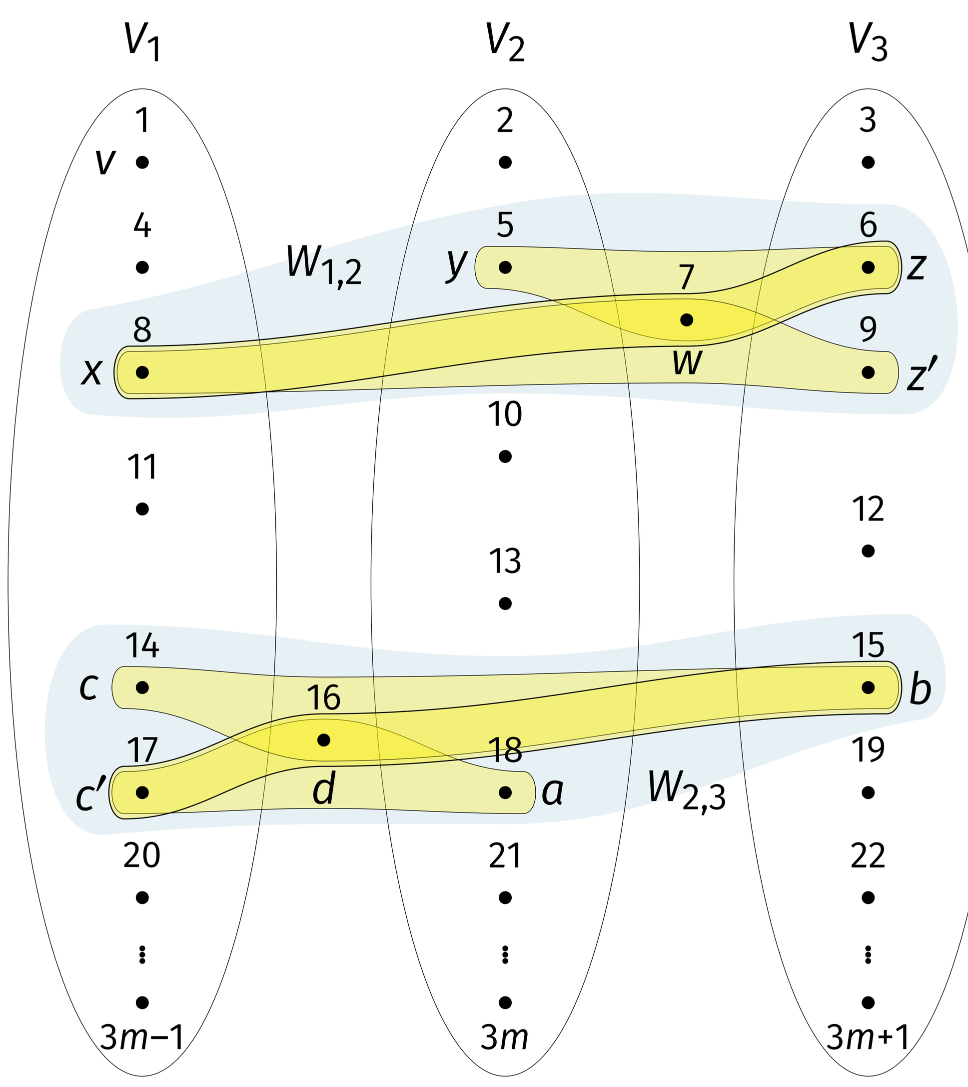
We give a sketch for $k = 3$. Let $\mathcal{H}$ be a 3-graph on n vertices with $\delta_2(\mathcal{H}) \ge (1/2 + o(1))n$, and let $v \in V(\mathcal{H})$ be any vertex. By Proposition 1, v is contained in a copy of $K^k(t)$ with vertex sets V_1, V_2 and V_3, for some fixed t to be defined later. Suppose $v \in V_1$.

Definition

A $\{1,2\}$**-gadget** is $W_{1,2} = \{x, y, z, z', w\}$ such that $x \in V_1$, $y \in V_2$, and $z, z' \in V_3$ and $w \in V(\mathcal{H}) \setminus (V_1 \cup V_2 \cup V_3)$ and $yzwxz'$ is a tight path in $\mathcal{H}$. Similarly, a $\{2,3\}$**-gadget** is $W_{2,3} = \{a, b, c, c', d\}$ such that $a \in V_2$, $b \in V_3$, $c, c' \in V_1$, $d \in V(\mathcal{H}) \setminus (V_1 \cup V_2 \cup V_3)$ and $acdc'b$ is a tight path in $\mathcal{H}$.

By the codegree condition, every pair in $V_1 \times V_3$ or $V_2 \times V_3$ has at least $(1/2 + o(1))n$ neighbours. By averaging, we find a vertex $w \notin V_1 \cup V_2 \cup V_3$ such that $N(w)$ contains at least $(1 + o(1))t^2$ pairs in $(V_1 \times V_3) \cup (V_2 \times V_3)$. If t is large enough, we can find x, y, z, z' as in the definition of a $\{1,2\}$-gadget. The existence of a $\{2,3\}$-gadget is proved analogously.

By adding these two gadgets we find a tight cycle covering v of any length $3m + 1 \ge 2k^2 + 1 = 19$, by following the vertices in the order shown in the diagram below.



Future work: tiling thresholds

We say that $\mathcal{H}$ has a C^k_s**-factor** if there are vertex-disjoint copies of C^k_s covering every vertex of $\mathcal{H}$. Define

$$t(n, C^k_s) = \max\{\delta_{k-1}(\mathcal{H}) : |V(\mathcal{H})| = n \text{ and } \mathcal{H} \text{ is a } k\text{-graph without a } C^k_s\text{-factor}\}.$$

We call this value the **minimum codegree tiling threshold for** C^k_s.
When $s \equiv 0 \bmod k$, the asymptotic value of $t(n, C^k_s) = (1/2 + o(1))n$ follows from a result of Mycroft [3], who studied K-factors for complete k-partite k-graphs K. We give lower bounds for other cases.

Proposition 2

For $3 \le k < s \le n$ with $s | n$, $t(n, C^k_s) \ge (1/2 + o(1))n$. Moreover, if $\gcd(s, k) = 1$,

$$t(n, C^k_s) \ge \begin{cases} \left\lfloor \left(\frac{1}{2} + \frac{1}{2s}\right) n \right\rfloor - k & \text{if } k \text{ is even,} \\ \left\lfloor \left(\frac{1}{2} + \frac{k}{4s(k-1) + 2k}\right) n \right\rfloor - k & \text{if } k \text{ is odd.} \end{cases}$$

As a work in progress, we think we can prove that for every $s \not\equiv 0 \bmod k$, $s \ge 2k^2$, $n \equiv 0 \bmod s$,

$$t_{k-1}(n, C^k_s) \le \left(\frac{1}{2} + \frac{1}{2s} + o(1)\right) n,$$

which would be asymptotically tight for even k and $\gcd(s, k) = 1$.

References

[1] Paul Erdős. "On extremal problems of graphs and generalized graphs". In: *Israel J. Math.* 2.3 (1964), pp. 183–190.
[2] Victor Falgas-Ravry and Yi Zhao. "Codegree thresholds for covering 3-uniform hypergraphs". In: *arXiv preprint arXiv:1512.01144* (2015).
[3] Richard Mycroft. "Packing k-partite k-uniform hypergraphs". In: *J. Combin. Theory Ser. A* 138 (2016), pp. 60–132.